# ON THE APPROXIMATE SOLUTIONS OF THE NONLINEAR LANGEVIN EQUATIONS

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## 1. Introduction

Langevin approach in study of fluctuations in nonlinear systems consists in adding a fluctuating term $\beta(x)\ \eta(t)$ to the deterministic equation $\dot{x} = C_1(x)$ to obtain

$$\dot{x} = C_1(x) + \beta(x)\ \eta(t) \tag{1}$$

It is conventional to take the noise $\eta(t)$ to be a Gaussian white noise with zero mean and

$$\langle \eta(t)\ \eta(t') \rangle = 2\epsilon\ \delta(t-t') \ . \tag{2}$$

When $\beta(x)$ is a non-constant function, (1) models fluctuations arising due to the coupling of the original subsystem to external surroundings. In this case the magnitude of fluctuations in the driving noise $\eta(t)$ is amplified by an amount proportional to the strength of coupling $\beta(x)$ depending on the state variable x. When $\beta(x)$ is constant, it refers to internal fluctuations [1] (often, however, the distinction between the subsystem of interest and external surroundings is fuzzy). In the following, we will consider the case when $\beta(x) = 1$. We will discuss a few approximate methods of solutions which serve to represent fluctuations adequately in different situations.

In the following when we refer to a nonlinear Langevin equation (NLE), we will also use the equivalent Fokker-Planck equation.

The first method we consider is the system size expansion method [2] which corresponds to calculating fluctuations around the deterministic solution of (1). In this case, fluctuations are expected to be small so that one can expand

$$C_1(x) = C_1(\langle x \rangle) + (x - \langle x \rangle\ C_1'(\langle x \rangle) \tag{3}$$

and

$$\frac{dz}{dt} = z\ C_1'(\langle x \rangle) + \eta(t) \quad ; \quad z = x - \langle x \rangle$$

where z is the fluctuating part. This method holds good as long as the extensivity ansatz is not violated [3]. When the system evolves from an intrinsically unstable state, this property has to be replaced by a more general one namely, $P(X,t) = \exp [\ \Omega^{-1}\ \phi_0(x,t) + \phi_1(x,t)+\ldots]$ where X is the extensive random variable and $\Omega$ is the size of the system, and $x = \frac{X}{\Omega}$. For this case, the scaling theory [4-7] demonstrates that for large times the second term becomes as dominant as the first.

The plan of the lecture is as follows. We first outline the $\Omega$ - expansion. Drawing on results of a specific example, the limitations of the method are discussed. An alternate scheme which has essentially the same features as the $\Omega$-expansion is discussed [8,9]. The method is shown to be useful even when multiple steady states are allowed. Finally scaling theory for decay from unstable states due to Suzuki is discussed with a specific example.

## 2. $\Omega$ - Expansion

Following the Kramers-Moyal expansion it is straight forward to derive the Fokker-Planck equation [10]

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} C_1(x)\, P(x,t) + \frac{\epsilon}{2} \frac{\partial^2}{\partial x^2} C_2(x)\, P(x,t) \tag{4}$$

where $C_1(x)$ and $C_2(x)$ are the first and the second jump moments. The system size expansion depends on the fact that the fluctuating part is of the order of $\epsilon^{1/2}$ compared to the mean. This can be easily seen by considering the linear Langevin equation [7]. This means that we could express x and P(x,t) as follows :

$$x = \phi + \epsilon^{1/2}\xi \;;\; \Pi(\xi,t)d\xi = P(x,t)dx \;;\; \epsilon = \Omega^{-1} \;. \tag{5}$$

where $\Pi(\xi,t)$ corresponds to the distribution for $\xi$. The corresponding linearized equation can be obtained by using (5) in (4). By equating terms of $\epsilon^0$ and $\epsilon^{1/2}$ we get

$$\dot{\phi}(t) = -C_1(\phi(t)) \tag{6}$$

and

$$\frac{\partial \Pi}{\partial t} = -C_1'(\phi(t)) \frac{\partial \xi \Pi}{\partial \xi} + \frac{1}{2} C_2(\phi(t)) \frac{\partial^2 \Pi}{\partial \xi^2} \tag{7}$$

The solution of (7) with the initial condition $\lim_{t \to o} \Pi(\xi,t) \to \delta(\xi)$ is

$$\Pi(\xi,t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp -\frac{\xi^2}{2\sigma^2(t)} \tag{8}$$

where $\sigma^2$ is determined by

$$\frac{d\sigma^2}{dt} = -2C_1'(\phi(t))\,\sigma^2(t) + C_2(\phi(t)) \;. \tag{9}$$

It can be ascertained that $\langle\xi\rangle$ satisfies the variational equation of the deterministic solution (6). This means that the dependence of $\langle\xi\rangle$ and $\sigma^2$ are controlled by $C_1'(\phi(t))$. If the system is allowed to relax from an arbitrary initial unstable point (i.e. $C'(0))>0$), then fluctuations are initially amplified. The maximum amplification can be shown to be

$$\sigma_m^2 \simeq \left(\sigma_o^2 + \frac{C_2(\phi^*)}{2\gamma}\right) \frac{C_1'^2(\phi_m)}{\delta^2 \gamma^2} \tag{10}$$

where $\phi^* = \phi(0)$, $C_1(\phi) = \gamma > 0$, $C_1'(\phi_m) = 0$ and $\delta = \phi^* - \phi_u$. Here $\phi_u$ is the unstable point of $C(\phi) = 0$. Thus the fluctuation enhancement is a $O(\delta^{-2})$ and the fluctuations are thus anomalously large [3].

$\Omega$-expansion holds good as long as Gaussian representation is adequate. This however is not the case when multiple steady states are accessible from an initial unstable state. In this case the extensitivity property [3] which is implicit in $\Omega$-expansion is violated.

To see the limitations of $\Omega$-expansion consider

$$\frac{dn}{dt} = (\beta - \alpha)n - \frac{\gamma}{\Omega} n^2 + \eta(t) \tag{11}$$

where $\Omega$ is the size of the system. This model represents Malthus-Verhulst equation where the range of n is $[o, \infty]$. The steady states are $n_1 = o$ and $n_2 = \Omega(\beta - \alpha)/\gamma$ of

which $n_1$=o is unstable and the other is stable. We shall consider a special case when the states are very close to each other namely $n_2$ is close to $n_1$ i.e., $\beta - \alpha \simeq 0$. Choose $\beta - \alpha = \Omega^{-1/2} \Delta$ , $\beta + \alpha = 1$ and $\gamma = 1$. In this case, one can show that the equation satisfied by $\langle \xi \rangle$ is no longer the variational equation of the deterministic solution of n. ($n = \Omega \phi + \Omega^{1/2} \xi$ is the choice here). For $t \to \infty$, $\langle \xi \rangle \to \frac{\Delta}{2}$ and $\langle \xi^2 \rangle - \langle \xi \rangle^2 \to \frac{1}{4}$. This means that the position of the peak (which in the present case is determined by $\langle \xi \rangle$) is of the same order as the variance. (See ref.3 for details). Since negative values of n are unphysical, the $\Omega$-expansion is valid only for

$$\phi(t) = \frac{\phi(0)}{1+t\phi(0)} >> \Omega^{-\frac{1}{2}} [\langle \xi^2 \rangle]^{\frac{1}{2}}$$

This means that the $\Omega$-expansion is valid only for $t \ll \epsilon^{1/2}$. Now suppose we allow negative values of n also and pose the question slightly differently, namely for what values of $\phi(0)$ is the $\Omega$-expansion valid for all t ? Then, we see that

$$\phi(0) >> \Omega^{1/2} = \epsilon^{1/2} \tag{12}$$

Although, this result has been derived when the two steady states are close, the central point that has emerged namely, the mean square of the position should be much larger than the variance for the $\Omega$-expansion to be valid, is a general result. Equivalently if $\delta \ll \epsilon^{1/2}$, the approximation breaks down since the spread becomes large to allow for the two possible steady states. In such a situation the initial time development is slow and the effect of the random force is important. For the intermediate time regime nonlinearity plays a crucial role (leading to strong non-Gaussian features) and the effect of fluctuations can be ignored. In the final regime, again the effect of fluctuations cannot be ignored. Thus, we see that there are qualitatively two different kinds of time dependence for the two cases $\delta \gg \epsilon^{1/2}$ and $\delta \ll \epsilon^{1/2}$.

## 3. Generalized Statistical Linearization Scheme

The basic idea of this method is to replace the original NLE by an equivalent linear equation [8,11,12]

$$\dot{x} = \tilde{\gamma}(t)x + \tilde{C}(t) + \eta(t) \tag{13}$$

With this replacement we should not expect that the non-Gaussian features of (1) to be retained. However, we can obtain the first and the second moments of x(t) in some optimal sense by making use of the arbitrary functions $\tilde{\gamma}(t)$ and $\tilde{C}(t)$. This is done by demanding that the ensemble average of the error due to the replacement of (1) by (13) be minimum [11,12]. This leads to an optimal choice of $\gamma(t)$ and C(t). For the example $C_1(x) = \gamma x - gx^3$, after substituting for $\gamma(t)$ and C(t), we get

$$\frac{d}{dt}\langle x \rangle = \gamma \langle x \rangle - g\langle x^3 \rangle , \tag{14}$$

and

$$\frac{d}{dt}\langle x^2 \rangle = 2[\gamma \langle x^2 \rangle - g\langle x^4 \rangle + \epsilon ]. \tag{15}$$

[See ref.8, for details.] These equations are identical to the equations obtained starting from the original NLE. Since (13) is linear, the distribution is Gaussian. Hence a good approximation is obtained if we use a Gaussian decoupling of the higher moments (than two). Obviously, this approximation is meaningful only when $\delta \gg \epsilon^{1/2}$ (i.e. the extensive regime where $\Omega$-expansion is valid).

Since both the steady states become accessible for ($\delta \ll \epsilon^{1/2}$, the intrinsically unstable regime) the method fails. However, for $\phi(0) = 0$, both the steady states are equally probable and $\phi(t)=0$ for all t, it is possible to get a reasonable representation of fluctuations by choosing a bimodal Gaussian distribution for decoupling higher order moments. The distribution we choose is [8]

$$P(x,t) = K\left[H(-x)\exp - \frac{(x+x_1)^2}{2\sigma_1^2} + H(x)\exp - \frac{(x-x_1)^2}{2\sigma_1^2}\right] \tag{16}$$

where

$$K = 1/\sqrt{2\pi}\,\sigma_1\left[1 + \operatorname{erf}(x_1/\sqrt{2}\,\sigma_1)\right]$$

Here, H(x) is the step function, $x_1$ is the position of the peak and $\sigma_1^2$ is the variance as defined for one part of the distribution. Using (16) we get

$$\frac{d}{dt}\langle x^2\rangle = 2\left[\gamma\langle x^2\rangle - g\langle x^2\rangle^2 + \epsilon - 2g\,\sigma_1^2\left\{2\langle x^2\rangle - \sigma_1^2 - Kx_1\langle x^2\rangle\exp\left(-\frac{x_1^2}{2\sigma_1^2}\right)\right\}\right] \tag{17}$$

since $\sigma_1^2 = \langle x^2\rangle_s - \langle x\rangle_s^2$ (where s refers to one segment), we expect it to be small. Hence to the first approximation we can retain the first three terms, which is Suzuki's self consistent linearization scheme. (He uses this scheme to approximately demonstrate the scaling behaviour). One can do better by writing a similar equation for the fourth moment and determine $x_1$ and $\sigma_1$ self consistently.

The Gaussian decoupling scheme can be effectively used in higher dimensions whenever only the first two moments are of interest. As an example we consider a model which has a basis in plastic flow [13]. The coupled equations are

$$\dot{x} = xy - x^2 + \eta_1 \quad , \tag{18}$$

$$\dot{y} = b_o - b_1x^2y + b_2x^3 + \eta_2 \quad , \tag{19}$$

where $\eta_1$ and $\eta_2$ are taken to be Gaussian white noise with zero mean and

$$\langle \eta_i(t)\ \eta_j(t')\rangle = 2\epsilon_i\ \delta_{ij}\ \delta(t-t') \ . \tag{20}$$

In the physical problem x is a dimensionless variable related to the square root of dislocation density and y is related to dimensionless stress. There is only one steady state given by $x = y = [b_o/(b_1-b_2)]^{1/3}$ and the constants $b_i$'s are material parameters.

The moment equations can be easily obtained to be

$$\frac{d}{dt}\langle x\rangle = \langle xy\rangle - \langle x^2\rangle \tag{21}$$

$$\frac{d}{dt}\langle y\rangle = b_o - b_1\langle x^2y\rangle + b_2\langle x^3\rangle \tag{22}$$

$$\frac{d}{dt}\langle x^2\rangle = 2\left[\langle x^2y\rangle - \langle x^3\rangle + \epsilon_1\right] \tag{23}$$

$$\frac{d}{dt}\langle y^2\rangle = 2\left[b_o\langle y\rangle - b_1\langle x^2y^2\rangle + b_2\langle x^3y\rangle + \epsilon_2\right] \tag{24}$$

$$\frac{d}{dt}\langle xy\rangle = \langle xy^2\rangle - \langle x^2y\rangle + b_o\langle x\rangle - b_1\langle x^3y\rangle + b_2\langle x^4\rangle \tag{25}$$

where we have used $\langle x\,\eta_1\rangle = \epsilon_1$, $\langle y\,\eta_2\rangle = \epsilon_2$, and $\langle x\,\eta_2\rangle = \langle y\,\eta_1\rangle = 0$, in the linear approximation. Using a bivariate Gaussian distribution, we express higher order moments occuring in (21-25) in terms of the first two moments. These coupled equations can be solved on a computer. The results obtained for the variance agrees very well with the Monte Carlo results for 4900 tracks as can be seen from Fig.1. (See for details ref.8).

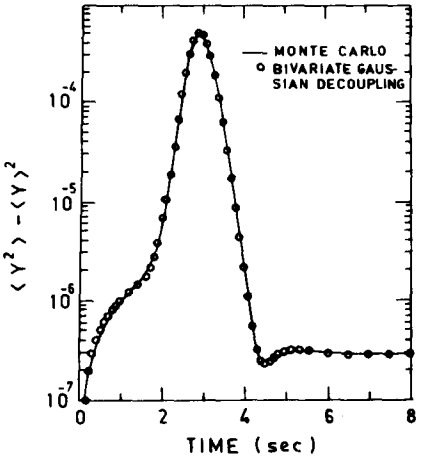


Fig 1. $\sigma_y^2$ as a function of time as obtained by Gaussian decoupling and Monte Carlo.

4. Scaling Theory

We shall briefly return to the self consistent linearization scheme, since it brings out the salient feature of the scaling theory. Suzuki linearizes the equation by writing $x^3 = \langle x^2\rangle x$ and then determines $\langle x^2\rangle$ self consistently. The solution can be expressed in terms a scaled variable $\tau$ given by

$$\langle x^2(t)\rangle = \langle x^2\rangle_{st}\,\frac{\tau}{1+\tau} \tag{26}$$

where

$$\tau = \frac{g}{\gamma}\left[\langle x^2(o)\rangle + \frac{\epsilon}{\gamma}\right]\exp\left(2\gamma t\right) \tag{27}$$

and $\langle x^2\rangle_{st} = \gamma/g + 0(\epsilon)$. In the initial regime since $\tau \sim \epsilon$, $\langle x^2\rangle \sim \epsilon$, whereas for the intermediate time $\langle x^2(t)\rangle \sim 1$. This means fluctuation enhancement of the order $\epsilon^{-1}$. (This should be contrasted with the fluctuation enhancement of $\delta^{-2}$ for the extensive regime $\delta^2 \gg \epsilon$). This fluctuation enhancement can be looked upon as formation of order. This time is

$$t = \frac{1}{2\gamma} \ln \frac{g}{\gamma}\left[\langle x^2(o)\rangle + \frac{\epsilon}{\gamma}\right]^{-1}$$

Even within the framework of the simple decoupling scheme we see the utility of the scaling variable. Although the decoupling scheme appears ad hoc the analysis presented in the previous section offers some justification [8].

We shall now outline the scaling theory via a nonlinear time independent transformation which essentially transforms the NLE into a linear Langevin equation in terms of the transformed variable.

Consider

$$\frac{dx}{dt} = C_1(x) + \eta(t) \tag{28}$$

For an unstable state, $\gamma = C_1'(0) > 0$, where we have assumed $x = 0$ is the unstable point of $C_1(x)$.

Define

$$\xi = F(x) = \exp \int_{a_o}^{x} \frac{\gamma}{C_1(y)}\, dy \tag{29}$$

where $a_o$ is taken such that $F'(o) = 1$. For the special case $C_1(x) = \gamma x - gx^3$ we have

$$\xi = F(x) = x\left[1 - \frac{g}{\gamma} x^2\right]^{-1/2} \tag{30}$$

and

$$x = F^{-1}(\xi) = \xi\left[1 + \frac{g}{\gamma} \xi^2\right]^{-1/2} \tag{31}$$

Then

$$\frac{d\xi}{dt} = \gamma\xi + \frac{\gamma \xi\, \eta(t)}{C_1(F^{-1}(\xi))} \tag{32}$$

Define

$$\frac{\gamma\xi}{C_1(F^{-1}(\xi))} = 1 + f(\xi) \tag{33}$$

For small $\xi$, $f(\xi) \to \xi$.

Then

$$\frac{d\xi}{dt} = \gamma\xi + \eta(t)\,(1 + f(\xi)) \tag{34}$$

Dropping $f(\xi)$ (in the scaling approximation), we get

$$\frac{d\xi}{dt} = \gamma\xi + \eta(t) \tag{35}$$

This has a Gaussian structure in $\xi$. Therefore we have

$$\xi_{sc}(t) = e^{\gamma t} \int_o^t e^{-\gamma t'} \eta(t')dt' + e^{\gamma t}\, \xi(o) \tag{36}$$

and

$$\langle \xi_{sc}^2 \rangle = \left[\langle \xi^2(o)\rangle + \frac{\epsilon}{\gamma}\right] e^{2\gamma t} - \frac{\epsilon}{\gamma} \tag{37}$$

From this we can calculate $\langle x_{sc}^2 \rangle$. We shall do it for the particular example considered.

$$\langle x^2(t)\rangle_{sc} = \langle \xi_{sc}^2(t)\left[1 + \left(\frac{g}{\gamma}\right)\xi_{sc}^2(t)\right]^{-1}\rangle = \sum_{n=1}^{\infty}\left(-\frac{g}{\gamma}\right)^{n-1}\langle \xi_{sc}^{2n}(t)\rangle$$

$$= \sum_{n=1}^{\infty}\left(-\frac{g}{\gamma}\right)^{n-1} e^{2n\gamma t}\int_0^t dt_1\, e^{-\gamma t_1} \cdots \int_0^t dt_{2n}\, e^{-\gamma t_{2n}}\langle \eta(t_1)\cdots\eta(t_{2n})\rangle$$

$$= \sum_{n=1}^{\infty}\left(-\frac{g}{\gamma}\right)^{n-1}(2n-1)!!\; e^{2n\gamma t}\left[\int_0^t dt_1 \int_0^t dt_2\, e^{-\gamma(t_1+t_2)}\langle \eta(t_1)\,\eta(t_2)\rangle\right]^n$$

$$\langle x^2(t)\rangle_{sc} = \sum_{n=1}^{\infty}\left(-\frac{g}{\gamma}\right)^{n-1}(2n-1)!!\left[\frac{\epsilon}{\gamma}\left(e^{2\gamma t}-1\right)\right]^n \tag{38}$$

Using

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\xi^2}\,\xi^{2n}\,d\xi = (2n-1)!! \tag{39}$$

in (38), and switching order of summation and integration, we get

$$\langle x^2(t)\rangle_{sc} = \langle x^2\rangle_{st}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\xi^2/2}\,\frac{\xi^2\tau}{1+\xi^2\tau}\,d\xi\,, \tag{40}$$

where

$$\tau = \frac{g\epsilon}{\gamma^2}\left(e^{2\gamma t}-1\right) \tag{40a}$$

If $\langle x^2(o)\rangle \neq 0$, then

$$\tau = \frac{g}{\gamma}\left[\frac{\epsilon}{\gamma}\left(e^{2\gamma t}-1\right) + \langle x^2(o)\rangle\, e^{2\gamma t}\right] \tag{40b}$$

To see how good the Gaussian decoupling scheme (where the first two non-vanishing moments are considered) and the self consistent linearization scheme perform, we have plotted in Fig.2, the variance as calculated by these methods along with the scaling result. Also shown is the variance calculated by Monte Carlo method (for 4900 tracks), and numerical solution of Fokker-Planck equation [9]. As expected the scaling theory performs the best (compare with Monte Carlo result which is numerically exact). In spite of the fact that the bimodal Gaussian decoupling is a simple representation of the actual non-Gaussian process it does perform reasonably well.

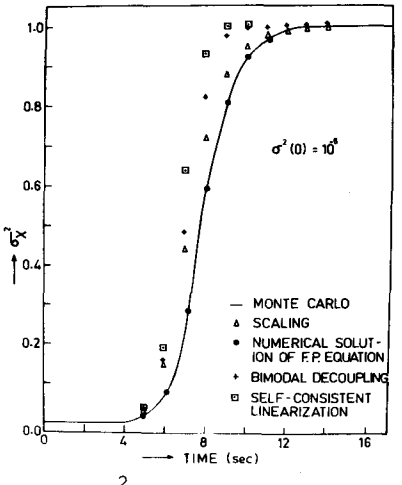


Fig. 2 $\sigma_x^2$ as a function of time as obtained by various methods.

The results of scaling theory hold good only for intermediate time regime where nonlinearities play an important role. If one wishes to consider the evolution of the system for $t \to \infty$ limit, we have to include the effect of fluctuations. Such an extention has been carried out by Suzuki [6].

The problem of relaxation from an intrinsically unstable state has been studied by a number of other authors [14-20]. For example de Pasquale and co-workers [16-18] introduce a time dependent nonlinear transformation given by

$$\xi = F^{-1} (e^{-\gamma t} F(x))$$

It represents the characteristic curve of the stochastic process and a constant along the deterministic path $\frac{dx}{dt} = C_1(x)$. The results via this transformation are not significantly different. The relation of some of these apparently different methods to Suzuki's work has been analysed in detail by Suzuki in his recent reviews on the subject[6,7]. For an interested reader we refer to these reviews.

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