MASSIVE GAUGE THEORIES IN THREE DIMENSIONS (= AT HIGH TEMPERATURE)

R. Jackiw
Center for Theoretical Physics
Laboratory for Nuclear Science and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

I. INTRODUCTION

Gauge theories in three dimensions provide interesting case studies of gauge invariant quantum field theoretic phenomena. Also they are physically relevant: a field theory on a 3-dimensional [Euclidean] space summarizes the high-temperature behavior of a theory in four-dimensional space-time.

Recently it has been established that the special topological properties of odd-dimensional space allow the construction of a gauge invariant mass term in three dimensions.[1] Moreover, for non-Abelian gauge groups, the configuration space of the quantum theory is not simply connected, and as a consequence the mass must be quantized, somewhat analogously to Dirac's monopole quantization condition. While all this appears to be a peculiar feature of 3-dimensional theories, it is an interesting phenomenon, whose certain aspects have 4-dimensional analogs. Also there may be a direct physical [high temperature] significance to the mass. For these reasons it is profitable to study the subject, and I shall report here on this research.[2]

II. ABELIAN THEORY

Consider the following Lagrange density in 3-dimensional space-time.

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\mu}{4} \varepsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.1}$$

Dimensional arguments show μ has dimension of mass. Although the Lagrange density is not gauge invariant, the equation of motion which follows from (2.1) is

$$\partial_\mu F^{\mu\nu} + \mu {*F}^\nu = 0 \tag{2.2}$$

Here we have defined the dual field, which in three dimensions is a vector.

$$*F^\mu = \frac{1}{2} \varepsilon^{\mu\alpha\beta} F_{\alpha\beta}$$

$$F^{\alpha\beta} = \varepsilon^{\alpha\beta\mu} {*F}_\mu \tag{2.3}$$

Note that the dual field is identically conserved.

$$\partial_\mu {*F}^\mu = 0 \tag{2.4}$$

This Bianchi identity is a consequence of the definitions of $F^{\mu\nu}$ and $*F^\mu$; alternatively, it follows from the equation of motion (2.2), owing to the antisymmetry of $F^{\mu\nu}$. Under a gauge transformation the Lagrange density changes by a total derivative.

$$A_\mu \rightarrow A_\mu + \partial_\mu \theta \tag{2.5a}$$

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{\mu}{2} \partial_\mu ({*F}^\mu \theta) \tag{2.5b}$$

This is why the equation of motion is gauge invariant.

While it is clear that μ has dimension of mass, it still remains to be established that it is indeed a mass term for the field. This is most easily done by writing the field equation (2.2) in terms of the dual tensor (2.3). Eq. (2.2) is equivalent to

$$(\mu g^{\mu\alpha} + \varepsilon^{\mu\alpha\beta} \partial_\beta) {*F}_\alpha = 0 \tag{2.6a}$$

Multiplying this with the differential operator $(\mu g^{\nu\mu} - \varepsilon^{\nu\mu\gamma}\partial_\gamma)$ yields

$$(\Box + \mu^2)\, {}^*F^\nu = 0 \tag{2.6b}$$

which demonstrates clearly that the gauge field excitations are massive.

An analysis of the kinematics shows that the massive vector meson carries non-vanishing spin $\mu/|\mu| = \pm 1$. The existence of a single excitation with only one value of the spin -- as opposed to two, each differing in sign from the other -- signals reflection non-invariance. Of course, the lack of this symmetry is already evident from the Lagrangian, which contains the reflection non-invariant structure $\varepsilon^{\alpha\beta\gamma}$. One may regain a P and T conserving system by working with a doublet of models, one with mass μ, the other with $-\mu$, and defining parity and time inversion to include a field interchange.

It is gratifying that μ^2 occurs in (2.6b) with the correct sign for a propagating particle. Although we have no <u>a priori</u> control over this sign [$\mathcal{L}$ is linear in μ], we may understand that it must emerge the way it does by considering the energy-momentum tensor $\theta^{\mu\nu}$. When coupling our theory to an external metric, $(\mu/4)\int d^3x\ \varepsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha$ is already a coordinate invariant world scalar, without additional metric factors. Hence the variation of action $I = \int d^3x$ with respect to the metric [this variation defines the energy-momentum tensor] does not see the mass term. Consequently $\theta^{\mu\nu}$ has its conventional Maxwell form.

$$\theta^{\mu\nu} = -F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \tag{2.7}$$

In particular the energy $\mathcal{E}$ is a positive definite quantity,

$$\mathcal{E} = \frac{1}{2} \int d^2\vec{x}\ (\vec{E}^2 + B^2) \tag{2.8a}$$

$$\vec{E} = -\vec{\nabla} A^0 - \dot{\vec{A}}, \qquad B = -\frac{1}{2}\varepsilon^{ij} F_{ij} = \vec{\nabla}\times\vec{A} \tag{2.8b}$$

and the system's excitations cannot be tachyonic. Of course $\theta^{\mu\nu}$ remains conserved in our theory, as a consequence of the field equation (2.2).

The fact that the action associated with our mass term is a world scalar is evidence for its topological nature. This will have profound implication for the quantum theory of the non-Abelian generalization, which we shall discuss later. Here I want to record another curious topological property. Consider the time component of the field equation (2.2), in the presence of an external charge density ρ. This is the analog of Gauss' law; in our theory it reads

$$\vec{\nabla} \cdot \vec{E} - \mu B = \rho \tag{2.9a}$$

Upon integrating (2.9a) over all space, the first term vanishes, since the fields, being massive, decrease exponentially at large distances. One is left with

$$-\mu \int d^2\vec{x} B = \int d^2\vec{x} \rho = Q \tag{2.9b}$$

The magnetic flux passing out of our 2-dimensional space is proportional to the external charge Q. Correspondingly, the magnetic potential is long range, even though the magnetic field is short range.

$$\vec{A} \underset{r\to\infty}{\longrightarrow} -\vec{\nabla} \frac{Q}{2\pi\mu} \tan^{-1} y/x \tag{2.10}$$

This is similar to the electromagnetic configuration supported by vortices in the Higgs model.

Let us note that apart from a total derivative, $\mathcal{L}$ may be written in a gauge invariant form, which, however, is spatially non-local, and Lorentz non-invariant. This follows from the identity

$$\frac{\mu}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha = \frac{\mu}{2}B\frac{\vec{\nabla}}{\nabla^2}\cdot\vec{E} - \frac{\mu}{2}\vec{E}\cdot\frac{\vec{\nabla}}{\nabla^2}B - \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}\,\partial_\alpha(F_{\mu\nu}\frac{\vec{\nabla}}{\nabla^2}\cdot\vec{A})$$

Such an explicitly gauge invariant formulation is not available for the non-Abelian generalization.

An interesting interacting generalization involves coupling fermions to (2.1). Their gauge invariant Lagrange density is

$$\mathcal{L}_F = i\,\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\bar{\psi}\psi \tag{2.11}$$

In three space-time dimensions, the Dirac algebra may be realized by 2x2 [Pauli] matrices, and the fermion field is a 2-component spinor, describing a particle [and an anti-particle] with spin $\frac{1}{2}\frac{m}{|m|} = \pm\frac{1}{2}$.

Correspondingly, the mass term violates P and T symmetries and belongs in a Lagrange density which contains the reflection non-invariant vector meson mass term. Indeed in a theory with only one mass term, the other will be induced by radiative corrections. [Reflection invariance is regained by supplementing (2.11) with a Lagrange density for another fermion field, with a mass term of opposite sign to (2.11). Reflection transformations are again defined to include field exchange, and the model becomes equivalent to a 4-component Dirac theory.]

Feynman-Dyson perturbation theory is straight-forwardly carried out. It is both infrared and ultraviolet finite in the Landau gauge, where the free boson and fermion propagators read, respectively

$$D_{\mu\nu}(p) = \frac{-i}{p^2-\mu^2+i\varepsilon} [P_{\mu\nu}(p)-i\mu\varepsilon_{\mu\nu\alpha}p^\alpha/p^2] \tag{2.12a}$$

$$S(p) = \frac{i}{\not p - m} \tag{2.12b}$$

Consequently, this is the only non-trivial field theory which is known to possess a perturbation expansion entirely free of divergences -- not even normal ordering need be performed, provided Lorentz and gauge invariance are maintained.

III. NON-ABELIAN THEORY

The 3-dimensional mass term can be generalized to a non-Abelian gauge theory. The gauge field Lagrange density is

$$\mathcal{L} = \frac{1}{2g^2} \operatorname{tr} F^{\mu\nu}F_{\mu\nu} - \frac{\mu}{2g^2} \varepsilon^{\mu\nu\alpha} \operatorname{tr}(F_{\mu\nu}A_\alpha - \tfrac{2}{3}A_\mu A_\nu A_\alpha) \tag{3.1}$$

We use a matrix notation

$$A_\mu = gT^aA^a_\mu$$

$$F_{\mu\nu} = gT^aF^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{3.2}$$

which employs the representation matrices of the group.

$$[T^a, T^b] = f^{abc} T^c \tag{3.3}$$

The coupling constant is g, while μ/g^2 is dimensionless. The field equations which follow from (3.1) are gauge covariant,

$$\mathcal{D}_\mu F^{\mu\nu} + \frac{\mu}{2} *F^\nu = 0 \tag{3.4a}$$

$$\mathcal{D}_\mu = \partial_\mu + [A_\mu, \quad] \tag{3.4b}$$

and from our previous consideration of the non-interacting limit (g=0), we know that μ indeed provides a mass for the field. The dual field

$$*F^\mu = \frac{1}{2} \varepsilon^{\mu\alpha\beta} F_{\alpha\beta}$$

$$F^{\alpha\beta} = \varepsilon^{\alpha\beta\mu} *F_\mu \tag{3.5}$$

satisfied the Bianchi identity

$$\mathcal{D}_\mu *F^\mu = 0 \tag{3.6}$$

as a consequence of the definitions (3.2) and (3.5), or alternatively as a consequence of the field equation (3.4a). The dual of (3.4a) is

$$\mathcal{D}_\alpha *F_\beta - \mathcal{D}_\beta *F_\alpha - \mu F_{\alpha\beta} = 0 \tag{3.7a}$$

and another covariant divergence converts this, with the help of (3.14) and the Ricci identity $[\mathcal{D}_\alpha, \mathcal{D}_\beta] = [F_{\alpha\beta}, \quad]$ into

$$(\mathcal{D}_\alpha \mathcal{D}^\alpha + \mu^2) *F^\mu = \varepsilon^{\mu\alpha\beta} [*F_\alpha, *F_\beta] \tag{3.7b}$$

which is the non-Abelian analogue of (2.6b).

$\mathcal{L}$ is not invariant against gauge transformations; rather it changes by a total derivative. Consider a finite transformation

$$A_\mu \rightarrow U^{-1}A_\mu U + U^{-1}\partial_\mu U \tag{3.8a}$$

The response of the action to the gauge transformation (3.8a) is

$$\begin{aligned}\int d^3x\mathcal{L} \rightarrow \int d^3x\mathcal{L} + \frac{\mu}{g^2}\int d^3x\ \varepsilon^{\alpha\beta\mu}\ \partial_\alpha\ \mathrm{tr}\ [\partial_\beta UU^{-1}A_\mu] \\ + \frac{\mu}{3g^2}\int d^3x\ \varepsilon^{\alpha\beta\gamma}\ \mathrm{tr}\ (\partial_\alpha UU^{-1}\ \partial_\beta UU^{-1}\ \partial_\gamma UU^{-1})\end{aligned} \tag{3.8b}$$

The second term on the right-hand side, which is manifestly a total divergence, is the analogue of the Abelian term (2.5b). We shall only consider gauge transformations which tend to the identity at temporal and spatial infinity.

$$U(x) \xrightarrow[x\to\infty]{} \pm I \tag{3.9}$$

This restriction is made to avoid convergence problems in (3.8). Also, it reflects our assumption of asymptotic space-time uniformity. With Eq. (3.9), we may conclude that the A-dependent surface integral in (3.8b) vanishes. The last term in (3.8b), which has no Abelian analog, can also be converted to a surface integral once the integrand is re-written as a total derivative. This can be made manifest by introducing an explicit parametrization for U. We choose the gauge group to be SU(2) [more generally, we consider a SU(2) subgroup of the gauge group] and make use of the exponential parametrization.

$$\begin{aligned}U(x) &= \exp T^a\ \theta^a(x) \\ T^a &= \sigma^a/2i \\ \theta^a &= \hat{\theta}^a|\theta|\end{aligned} \tag{3.10}$$

It follows that

$$\int d^3x\mathcal{L} \rightarrow \int d^3x\mathcal{L} + \mu\frac{8\pi^2}{g^2}\ w(U) \tag{3.11}$$

where we have introduced the "winding number" of the gauge transformation U.

$$w(U) = \frac{1}{24\pi^2} \int d^3x \; \varepsilon^{\alpha\beta\gamma} \; tr[\partial_\alpha UU^{-1} \partial_\beta UU^{-1} \partial_\gamma UU^{-1}]$$

$$= \frac{-1}{16\pi^2} \int d^3x \; \varepsilon^{\alpha\beta\gamma} \; \varepsilon^{abc} \; \partial_\alpha \; [\hat{\theta}^a \partial_\beta \hat{\theta}^b \; \partial_\gamma \hat{\theta}^c \; x(|\theta|-\sin|\theta|)] \qquad (3.12)$$

This quantity, though given by a surface integral, is not in general zero. It vanishes only for homotopically trivial U's -- those continuously deformable to I. However, as a consequence of the fact that

$$\Pi_3(SU(2) = \Pi_3(S_3) = \text{group of all integers} \qquad (3.13)$$

there are U's which are not continuously deformable to the identity. Indeed, the gauge functions U can be arranged into homotopically inequivalent classes, labelled by the integers, and w(U) equals precisely that integer.[3] These considerations are, of course, familiar from the analysis of topological structure in 4-dimensional Yang-Mills theory.[3] That they should reappear in the 3-dimensional theory is not surprising, in view of the further mathematical/topological connections which we shall draw later.

We conclude that the action is not gauge invariant, but changes by $\mu(8\pi^2/g^2)\; W(U) = \mu(8\pi^2/g^2) \times$ integer. While the action is of no particular consequence in classical mechanics, in quantum mechanics the exponential of the action, $\exp\; i\int d^3x\mathcal{L}$, determines probability amplitudes and must be gauge invariant (see also below). Hence a change in the action, can be tolerated only if it is an integral multiple of 2π. Consequently the requirement of gauge invariance gives a quantization condition on the dimensionless ratio $4\pi\mu/g^2$.

$$4\pi \frac{\mu}{g^2} = n \qquad n = 0, \pm 1, \ldots \qquad (3.14)$$

A Euclidean formulation leads to the same conclusion. The functional integral requires $\exp - \int d^3x\mathcal{L}$ to be gauge invariant, but the mass term's contribution to the action is purely imaginary; a factor of i appears when the continuation to imaginary time [Euclidean space] is performed. The winding number is a world scalar; hence it takes the same integer value regardless of the space's signature. The quantization condition (3.14) follows as before; it is entirely due to the internal group.

The way that gauge transformations act on the mass term may also be appreciated once it is recognized that its action is a well-known mathematical entity -- the "Chern-Simons secondary class characteristic."[4]

This is defined in the following way. In even number of dimensions one can construct a gauge invariant quantity -- the "Pontryagin density" P_{2n} -- whose integral over the even-dimensional space is an invariant that characterizes the topological class to which the gauge field belongs. Examples in two and four dimensions are

$$P_2 = \frac{1}{2\pi} {*F} = \frac{1}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} \tag{3.15a}$$

$$P_4 = -\frac{1}{16\pi^2} \mathrm{tr}\, {*F}^{\mu\nu} F_{\mu\nu} \tag{3.15b}$$

[The 2-dimensional Pontryagin density arises in 2-dimensional massless QED as the anomalous divergence of the axial vector current, and is responsible for mass generation in that model. The 4-dimensional Pontryagin density is associated with θ-vacua in 4-dimensional Yang-Mills theories.] Since the integral is a topological invariant, this gauge invariant Pontryagin density can also be written as a total derivative of a gauge variant quantity.

$$P_{2n} = \partial_\mu X^\mu_{2n} \tag{3.16}$$

The formulas corresponding to (3.15) are

$$X^\mu_2 = \frac{1}{2\pi} \varepsilon^{\mu\nu} A_\nu \tag{3.17a}$$

$$X^\mu_4 = -\frac{1}{16\pi^2} \varepsilon^{\mu\alpha\beta\gamma} \mathrm{tr}(F_{\alpha\beta}A_\gamma - \tfrac{2}{3} A_\alpha A_\beta A_\gamma) \tag{3.17b}$$

On odd-dimensional spaces Pontryagin classes do not exist. But one may construct another topological quantity, the Chern-Simons secondary characteristic class. This is gotten by integrating one component of X^μ_{2n} over the 2n - 1 dimensional space which does not include that component. The integral is gauge invariant against homotopically trivial gauge transformations; otherwise, it changes by the winding number of the transformation.

We recognize that our mass term action is proportional to the 3-dimensional Chern-Simons structure.

This observation about the mass term in Yang-Mills theory has an immediate parallel in the construction of a topological mass for 3-dimensional gravity from the 4-dimensional *RR Hirzebruch-Pontryagin density. But this subject is outside the scope of my lectures, hence those interested are referred to the literature.[5]

Let us explore further the need for the mass quantization condition, and show that the theory is inconsistent without it. The functional integral for a 3-dimensional, massive gauge theory [in Euclidean space] is given by

$$Z = \int \mathcal{D}A^\mu \exp -\{ \int d^3x \, (-\tfrac{1}{2} \operatorname{tr} F^{\mu\nu}F_{\mu\nu}) - i\mu \frac{8\pi^2}{g^2} W(A)\}$$

$$W(A) = -\frac{1}{16\pi^2} \int d^3x \, \varepsilon^{\mu\nu\alpha} \operatorname{tr} (F_{\mu\nu}A_\alpha - \tfrac{2}{3} A_\mu A_\nu A_\alpha) \qquad (3.18)$$

As the functional integration ranges over all gauge potentials, for any given A^μ it also encounters its gauge copies A'^μ.

$$A'_\mu = U^{-1}A_\mu U + U^{-1}\partial_\mu U \qquad (3.19)$$

where the gauge functions U fall into homotopically distinct classes labelled by the integers n. Now the usual gauge fixing prescriptions, which are [implicitly] contained in $\mathcal{D}A^\mu$, remove gauge copies arising from the homotopically trivial gauge transformations. [Recall that Faddeev-Popov procedures are formulated in infinitesimal terms.] However, there seems to be no way of removing gauge copies in (3.18) arising from non-trivial, large gauge transformations. Thus we may write Z as

$$Z = \sum_{n=-\infty}^{\infty} Z_n \qquad (3.20)$$

Here Z_0 results by performing an integration over vector potentials, with no gauge copies; Z_1 results from integrating over vector potentials related by a gauge transformation in the first homotopy class to those occurring in the integral for Z_0; <u>etc</u>. But it is clear that once we have determined Z_0, Z_n for $n \neq 0$ may be evaluated by changing variables in the functional integral from A^μ to A'^μ, which is defined to be the gauge transform of A^μ with a large gauge function of the nth homotopy class. Such a change of variables does not affect the measure nor the usual action, since both are gauge invariant. In the mass term, W(A) changes by an integer, so we get

$$Z = Z_0 \sum_{n=-\infty}^{\infty} e^{in\mu \frac{8\pi^2}{g^2}} \qquad (3.21)$$

Now we see that if the mass term is not quantized, the infinite sum vanishes, by destructive interference. On the other hand, when the quantization condition holds, the sum becomes $\sum_{n=-\infty}^{\infty} 1$, which, though infinite, may be harmlessly cancelled by a normalizing denominator in the definition of Z.

Thus the result: the massive gauge theory vanishes in the absence of mass quantization. The same conclusion may be established by canonical reasoning.[6] Lack of time and space prevent giving details, but the essential steps are two in number. First, one realizes that the homotopy formula (3.13) implies that at fixed time the canonical configuration space consisting of spatial components of the vector potentials, is not simply connected. Second, one finds that Gauss' law, which in the canonical formalism is a constraint on physical states, i.e. a [functional] differential equation that physical wave functionals satisfy, cannot be globally integrated unless the mass term is quantized.[7]

In conclusion, let me speculate concerning the physical significance of the mass term. As I have already remarked, finite temperature perturbation theory for simple models suggests that in the high temperature limit a 3-dimensional version of that same model comes into play. Of course such a dimensional reduction makes no reference to non-perturbative phenomena. Moreover, for a non-Abelian gauge theory, which we know to be rich in non-perturbative effects, neither the 4-dimensional finite temperature perturbation theory, nor zero temperature perturbation theory in three dimensions makes sense owing to infrared divergences. The discovery of the 3-dimensional mass term allows the conjecture that the high temperature limit of a non-Abelian, 4-dimensional gauge theory is governed by a 3-dimensional, massive yet gauge invariant Yang-Mills theory, of the type here described.

There is no derivation of this fact; but neither can it be falsified, since naive perturbation theory does not exist and we do not have sufficient control over the formalism to extract non-perturbative behavior. Confronting such a hiatus, we invoke the principle of "naturalness" to aid in constructing the effective Lagrangian. The 3-dimensional effective Lagrangian should possess all terms with quantum numbers of the 4-dimensional theory, whose high temperature asymptote is under discussion. According to this criterion, the gauge invariant mass should be present, since its reflection non-invariance mirrors the reflection non-invariance of the 4-dimensional θ-vacua. Indeed, the topological setting of our mass term puts into evidence an intimate mathematical connection between it and the quantity responsible for the θ vacua. However, it is not known at present whether this mathematical relationship can be the basis for a physical derivation.

If we accept the gauge invariant mass as a proper term in the effective Lagrangian which summarizes high temperature behavior of physical non-Abelian gauge theories, we get another bonus, beyond infrared regularity. Owing to the quantization condition (3.14), the mass becomes evaluated in terms of the coupling constant. Recalling that in a high-temperature effective Lagrangian, the dimensionful 3-dimensional coupling constant g is related by a power of the temperature T to the dimensionless 4-dimensional coupling e, we find

$$\mu = \frac{g^2}{4\pi} n = \frac{e^2 T}{4\pi} n = \alpha T n \qquad (3.22)$$

The integer structure to μ is most fascinating. A non-vanishing mass presumably arises from a non-vanishing θ, and discontinuities in the former for the 3-dimensional model are suggestive of different phases in the latter for the 4-dimensional theory. That different values of θ correspond to different phases has been occasionally suggested. Clearly it would be most satisfying if more understanding of these speculative ideas could be obtained.

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2. Other reviews are R. Jackiw in "Asymptotic Realms of Physics" (A. Guth, K. Huang, and R. Jaffe, editors), MIT Press, Cambridge, MA, 1983 and Arctic Summer School Proceedings (1982); S. Deser, DeWitt Festschrift, to appear.
3. R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976); R. Jackiw, Rev. Mod. Phys. 52, 661 (1980).
4. S. Chern, "Complex Manifolds without Potential Theory", 2 ed. Springer Verlag, Berlin, 1979.
5. See Deser, Jackiw and Templeton, Ref. 1; Deser, Ref. 2.
6. The canonical description is due to J. Goldstone and E. Witten unpublished; for details see Jackiw, Ref. 2 (second cited work).
7. An analogous quantization condition has been obtained by E. Witten in a 4-dimensional SU(2) gauge theory, Princeton University preprint (unpublished). One begins with the observation that $\Pi_4(SU(2)) = \Pi_4(S_3)$ = cyclic group of two integers, to conclude that the 4-dimensional gauge functions $U(t,\vec{x})$ fall into two homotopically distinct classes. Next one finds that when N species of left-handed

Weyl fermions in the fundamental [doublet] representation are coupled to the SU(2) gauge field, their functional [fermionic] determinant is not invariant against homotopically non-trivial gauge transformations. Rather it changes by the factor $(-1)^N$; hence N must be even.