A Geometrical Foundation of a Unified Field Theory

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Unfortunately the authors were unable to attend the Symposium

I. Introduction

In a series of two not well known papers Einstein and Mayer [1] proposed a formalism by which they were able to obtain a theory of gravitation and electromagnetism similar to that of Kaluza and Klein. [2] Instead of assuming, as these authors did, the existence of a five-dimensional continuum they assumed that at each point of space-time, regarded as a Riemannian space, there exists a five-dimensional vector space. The purpose of this work is to generalize the approach of Einstein and Mayer to N-dimensions, and to lay the geometrical foundation of a possible unified field theory of gravitation with other fields. [3]

Accordingly, we assume the existence of a four-dimensional Riemannian base space, characterized by coordinates x^i $(i = 1,..4)$ and metric g_{ij}. At each point (of the base space) there is a linear vector space of N dimensions (N 5), vectors in which would have components a_μ, b^ν $(\mu,\nu = 1,2..N)$. Quantities in the two spaces are connected a mixed tensor or projector $h_\mu{}^k$ so that

$$a^k = h_\mu{}^k \, a^\mu \tag{1.1}$$

For a given a^μ (1.1) determines a^k uniquely, but the reverse is not the case. In particular, for a vector $A^k = 0$ we can write

$$A^k = h_\mu{}^k \, A^\mu = 0 \tag{1.2}$$

which will have $n = N-4$ independent solutions, if the matrix $(h_\mu{}^k)$ is of rank 4. Labelling these solutions with an index P $(P = 1,2,..n)$ we can define a metric g_{PQ}

$$A_P{}^\mu \, A_{Q\mu} = g_{PQ} \tag{1.3}$$

In general, g_{PQ} will be functions of coordinates y^m $(m = 1,2,..N-4)$ in this subspace, but for the present discussion[4] we shall assume that g_{PQ} are constants, which we can take as

$$g_{PQ} = \delta_{PQ} \tag{1.4}$$

However, to keep the notation more uniform we shall replace the A's by quantities such as $h_\mu^{\ P}$, $h_{\mu P}$, $h^\mu_{\ P}$ etc. In the N-dimensional space we define a metric tensor $f_{\mu\nu}$ related to g_{ij} through

$$g_{ij} = f_{\mu\nu}\, h^\mu_{\ i}\, h^\nu_{\ j} \quad \text{or} \quad g^{ij} = f^{\mu\nu}\, h_\mu^{\ i}\, h_\nu^{\ j} \tag{1.5}$$

and by raising (or lowering) indices

$$h_\lambda^{\ i}\, h^\lambda_{\ j} = \delta^i_j \tag{1.6}$$

A simple calculation using (1.2) and (1.6) gives

$$f_{\mu\nu} = h_\mu^{\ i}\, h_\nu^{\ j}\, g_{ij} + h_\mu^{\ P}\, h_\nu^{\ Q}\, g_{PQ} \tag{1.7}$$

which can be considered to be the inverse of the relation given by (1.5).

II. Curvature tensor

Let us now consider covariant differentiation, which involves the ordinary Christoffel symbols in base space, but a number of connections or three-index symbols in vector-space. For example, the covariant derivative of $S^\mu_{\ k}$ is

$$S^\mu_{\ k\|j} = S^\mu_{\ k,j} + \Gamma^\mu_{\ \lambda j}\, S^\lambda_{\ k} - \left\{ {m \atop k\,j} \right\} S^\mu_{\ m} \tag{2.1}$$

Thus, in particular

$$a^k_{\ \|j} = a^k_{\ ;j} \tag{2.2}$$

where a semi-colon denotes a Riemannian covariant derivative. Furthermore,

$$g_{ij\|k} = g_{ij;k} = 0 \tag{2.3}$$

as usual, and we shall also assume

$$f_{\mu\nu\|k} = 0 \tag{2.4}$$

In order to determine the form of the three-index symbol $\Gamma^\lambda_{\ \mu k}$ consider a simpler quantity $\overset{\circ}{\Gamma}{}^\lambda_{\ \mu k}$ involved in covariant derivatives denoted by a single bar | and defined such that

$$h_\mu^{\ j}{}_{|k} = h_\mu^{\ J}{}_{|k} = 0 \tag{2.5}$$

One finds then from (2.5)

$$\overset{\circ}{\Gamma}{}^\lambda_{\ \mu k} = h^\lambda_{\ j}\, h_\mu^{\ j}{}_{,k} + h^\lambda_{\ P}\, h_\mu^{\ P}{}_{,k} + h^\lambda_{\ j}\, h_\mu^{\ m} \left\{ {j \atop m\,k} \right\} + h^\lambda_{\ P}\, h_\mu^{\ Q}\, \Gamma^P_{\ Qk} \tag{2.6}$$

where $\Gamma^P_{\ Qk}$ is still undetermined.[5] If we now write

$$\Gamma^\lambda_{\ \mu k} = \overset{\circ}{\Gamma}{}^\lambda_{\ \mu k} + V^\lambda_{\ \mu k} \tag{2.7}$$

it follows from (2.4) that

$$V_{\mu\nu k} + V_{\nu\mu k} = 0 \tag{2.8}$$

Consequently, we can take $V_{\mu\nu k}$ in the form

$$V_{\mu\nu k} = h_\mu{}^i h_\nu{}^j W_{ijk} + (h_\mu{}^P h_\nu{}^j - h_\nu{}^P h_\mu{}^j) F_{Pjk} + h_\mu{}^P h_\nu{}^Q U_{PQk} \tag{2.9}$$

where

$$W_{ijk} = - W_{jik} \quad \text{and} \quad U_{PQk} = - U_{QPk}$$

and the functions appearing in (2.9) are to be determined.

The curvature tensor in the base-space is just the Riemann-Christoffel tensor $R^i{}_{jkn}$. In the vector space we can define the curvature tensor

$$P^\lambda{}_{\mu jk} = - \Gamma^\lambda{}_{\mu j,k} + \Gamma^\lambda{}_{\mu k,j} + \Gamma^\lambda{}_{\sigma j} \Gamma^\sigma{}_{\mu k} - \Gamma^\lambda{}_{\sigma k} \Gamma^\sigma{}_{\mu j} \tag{2.10}$$

From $P^\lambda{}_{\mu jk}$ one can form tensors of lower order

$$P_{\mu j} = P^\lambda{}_{\mu jk}\, h_\lambda{}^k \quad \text{and} \quad P = P_{\mu j}\, h^{\mu j} \tag{2.11}$$

being the analogues of the Ricci tensor and invariant curvature.

From the usual anti-commutation relations we obtain

$$K_{\mu j} = h_\mu{}^k{}_{\| j \| k} - h_\mu{}^k{}_{\| k \| j} = P_{\mu j} - h_\mu{}^m R_{mj}$$

Multiplication by $h^{\mu j}$ then gives

$$P - R = h^{\mu j} K_{\mu j} \tag{2.12}$$

The right hand side can be evaluated by making use of (2.5), (2.7) and

$$h_\mu{}^i{}_{\| k} = - h_\sigma{}^i V^\sigma{}_{\mu k}$$

where $V^\sigma{}_{\mu k}$ is given by (2.9). Carrying out the indicated calculations we find

$$P = R + W^{ijk} W_{ijk} + F^{Pjk} F_{Pjk} \tag{2.13}$$

III. Field equations

To obtain the field equations it is convenient to make use of a variational principle. Since for constant $g_{PQ} = \delta_{PQ}$ the only scalar at our disposal in vector space is P given by (2.13) we shall take the variational functional as

$$\delta I = \int P \, (-g)^{\frac{1}{2}} \, d^4x = 0 \tag{3.1}$$

where $g = \det|g_{ij}|$. Varying (3.1) with respect to g_{ab} we then obtain

$$\begin{aligned} R^{ab} - \tfrac{1}{2} g^{ab} R = & - 2(F_Q{}^{aj} F^{Qb}{}_j - \tfrac{1}{4} g^{ab} F^{QPq} F_{QPq}) \\ & - 3(W^{arp} W^b{}_{rp} - \tfrac{1}{6} g^{ab} W^{rpq} W_{rpq}) \end{aligned} \tag{3.2}$$

Varying (3.1) with respect to W_{ijk} and F_{Pjk} gives $W^{ijk} = F^{Pjk} = 0$, as can be seen from (2.13). In order to avoid this, let us express these functions as potentials. Let us first assume [6)]

$$F_{Pjk} = - F_{Pkj} \quad \text{and} \quad F_{Pjk} = F_{Pj|k} - F_{Pk|j} \tag{3.3}$$

Varying now the potentials F_{Pj} we obtain the field equations

$$F_P{}^{jk}{}_{|k} = F_P{}^{jk}{}_{;k} - \Gamma^Q{}_{Pk} F_Q{}^{jk} = 0 \tag{3.4}$$

Assume also

$$W_{ijk} = - W_{ikj}$$

so that now W_{ijk} is completely antisymmetric upon interchange of any two indices. Two possibilities now suggest themselves:

a) For

$$W_{ijk} = (-g)^{\frac{1}{2}} e_{ijkm} g^{mn} \Psi_{,n} \tag{3.5}$$

where e_{ijkm} is the completely antisymmetric Levi-Civita symbol. Varying now Ψ in (3.1) results in the wave equation

$$g^{jk} \Psi_{;jk} = 0 \tag{3.6}$$

b) Alternately,

$$W_{ijk} = w_{ij,k} + w_{jk,i} + w_{ki,j} \quad \text{with } w_{ij} = - w_{ji} \tag{3.5'}$$

If we now vary with respect to w_{ij} we find from (3.1)

$$W^{ijk}{}_{;k} = 0 \tag{3.6'}$$

It should also be noted that no equations have been obtained for U_{PQk}.

IV. Gauge fields

So far we have considered the vectors $h_\mu{}^P$ satisfying (1.2) and (1.3) as having permanent identities. However, we can get a further generalization by taking into account the possibility of replacing them by linear combinations. If under the transformation

$$h_\mu{}^P \longrightarrow h_\mu{}^{P'} = S^P{}_Q h_\mu{}^Q \tag{4.1}$$

where $S_{PQ} = (S^{-1})_{QP}$ is an orthogonal matrix, the vector Ψ_μ is invariant

$$\Psi_\mu = \Psi_P h_\mu{}^P = \Psi_P{}' h_\mu{}^{P'} \tag{4.2}$$

then

$$\Psi_P{}' = S_P{}^Q \Psi_Q \quad \text{or, in matrix notation,} \quad \underset{\sim}{\Psi}' = \underset{\sim}{S} \underset{\sim}{\Psi} \tag{4.3}$$

If we now define the covariant derivative

$$\underset{\sim}{\Psi}_{|j} = \underset{\sim}{\Psi}_{,j} - \underset{\sim}{B}_j \underset{\sim}{\Psi} \quad \text{so that} \quad \underset{\sim}{\Psi}'_{|j} = \underset{\sim}{S} \underset{\sim}{\Psi}_{|j} \tag{4.4}$$

we obtain the transformation law for B_j

$$\underset{\sim}{B}'_j = \underset{\sim}{S}\,\underset{\sim}{B}_j\,\underset{\sim}{S}^{-1} + \underset{\sim}{S}_{,j}\,\underset{\sim}{S}^{-1} \tag{4.5}$$

Thus,

$$\underset{\sim}{B}_{jk} = \underset{\sim}{B}_{j,k} - \underset{\sim}{B}_{k,j} + [\underset{\sim}{B}_j\ ,\ \underset{\sim}{B}_k] \tag{4.6}$$

transforms according to the relation

$$\underset{\sim}{B}'_{jk} = \underset{\sim}{S}\,\underset{\sim}{B}_{jk}\,\underset{\sim}{S}^{-1} \tag{4.7}$$

We see that we have here the gauge-field formalism. Writing out (4.4) in terms of matrix elements gives

$$\Psi_{P|j} = \Psi_{P,j} - B_P{}^Q{}_j\,\Psi_Q = \Psi_{P,j} - \Gamma^Q{}_{Pj}\,\Psi_Q \tag{4.8}$$

which shows that the matrix elements $B_P{}^Q{}_j$ are nothing else than the three-index symbols $\Gamma^Q{}_{Pj}$ we met previously (cf. 2.6), and thus our formalism does contain the seed of the gauge transformation. In particular, (4.6) written out is just the tensor (apart from an overall sign)

$$B^P{}_{Qjk} = \Gamma^P{}_{Qj,k} - \Gamma^P{}_{Qk,j} + \Gamma^P{}_{Rj}\,\Gamma^R{}_{Qk} - \Gamma^P{}_{Rk}\,\Gamma^R{}_{Qj} \tag{4.9}$$

We also note that the three-index symbol $\overset{o}{\Gamma}{}^\lambda{}_{\mu k}$ (2.6) is gauge-invariant as it stands. Moreover, the field equations for F_{Pjk} are gauge-invariant as can be seen from (3.4). Also, from (3.3) we obtain

$$F_{Pjk} = F_{Pj|k} - F_{Pk|j} = F_{Pj,k} - F_{Pk,j} - \Gamma^Q{}_{Pk}\,F_{Qj} + \Gamma^Q{}_{Pj}\,F_{Qk} \tag{4.10}$$

We, then, note that varying (3.1) with respect to $\Gamma^Q{}_{Pj}$ would impose a restriction on F_{Qj}. It is, therefore, suggestive to add to the Lagrangian in (3.1) a term involving these connections. Thus, we replace (3.1) by

$$\delta \int (P + B^P{}_{Qjk}\,B^Q{}_{Pmn}\,g^{mj}\,g^{nk})(-g)^{\frac{1}{2}}\,d^4x = 0 \tag{4.11}$$

Varying (4.11) with respect to g_{ab} adds to (3.2) a term on the r.h.s. of the form

$$-\,(B^P{}_Q{}^{ka}\,B^Q{}_{Pk}{}^b + B^P{}_Q{}^{kb}\,B^Q{}_{Pk}{}^a - \tfrac{1}{2}g^{ab}\,B^P{}_Q{}^{mn}\,B^Q{}_{Pmn}) \tag{4.12}$$

while varying with respect to $\Gamma^P{}_{Qk}$ gives

$$-\,4B_P{}^{Qjk}{}_{|k} + 2(F_P{}^{jk}\,F^Q{}_k - F^{Qjk}\,F_{Pk}) = 0 \tag{4.13}$$

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1) A. Einstein and W. Mayer, Sitzber. Preuss. Akad. Wiss. 1931, p. 541; 1932, p.130

2) Th. Kaluza, Sitzber. Preuss. Akad. Wiss. 1921, p. 966

3) Work along these lines is now in progress.

4) The general case will be presented in a separate publication elsewhere.

5) These symbols play the role of gauge fields (see section IV)

6) For P = 1 we get the Maxwell fields considered by Einstein and Mayer.