KALUZA-KLEIN TYPE THEORY

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I would like to discuss two problems related to the recently proposed unified theory of Kaluza-Klein[1] type by T. Kaneko and myself[2]. One is Mach's principle and the other is a classical solution to Einstein's equation which has a $[\delta(X)]^2$ singularity.

Let us start with Mach's principle. Mach stated this principle rather vaguely implying that the inertial mass may be determined by the distant matter or by the entire matter in the universe. I would like to show in this talk that this situation can be realized in a very simple model. I am aware of the existence of the long history of discussions on Mach's principle, mostly within the framework of Einstein theory[3]. I ask this learned audience to forget about these discussions for the time being and listen to may naive approach.

Suppose that our universe is composed of only two particles. How can we write down an action which incorporates Mach's principle in this simplified case? The conventional classical action which describes two particle system is;

$$L = \int L(t)dt = \int [\tfrac{1}{2}m_1(\dot{\vec{X}}_1)^2 + \tfrac{1}{2}m_2(\dot{\vec{X}}_2)^2 - V(\vec{X}_1, \vec{X}_2)]dt \qquad (1)$$

Here the masses m_1 and m_2 are free parameters and there is no way to relate these parameters to the motion of the particles. Let us consider, therefore, the following action which might look weird to most of you.

$$L = \int L_1(\dot{q}_1, q_1, q_2)dt \times \int L_2(\dot{q}_2, q_1, q_2)dt \quad , \qquad (2)$$

where the form of L_1 and L_2 will be fixed later. Here we restrict ourselves to one space dimension for simplicity. We also assume that there exists a natural mass unit which we take to be 1. We get the following set of Euler's equations from equation (2):

$$\frac{\partial L_1}{\partial q_1}\bar{L}_2 + \frac{\partial L_2}{\partial q_1}\bar{L}_1 - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial \dot{q}_1}\right)\bar{L}_2 = 0 \quad , \qquad (3a)$$

and

$$\frac{\partial L_2}{\partial q_2}\bar{L}_1 + \frac{\partial L_1}{\partial q_2}\bar{L}_2 - \frac{\partial}{\partial t}\left(\frac{\partial L_2}{\partial \dot{q}_2}\right)\bar{L}_1 = 0 \quad , \qquad (3b)$$

where

$$\bar{L}_1 = \int L_1(q_1', q_1, q_2)dt \tag{4a}$$

and

$$\bar{L}_2 = \int L_2(q_2', q_1, q_2)dt \quad . \tag{4b}$$

Let us now write down an explicit form of L_1 and L_2:

$$L_1 = \frac{1}{2} q_1'^2 - V(q_1-q_2) \quad , \tag{5a}$$

and

$$L_2 = \frac{1}{2} q_2'^2 - V(q_2-q_1) \quad . \tag{5b}$$

Then equations (3a) and (3b) reduce to the following forms respectively:

$$\bar{L}_2 q_1'' = - \frac{\partial V_1}{\partial q_1} \quad , \tag{6a}$$

and

$$\bar{L}_1 q_2'' = - \frac{\partial V_2}{\partial q_2} \quad , \tag{6b}$$

where

$$V_1 = V_2 = (\bar{L}_1 + \bar{L}_2)V \quad . \tag{7}$$

The meaning of equations (6a) and (6b) is clear: The mass of particle 1 is equal to $\bar{L}_2$ and the mass of particle 2 is equal to $\bar{L}_1$ implying that the mass of a particle is determined by the motion of the other particle which is interacting with it. By adding equation (6a) to equation (6b) and taking equation (7) into account we obtain

$$\bar{L}_2 \dot{q}_1 + \bar{L}_1 \dot{q}_2 = \text{constant} \quad , \tag{8}$$

which is nothing but momentum conservation. Equations (6a) and (6b) reduce to the following equation in terms of the relative coordinate $q = q_1 - q_2$:

$$mq'' = - \frac{\partial V(q)}{\partial q} \quad , \tag{9}$$

where

$$m = \bar{L}_1 \bar{L}_2 / (\bar{L}_1 + \bar{L}_2)^2 \quad , \tag{10}$$

and

$$V(q) = V_1 = V_2 \quad . \tag{11}$$

We obtain from equation (9) the following energy integral:

$$\frac{1}{2} mq'^2 + V(q) = E \quad . \tag{12}$$

From equation (12) and the definition of L_1 we get

$$L_1 = \frac{1}{2}\left(\frac{\bar{L}_1}{\bar{L}_1+\bar{L}_2}\dot{q}\right)^2 - V(q)$$

$$= \left(\frac{\bar{L}_1}{\bar{L}_2}\right)E - \left(1 + \frac{\bar{L}_1}{\bar{L}_2}\right)V(q) \quad . \tag{13}$$

we have, therefore,

$$\bar{L}_1 = \frac{\sqrt{k}}{1+k}\int [kE - (1+k)V]\frac{da}{\sqrt{2(E-V)}} \quad , \tag{14a}$$

and

$$\bar{L}_2 = \frac{\sqrt{1/k}}{1+k}\int [E - (1+k)V]\frac{dq}{\sqrt{2(E-V)}} \quad , \tag{14b}$$

with

$$k = \bar{L}_1/\bar{L}_2 \quad . \tag{15}$$

We get from equations (14a), (14b) and (15);

$$k = 1 \quad . \tag{16}$$

Let us now consider the special case when

$$V(q) = \frac{1}{|q| - a} \quad . \tag{17}$$

Two particles start separating from each other after the 'big bang' at $t = 0$ ($q(t=0) = 0$) and reach the largest separation distance at $t = \infty$ ($q(t=\infty) = a$). From equation (14a) and (14b) we obtain

$$m_1 = m_2 = \bar{L}_1 = \bar{L}_2 = \frac{1}{2\sqrt{2}}\int_o^a \left[\frac{2\sqrt{a}}{\sqrt{q(a-q)}} - \frac{1}{\sqrt{a}}\sqrt{\frac{a-q}{q}}\right]dq \quad . \tag{18}$$

We, therefore, conclude that the mass of a particle is directly related to the size of the universe. To show that the form given in equation (2) is not too peculiar I will prove that the ordinary Maxwell's theory with an unspecified value of the electric charge can be written in this form. We take scalar electrodynamics as an example:

$$L = \left[-\frac{1}{4}\int d^4X \; F_{\mu\nu}F^{\mu\nu}\right] \times \int [(D_\mu\phi)^\dagger(D^\mu\phi) - V(|\phi|)]d^4X \quad , \tag{19}$$

$$\equiv L_A \cdot L_\phi \quad ,$$

with $D_\mu = \partial_\mu - iA_\mu$ where A_μ is the usual vector potential multiplied by the charge e. ϕ is a complex scalar field.

The equation of motion we get for A_μ is:

$$\partial_\mu F^{\mu\nu} + \frac{L_A}{L_\phi} J^\nu(x) = 0 \quad , \tag{20}$$

where

$$J^\nu(x) = i(\phi^\dagger D^\nu\phi - (D^\nu\phi)^\dagger\phi) \quad . \tag{21}$$

The equation for scalar field is conventional. Multiplying $A_\nu(X)$ to equation (20) and integrating over the whole 4-dimensional space, we obtain

$$2L_\phi = -\int J^\nu A_\nu d^4X \quad . \tag{22}$$

The equation (20), therefore, can be rewritten as

$$\partial_\mu F^{\mu\nu} - \frac{2L_A}{\int J^\nu A_\nu d^4x} J^\nu = 0 \quad . \tag{23}$$

This equation shows that we have

$$e^2 = \frac{-2L_A}{\int J^\nu A_\nu d^4x} \quad . \tag{24}$$

We can easily check that the value of e^2 is completely arbitrary just like the boundary values to equation (23). We conclude that the action (19) is equivalent to scalar electrodynamics at least at the classical level with an arbitrary value for the fine structure constant. We do not discuss the quantum version in this talk. The action we considered in our paper[(2)] is the same kind of action as in equation (2) or in equation (19). It is the product of Kaluza's action and the action for the minimal 4-dimensional surface in the n-dimensional Riemannian space. For the detail see paper (2).

Let me now turn to the next topic which is the solution to Einstein's equation with a $[\delta(x)]^2$ singularity. Although our task is to obtain a solution to the Kaluza-Klein equation[(2)] I will show in this talk that the usual 4-dimensional Einstein equation with a single particle as the source has a solution with a $[\delta(x)]^2$ singularity.

Einstein's equation in this case reads

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}R + 8\pi G T^{\mu\nu} = 0 \quad , \tag{25}$$

with

$$T^{\mu\nu} = m\int d\tau \delta^4(x-x(\tau)) \frac{\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}{[-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}]^{\frac{1}{2}}} \frac{1}{\sqrt{g}} \quad , \tag{26}$$

where $g = \det(g_{\mu\nu})$. For $X^0 = f(\tau)$ and $X^i = 0$ we have

$$T^{\circ\circ} = m\delta^3(x) \frac{1}{[-g_{00}]^{\frac{1}{2}}} \frac{1}{\sqrt{g}} \quad , \tag{27}$$

and

$T^{ij} = T^{io} = 0$. We now make the ansatz

$$g_{\mu\nu} = \begin{pmatrix} -a(t) & \\ & b(t) \begin{pmatrix} \delta(x_1)^2 & & \\ & \delta(x_2)^2 & \\ & & \delta(x_3)^2 \end{pmatrix} \end{pmatrix} , \tag{28}$$

to solve equation (25). We then have

$$T^{oo} = m/\{a(t)b(t)^{\frac{3}{2}}\} , \tag{29}$$

and

$$\sqrt{g} = [a(t)b^3(t)]^{\frac{1}{2}}\delta(x_1)\delta(x_2)\delta(x_3) . \tag{30}$$

After some calculations we can rewrite equation (25) in the following form with the ansatz (28):

$$-\frac{d}{dt}(\frac{\dot{b}}{2a}) + 2(\frac{\dot{b}}{2a})(\frac{\dot{b}}{2b}) - \frac{\dot{b}}{2a}(\frac{\dot{a}}{2a} + \frac{3\dot{b}}{2b}) + \frac{b}{2}[\lambda + \frac{8\pi Gm}{\sqrt{b^3}}] = 0, \tag{31}$$

and

$$3\frac{d}{dt}(\frac{\dot{b}}{2b}) + 3(\frac{\dot{b}}{2b})^2 - \frac{\dot{a}}{2a}(\frac{3\dot{b}}{2b}) - \frac{a}{2}[\lambda - \frac{8\pi Gm}{\sqrt{b^3}}] = 0, \tag{32}$$

where we have added the cosmological term λ for the sake of completeness. We can easily see that $a = a_0 t^4$ and $b = b_0 t^4$ give a solution to equations (31) and (32) when $\lambda = 0$ if the following condition is satisfied:

$$4\pi G m a_0 / b_0^{\frac{3}{2}} = 6 \quad . \tag{33}$$

For this solution we have

$$\int T^{oo}\sqrt{g}\, d^3x = \frac{m}{\sqrt{a_0}}\frac{1}{t^2} \quad .$$

The physical meaning of this solution in the above 4-dimensional case is not very clear but it is rather straight forward in the Kaluza-Klein case as has been extensively discussed by T. Kaneko and myself in reference (2).

(references)

1. Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin, Math.-Phys. Kl, 966(1921)
2. T. Kaneko & H. Sugawara, to be published in Prog. Theor. Phys.
3. See, for example, paper by N. Rosen in "To Fulfill a Vision', Jerusalem Einstein Centennial Symposium, Edited by Y. Ne'eman and published by Addison-Wesley, Inc., (1981)