SPACE-TIME STRUCTURE OF GRAVITATIONAL SOLITONS

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The axially symmetric and stationary soliton solution of Einstein's equations in a vacuum has been found by means of the inverse scattering problem technique. This solution has metric of the form

$$ds^2 = g_{ab}dx^a dx^b + h(d\rho^2 + dz^2) \ , \tag{1}$$

where $a,b = t,\phi$. For the N-soliton metric on a flat background for which $g^o_{ab} = \mathrm{diag}(-1,\rho^2)$ and $h^o = 1$, we obtain the following expressions[1]:

$$\begin{aligned} g_{ab} &= \prod_{k=1}^{N} |\mu_k/\rho| (\chi(\lambda=0;\rho,z))_{ac} g^o_{cb} \\ &= \prod_{k=1}^{N} |\mu_k/\rho| \{ g^o_{ab} - \sum_{k,\ell=1}^{N} (\Gamma^{-1})_{k\ell} N_{ka} N_{\ell b} \} \ , \end{aligned} \tag{2}$$

$$h = c_o \rho^{N^2/2-N} \left(\prod_{k=1}^{N} |\mu_k/\rho|\right)^{N-1} \prod_{k>\ell}^{N} (\mu_k - \mu_\ell)^{-2} \det(\Gamma_{k\ell}) \ , \tag{3}$$

$$N_{kt} = -1, \ N_{k\phi} = c_k \mu_k^{-1} \rho^2 \ ,$$

$$\Gamma_{k\ell} = (\rho^2 c_k c_\ell - \mu_k \mu_\ell)/(\rho^2 + \mu_k \mu_\ell) \ ,$$

where c_o and c_k are any constants. The matrix function χ depends on a complex spectral parameter λ, and has N poles at the points $\lambda = \mu_k(\rho,z) = w_k - z + \varepsilon_k\{\rho^2 + (w_k - z)^2\}^{1/2}$, where w_k are any constants and $\varepsilon_k = \pm 1$. From the conditions of asymptotic flatness and $\det(g_{ab}) < 0$, we require that N is even and $\sum_{k=1}^{N} \varepsilon_k = 0$. Then, we subdivide all the μ_k into pairs with opposite signs ε_k, and rewrite the parameters w_k and c_k as follows,

$$\begin{aligned} &\mu^{\pm}_\gamma = w^{\pm}_\gamma - z \pm \{\rho^2 + (w^{\pm}_\gamma - z)^2\}^{1/2} \ , \quad \gamma = 1,2,\ldots,N/2, \\ &w^{\pm}_\gamma = z_\gamma \mp m_\gamma \cos\lambda_\gamma \ , \ c^{\pm}_\gamma = \cot((\alpha_\gamma \pm \lambda_\gamma)/2) \ . \end{aligned} \tag{4}$$

The N-soliton metric turns out to be a nonlinear superposition of N/2 Kerr-NUT metrics aligned along the common rotational axis (z-axis). Each soliton pair describes the Kerr-NUT metric with the Kerr and NUT rotation parameters λ_γ, α_γ and the mass m_γ located at the point $z = z_\gamma$. This stationary configuration of several Kerr-NUT masses will be realized as a result of the presence of any singularities in the space-time or a dynamical balance between the gravitational attraction and the rotational repulsion. If there exists no singular structure, great interest from the physical point of view attaches to the N-soliton metric, because it can represent a dynamical equilibrium of many interacting black holes. In this paper we investigate the space-time structure of the multi-soliton metric, by taking the two Kerr-NUT case (N=4) as a typical example.

The metric components can be derived from the Ernst potential ξ, which is given by[2)]

$$\xi \equiv \frac{N}{D} = \begin{vmatrix} S_1^- & S_1^+ & S_2^- & S_2^+ \\ 1 & 1 & 1 & 1 \\ w_1^- & w_1^+ & w_2^- & w_2^+ \\ (w_1^-)^2 & (w_1^+)^2 & (w_2^-)^2 & (w_2^+)^2 \end{vmatrix} \Bigg/ \begin{vmatrix} S_1^- & S_1^+ & S_2^- & S_2^+ \\ 1 & 1 & 1 & 1 \\ w_1^- & w_1^+ & w_2^- & w_2^+ \\ w_1^-S_1^- & w_1^+S_1^+ & w_2^-S_2^- & w_2^+S_2^+ \end{vmatrix} , \tag{5}$$

$$S_\gamma^\pm = \mp e^{-i\beta_\gamma^\pm}\{\rho^2 + (w_\gamma^\pm - z)^2\}^{1/2} , \quad \beta_\gamma^\pm = \alpha_\gamma \pm \lambda_\gamma \ (\gamma = 1,2) ,$$

for the 4-soliton metric. We can set

$$m_\gamma > 0 , \quad \cos\lambda_\gamma > 0 , \quad z_1 = -z_2 \equiv z_o > 0 .$$

If $w_1^+ = w_2^-$ ($2z_o = \sum_{\gamma=1}^{2} m_\gamma \cos\lambda_\gamma$), this metric reduces to the Kerr-NUT metric. For a double coincidence of the poles, i.e., $w_1^\pm = w_2^\pm$ ($z_o = 0$)[2),3)], we obtain the $\delta = 2$ Tomimatsu-Sato and Kinnersley-Chitre metrics. Therefore, we consider only the case that any $w_\gamma^\pm$ do not coincide. Then, we can classify the 4-soliton metric into two types;

(I) separated type of two Kerr-NUT metrics, i.e., $2z_o > \sum_{\gamma=1}^{2} m_\gamma \cos\lambda_\gamma$,

(II) overlapped type, i.e., $2z_o < \sum_{\gamma=1}^{2} m_\gamma \cos\lambda_\gamma$.

We construct diagrams (Fig. 1) that represent clearly such different types.[4)] We place on the horizontal axis (z-axis) four points $z = w_\gamma^\pm$. The axis is divided into five different regions (a,...,e). The vertical bars drawn from these points are directed up at the

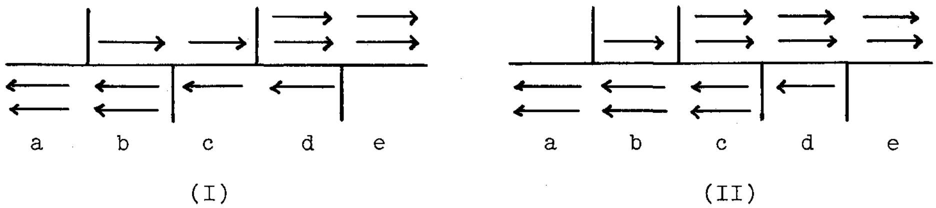

Fig. 1

points $z = w_\gamma^+$ and down at the points $z = w_\gamma^-$. An arrow along the axis means that one of the $\mu_\gamma^\pm$ at this region of the axis is of the order of $O(\rho^2)$.

For the static field[4], i.e., $\alpha_\gamma = \lambda_\gamma = 0$, the metric coefficient h becomes of the order of $O(\rho^{2(s-2)(s-3)})$ near each region of the axis (s is the number of arrows). Then, for the metric of type II, the invariant of the Riemann curvature tensor $\Lambda \equiv R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ $(\mu,\nu,\ldots = t,\phi,\rho,z)$ becomes infinite at the middle region (c).

For the rotational field, the invariant Λ does not increase without limit at the z-axis. In general, such a singularity appears at the point $\xi = -1$. In order to look for this point, we consider a simple case such as $m_1 = m_2 \equiv m$, $\lambda_1 = \lambda_2 \equiv \lambda$ $(\cos\lambda \equiv p)$, $\alpha_1 = \alpha_2 = 0$. Then, we have $c_1^+ = -c_1^- = c_2^+ = -c_2^- \equiv c$, $c^2 = (1+p)/(1-p) > 1$. The relation $\xi = -1$ means that $\mathrm{Re}(N+D) = \mathrm{Im}(N+D) = 0$. We see that $\mathrm{Im}(N+D)$ becomes an odd function of z, so the singular point will exist at the plane $z = 0$, where the metric has the form

$$g_{tt} = -B^{-1}[(\eta_+-\eta_-)^2(c^2\eta_+\eta_-+1)^2 - c^2(1-\eta_+^2\eta_-^2)^2] ,$$

$$g_{t\phi} = -\rho c(\eta_+-\eta_-)(1-\eta_+^2\eta_-^2)(\eta_+\eta_-B)^{-1}$$

$$\times [(\eta_+^2+\eta_-^2-\eta_+\eta_-)(c^2\eta_+\eta_-+1) - (c^2+\eta_+^3\eta_-^3)] , \tag{6}$$

$$g_{\phi\phi} = \rho^2(\eta_+^2\eta_-^2B)^{-1}[(\eta_+-\eta_-)^2(c^2+\eta_+^3\eta_-^3)^2 - c^2(1-\eta_+^2\eta_-^2)^2(\eta_+^2+\eta_-^2-\eta_+\eta_-)^2],$$

$$B \equiv (\eta_+-\eta_-)^2(c^2+\eta_+\eta_-)^2 - c^2(1-\eta_+^2\eta_-^2)^2 ,$$

$$\eta_\pm \equiv \rho^{-1}\mu_1^\pm(\rho,z=0) = \rho^{-1}[z_0 \mp mp \pm \{(z_0 \mp mp)^2 + \rho^2\}^{1/2}] .$$

All the metric components g_{ab} become infinite at the point $B = 0$, while the metric coefficient h vanishes at the same point ($B \propto$ det

$(\Gamma_{k\ell})$).

For the metric of type II, i.e., $z_o < mp$, $\eta_+(\rho)$ is monotonously increasing, and

$$(\eta_+-\eta_-)(c^2+\eta_+\eta_-) - |c|(1-\eta_+^2\eta_-^2) = \begin{cases} -|c| < 0 & \text{as } \rho \to 0, \\ 2(c^2-1) > 0 & \text{as } \rho \to \infty. \end{cases}$$

It can be proved that the algebraic equation $B = 0$ has only one root ρ_o (For example, when $z_o \to mp$, we obtain $\rho_o \simeq (mp-z_o)|\tan\lambda|$). This means that the metric of type II includes one ring-singularity around the region (c). On the other hand, for the metric of type I, $\eta_+(\rho)$ is monotonously decreasing. If the equation $B = 0$ holds, we have

$$|c| = [1-\eta_+^2\eta_-^2+\{(1-\eta_+^2\eta_-^2)^2 - 4\eta_+\eta_-(\eta_+-\eta_-)^2\}^{1/2}]/2(\eta_+-\eta_-) \ .$$

This relation turns out to be incompatible with the constraint $|c| > 1$. For any other choice of the parameters we expect that the presence of this kind of singularity is a crucial difference between the two types of the 4-soliton metric.

Next, we examine some properties of the z-axis of the 4-soliton metric of type I.[5),6)] We rewrite the metric as follows,

$$ds^2 = -f(dt-\omega d\phi)^2 + f^{-1}[\rho^2 d\phi^2 + e^{2\mu}(d\rho^2 + dz^2)] \ . \qquad (7)$$

On the z-axis we find that $\partial\omega/\partial z = \partial\mu/\partial z = 0$ except at the points $z = w_\gamma^\pm$, and $f \propto \prod_{\gamma=1}^{2}(z-w_\gamma^+)(z-w_\gamma^-)$. We denote the constant values of ω and μ at each region of the axis by ω_A and μ_A ($A = a,\ldots,e$). If the z-axis has a local Euclidean property of a line, for any infinitesimal spacelike circle around the z-axis the ratio of circumference to radius should be 2π, i.e., $g_{\phi\phi}/\rho^2 g_{\rho\rho} \to 1$ as $\rho \to 0$. This requires that $\omega_A = \mu_A = 0$ and gives some restrictions to a choice of the parameters. By a suitable choice of the coordinates (t,ϕ) and c_o (see Eq. (3)), we can always set $\omega_e = \mu_e = 0$. Then, the conditions $\omega_a = \mu_a = 0$ lead to

$$4z_o^2(m_1v_1 + m_2v_2) + 4m_1m_2z_o(q_2u_1 - q_1u_2) - (m_1^2p_1^2 - m_2^2p_2^2)(m_1v_1 - m_2v_2) = 0 \ , \qquad (8)$$

where $e^{i\alpha_\gamma} \equiv u_\gamma + iv_\gamma$ and $e^{i\lambda_\gamma} \equiv p_\gamma + iq_\gamma$.

At the regions (b) and (d), we cannot find any choice of the parameters satisfying the conditions $\omega_A = \mu_A = 0$ (A = b,d). Using the coordinate θ_γ ($\gamma = 1$ at the region (d) and $\gamma = 2$ at the region (b)) defined by $z - z_\gamma = m_\gamma p_\gamma \cos\theta_\gamma$ ($0 \leq \theta_\gamma \leq \pi$), we can prove that each region has the structure of a closed 2-sphere. Furthermore, there exists a Killing vector $\xi^\nu_{(A)} \equiv \omega_A \xi^\nu_{(t)} + \xi^\nu_{(\phi)}$ which becomes null on this closed surface Σ_A.[7] Here, $\xi^\nu_{(t)}$ and $\xi^\nu_{(\phi)}$ are the Killing vectors associated with the stationarity and axial symmetry respectively. We verify that $\xi^\nu_{(A)}$ lies in Σ_A[*] since it is orthogonal to the vector $n^{(A)}_\nu \equiv \xi_{(A)\sigma;\nu}\xi^\sigma_{(A)}$ which is normal to Σ_A. The vector $n^{(A)}_\nu$ becomes null when $\xi^\nu_{(A)}$ does, so the regions (b) and (d) can be regarded as two horizons separated by the region (c).

The structure of the middle region (c) between two black holes is important. If there is no artificial "support" between the masses, this region should have a regular structure of a line. Hence, the conditions $\omega_c = \mu_c = 0$ seem to assure a dynamical balance between the gravitational attraction and the rotational repulsion. These conditions lead to

$$\cos(\alpha_1 - \alpha_2) = 0 \ , \tag{9}$$

$$(4z_o^2 - m_1^2 p_1^2 + m_2^2 p_2^2)v_1 + 4m_2 z_o(q_2 u_1 + \sin(\alpha_1 - \alpha_2)) = 0 \ . \tag{10}$$

It is remarkable that due to Eqs. (8) $\sim$ (10) the distance between two masses, $2z_o$, is fixed by the masses and angular momenta. If $\omega_c \neq 0$, the metric has a pathological structure of causality violation (i.e., the existence of a closed timelike curve), since $g_{\phi\phi} \simeq -f\omega_c^2 < 0$ near the region (c). We conclude that only the 4-soliton metric of type I with the parameters satisfying the conditions (8) $\sim$ (10) will avoid any singular structures in the space-time.

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*) This surface should include the t-direction too.