SOLUTION OF FOKKER-PLANCK EQUATIONS USING TROTTER'S FORMULA

M.C. Valsakumar
Reactor Research Centre, Kalpakkam 603 102, India

1. Introduction

Fokker-Planck equations (FPE) with nonlinear drift terms are rarely amenable to closed form solutions [1,2]. The noteworthy methods of exact solution of FPE are the eigenfunction method [2,3] and the path integral formulation [4,5]. Both these methods, for their application, demand the FPE be first set in its self adjoint form. If the complete eigen spectrum of the corresponding Hermitian operator is obtainable, then the eigenfunction method can be used. The latter method, on the other hand, provides a formal Feynman path integral representation of the propagator of the above operator. In the present paper we show that the path integral solution to the FPE can be obtained without setting it in its self adjoint form. The method makes explicit use of the Trotter's product formula [6] widely used in perturbation theory [7]. The integral representation of the solution process is easily amenable to approximations. First order approximation gives the scaling solution [8], which has been demonstrated to be of remarkable success in the treatment of diffusion from intrinsically unstable states in a bistable potential [8-10].

2. The Method

We consider solving the FPE given by

$$\frac{\partial}{\partial t} P(x,t) = L\, P(x,t)\,; \quad L = -\gamma \frac{\partial}{\partial x} C(x) + \epsilon \frac{\partial^2}{\partial x^2} \tag{1}$$

for the initial condition

$$P(x,o) = \delta(x-y) = (2\pi)^{-1} \int_{-\infty}^{\infty} dk \; e^{-ik(x-y)} \tag{2}$$

Since the operator L is a sum of two noncommuting operators, the formal solution exp (tL) P(x,o) of eq.(1) cannot be of any immediate use. However, by exploiting the Trotter's formula [6], which reads as

$$e^{A+B} = \lim_{n\to\infty} \left[e^{A/n} \; e^{B/n} \right]^n , \tag{3}$$

we can represent the formal solution of eq.(1) in a more convenient form as

$$P(x,t) = \lim_{n\to\infty} \frac{\theta^n}{2\pi} \int_{-\infty}^{\infty} dk \; e^{-ik(x-y)} \; ; \quad \theta = e^{-b \frac{d}{dx} C(x)} \; e^{a \frac{d^2}{dx^2}} \tag{4}$$

In the above expression $a = \epsilon t/n$ and $b = \gamma t/n$.

We now illustrate the method with the Ornstein-Uhlenbeck (O-U) process for which $C(x) = -x$. Using the relations

$$e^{a\frac{d^2}{dx^2}} e^{-ikx} = e^{-[ak^2 + ikx]} \quad , \quad e^{b\frac{d}{dx}x} e^{-ikx} = e^{(b-ike^b x)} \quad , \tag{5}$$

eq. (4) for this stochastic process reduces to

$$P(x,t) = \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ak^2 \sum_{r=1}^{n} e^{2(r-1)b}} e^{nb-ik(xe^{nb}-y)} \tag{6}$$

Summing the series in the integrand, and recognising that

$\lim_{n\to\infty} a/(e^{2b} - 1) = \epsilon/(2\gamma)$, we get the O-U solution

$$P(x,t) = [2\pi \frac{\epsilon}{\gamma} (1-e^{-2\gamma t})]^{-\frac{1}{2}} e^{-(x-ye^{-\gamma t})^2 / \frac{2\epsilon}{\gamma}(1-e^{-2\gamma t})} \tag{7}$$

3. Solution for General Nonlinear Drift Term

The solution for a general $C(x)$ is not so simple as in the case of the O-U process. The reason for this is that the quantity $\theta\, e^{-ikx}$ does not go to the form $\alpha\, e^{-\beta x}$ where α and β are independent of x. In fact we have

$$e^{-b\frac{d}{dx}C(x)} f(x) = \frac{C[G(x)]}{C[x]} f[G(x)] \tag{8}$$

where

$$G(x) = \xi = F^{-1}(F(x)\, e^{-b}) ; \; F(x) = \exp[\int^{x} dx'/C(x')] \tag{9}$$

Therefore we get

$$\theta e^{-ikx} = \frac{C[G(x)]}{C[x]} e^{-[ak^2 + ikG(x)]} = H(k,x) \quad , \tag{10}$$

which in the Fourier representation reads

$$\theta e^{-ikx} = (2\pi)^{-1} \int_{-\infty}^{\infty} dk_1 e^{-ik_1 x} \int dx_1 e^{ik_1 x_1} H(k,x_1) \tag{11}$$

In eq.(11), the limits of x_1 integral is such that $G(x_1)$ is real. Repeating the process n times yields

$$P(x,t) = \lim_{n\to\infty} (2\pi)^{-n} \int ..\int dk \prod_{i=1}^{n-1} dk_i dx_i\, e^{ik_i x_i}\, e^{iky} H(k_{n-1},x)\ldots H(k,x_1) \tag{12}$$

Using the fact that

$$\frac{dG(x)}{dx} = \frac{\partial \xi}{\partial x} = \frac{C[G(x)]}{C[x]} \text{ and } P(\xi,t)d\xi = P(x,t)dx \, , \tag{13}$$

$$P(\xi,t) = \lim_{n\to\infty} (2\pi)^{-n} \int_{-\infty}^{\infty} ..\int_{-\infty}^{\infty} dk \prod_{i=1}^{n-1} dk_i\, d\xi_i\, e^{ik_i G^{-1}(\xi_i) - ak_i^2}$$
$$e^{iky-ak^2} e^{-i[k_{n-1}\xi + \ldots\ldots + k\xi_1]} \tag{14}$$

Performing the $\{k\}$ integrations gives the equivalent result

$$P(\xi,t) = \lim_{n\to\infty} (4\pi a)^{-n/2} \int_{-\infty}^{\infty}..\int_{-\infty}^{\infty} \prod_{i=1}^{n-1} d\xi_i \; e^{-\frac{1}{4a}\left\{[\xi - G^{-1}(\xi_{n-1})]^2 + \dots + [\xi_1 - y]^2\right\}} \tag{15}$$

These results [eqs.(14) and (15)] are equivalent to the path integral result [4,5].

4. Approximate Closed Form Solution

We repeat that the results given by eqs.(14) and (15) are only formal and they do not lead to simple closed form solutions for arbitrary drift terms. So approximations are imperative in the useage of these. An expansion in b (which is necessarily small) may be a choice. But one can easily satisfy himself, with explicit application of this procedure on the O-U process, that this will lead to erroneous results. In what follows, we show that an expansion of $G^{-1}(\xi)$ in powers of ξ gives very good results. We illustrate this with the well known model• for diffusion in a bistable potential for which

$$C(x) = x - \frac{g}{\gamma} x^3 \tag{16}$$

When the diffusion is from the extensive regime (initial point sufficiently far from the unstable steady state), the probability distribution function is practically unimodal. Hence the approximation schemes based on 'linearisation' works well [9-11]. On the other hand, when the evolution is from the intrinsically unstable states (initial point close to the unstable steady state), the distribution function is necessarily bimodal and hence the linearisation approximations fail. Only the scaling theory (and its equivalent ones) gives the best results [8,9].

For this problem, the transformation $\xi = G(x)$ and its inverse are

$$\xi = x\, e^{-b} [1 - \frac{g}{\gamma} x^2 (1 - e^{-2b})]^{-\frac{1}{2}} \tag{17}$$

$$G^{-1}(\xi) = \xi e^{b} [1 + \frac{g}{\gamma} \xi^2 (e^{2b} - 1)]^{-\frac{1}{2}} \tag{18}$$

To the first order in ξ

$$G^{-1}(\xi) = \xi e^{b} \tag{19}$$

Suzuki [8] has demonstrated this approximation to be good in the scaling limit (i.e. $\epsilon \to o$ and b fixed). On the substitution of this in eq.(14), the $\{\xi\}$ integrals yield delta function in $\{k\}$. Performing the $\{k\}$ integrals and changing the argument of ξ to the actual time variable of interest (γt), we get

$$P(\xi(\gamma t),t) = [2\pi \frac{\epsilon}{\gamma}(1-e^{-2\gamma t})]^{-\frac{1}{2}} \; e^{-(\xi-y)^2/\frac{2\epsilon}{\gamma}(1-e^{-2\gamma t})} \tag{20}$$

This is the same as the scaling solution of Suzuki [8]. Suzuki has shown that the approximation given by eq.(19) is valid for any C(x) with x as the leading term. Secondly it is the presence of the small diffusion constant ϵ that allows this approximation to be good. Hence under the same circumstances and because of the same reason, our formal results also lead to the scaling solution.

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