STABILITY OF STOCHASTIC SYSTEMS

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1. Introduction

The analysis of stability plays a crucial role in the study of parametrically excited linear systems. Similar problems also appear during the investigation of the local stability of the equilibrium states of nonlinear systems under non-parametric excitation. The concepts of stability and asymptotic stability of deterministic systems were rigorously defined by Lyapunov. But there is no unique way of extending these concepts of stability to systems with stochastic inputs.

This article presents a brief survey of the literature on stochastic stability. Various definitions of stability are given in Section 2. A sufficient condition for almost sure asymptotic stability is derived in Section 3. Necessary and sufficient conditions for almost sure asymptotic stability are discussed in Section 4. A new result concerning the necessary and sufficient condition for almost sure asymptotic stability of a second order system with narrowband excitation is derived in Section 5.

2. Definitions of stability

Consider the linear system represented by the equation

$$\frac{dx}{dt} = \left[A + F(t)\right] x \quad , \tag{1}$$

where $x = x(t)$ is an n-dimensional vector

$$x^T = \left\{x_1, x_2 \ldots\ldots x_n\right\} \quad , \tag{2}$$

and A and F(t) and n x n matrices. We are interested in the stability of the equilibrium solution $x(t) = 0$.

Stability in the Lyapunov sense is uniform convergence of the solution with respect to the initial conditions. Let $x(t;x_o,t_o)$ denote the solution at time t corresponding to an initial state x_o at time t_o, and let $\|x\|$ denote a suitable norm, e.g.,

$$\|x\| = (x_1^2 + x_2^2 + \ldots\ldots\ldots + x_n^2)^{\frac{1}{2}} \quad . \tag{3}$$

For deterministic systems, i.e., when all elements for F(t) are deterministic functions of t, definitions of Lyapunov stability are stated as follows [1] :

1. Lyapunov stability

The equilibrium solution is stable if, given $\varepsilon > 0$, there exists $\delta(\varepsilon, t_o) > 0$ such that

$$\|x_o\| < \delta \text{ implies } \sup_{t \geq t_o} \|x(t;x_o,t_o)\| < \varepsilon \ .$$

2. Asymptotic Lyapunov stability

The equilibrium solution is asymptotically stable if (i) it is stable and (ii) there exists $\delta > 0$ such that

$$\| x_o \| < \delta \text{ implies } \lim_{t\to\infty} \| x(t;x_o,t_o) \| = 0 \quad .$$

In the case of stochastic systems, one or more elements of F(t) are random functions of t, and the definition of stability varies depending on the chosen mode of convergence. Application of the three commonly employed modes of stochastic convergence [2], viz. (i) convergence in probability, (ii) convergence in the mean, and (iii) almost sure convergence (or convergence with probability 1), leads to the following definitions of stochastic stability [3] :

I_p. Lyapunov stability in probability

The equilibrium solution possesses stability in probability if, given $\varepsilon, \varepsilon' > 0$, there exists $\delta(\varepsilon, \varepsilon', t_o) > 0$ such that

$$\| x_o \| < \delta \text{ implies } P\left[\sup_{t \geq t_o} \| x(t;x_o,t_o) \| > \varepsilon'\right] < \varepsilon .$$

I_m. Lyapunov stability in the mean

The equilibrium solution possesses stability in the mean if, given $\varepsilon > 0$, there exists $\delta(\varepsilon, t_o) > 0$ such that

$$\| x_o \| < \delta \text{ implies } E\left[\sup_{t \geq t_o} \| x(t;x_o,t_o) \| \right] < \varepsilon .$$

$I_{a.s.}$ Almost sure Lyapunov stability

The equilibrium solution possesses almost sure stability if

$$P\left[\lim_{\|x_o\| \to 0} \sup_{t \geq t_o} \| x(t;x_o,t_o) \| = 0\right] = 1 .$$

II_p. Asymptotic Lyapunov stability in probability

The equilibrium solution is asymptotically stable in probability if I_p holds and there exists $\delta > 0$ such that

$$\| x_o \| < \delta \text{ implies } \lim_{T\to\infty} \left[\sup_{t \geq T} \| x(t;x_o,t_o) \| > \varepsilon\right] = 0$$

for any $\varepsilon > 0$.

II_m. Asymptotic Lyapunov stability in the mean

The equilibrium solution is asymptotically stable in the mean if I_m holds and there exists $\delta > 0$ such that

$$\| x_o \| < \delta \text{ implies } \lim_{T\to\infty} E\left[\sup_{t \geq T} \| x(t;x_o,t_o) \|\right] = 0 .$$

$II_{a.s}$ Almost sure asymptotic Lyapunov stability

The equilibrium solution has almost sure asymptotic stability if $I_{a.s}$ holds and if

$$\lim_{T\to\infty} P\left[\sup_{t\geq T} \|x(t;x_o,t_o)\| = 0\right] = 1 \quad .$$

In all these definitions of stochastic stability, it is the random variable $\sup_{t\geq t_o} \|x(t;x_o,t_o)\|$ whose convergence is tested relative to the parameter x_o, where $x(t;x_o,t_o)$ now represents a sample solution. This random variable depends on the behaviour of the sample solution over the entire interval (t_o,∞).

In the early stages of development of the subject, most studies were concerned with the stability of various moments of the solution. For instance, the following definition of stability was used extensively.

III. Stability of the mean

The equilibrium solution possesses stability of the mean if the mean exists and if

$$\lim_{\|x_o\|\to 0} E\left[\|x(t;x_o,t_o)\|\right] = 0 \quad \text{for all } t \geq t_o.$$

Similar definitions can be formulated for stability of moments of higher order. Investigation of this type of stability involves the study of statistical behaviour which is simpler than the study of sample behaviour. But when we test a real system subjected to random excitation, we observe a sample solution. Hence, it is more appropriate to study sample stability rather than moment stability. Most of the work on stochastic stability during the last twenty years is concerned with almost sure sample stability as defined by $I_{a.s}$ or $II_{a.s.}$

3. Sufficient conditions for almost sure Asymptotic Stability

We consider, once again, a system governed by equation (1). in which all non-identically zero elements of the matrix $F(t)$ are assumed to be sample functions of ergodic random processes. Choosing the norm

$$\|x\|_p = x^T Px \quad , \tag{4}$$

where P is a positive definite constant matrix, it can be shown that

$$\|x(t)\|_p = \|x(0)\|_p \exp\left(\int_0^t g(s)\, ds\right) \quad , \tag{5}$$

where

$$g(t) = \frac{x^T\left\{\left[A + F(t)\right]^T P + P\left[A + F(t)\right]\right\} x}{x^T P x} \tag{6}$$

It can also be shown that

$$\max_x g(t) = \lambda(t) \quad , \tag{7}$$

where $\lambda(t)$ is the maximum eigenvalue of the matrix $\left\{\left(A+F(t)\right)^T P+P\left(A+F(t)\right)\right\} P^{-1}$.

From equations (5) and (7) we get

$$\|x(t)\|_p < \|x(0)\|_p \exp\left(\int_0^t \lambda(s)\, ds\right) \quad . \tag{8}$$

Since F(t) is a matrix of ergodic processes, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda(s) = E\left[\lambda(t)\right] \quad . \tag{9}$$

Hence, a sufficient condition for almost sure asymptotic stability is

$$E\left[\lambda(t)\right] < 0 \quad . \tag{10}$$

The stability condition (10) was first derived by Kozin [4]. A lot of effort [5-7] has gone into sharpening the sufficient condition for stability through a proper choice of the matrix P. Kozin and Wu [8] have exploited the first order probability density functions of the co-efficient processes to further sharpen the sufficient condition for stability. Sufficient conditions for almost sure stability employing the spectral density of the parametric excitation have been determined by Gray [9] and Ariaratnam [10].

4. Necessary and Sufficient Conditions for Almost Sure Asymptotic Stability

It is evident from equation (5) that $\lim_{t\to\infty} \|x(t)\| = 0$ with probability 1 if

$$P\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t g(s)\,ds < 0\right) = 1. \tag{11}$$

Hence, if equation (11) is satisfied, we have almost sure asymptotic stability of the equilibrium solution. On the other hand, if the limit in equation (11) is positive with probability 1, we have instability. Hence, equation (11) is a necessary and sufficient condition for almost sure asymptotic stability. It is therefore necessary to establish the existence of the limit in equation (11) with probability 1, and then to evaluate this limit. Khasminskii [11] has shown that such limits can be evaluated for linear systems with Gaussian white noise coefficients. Kozin and Prodromou [12] have applied this technique to a second order linear system with Gaussian white noise coefficients. But the problem of determining the necessary and sufficient stability conditions in the case of 'coloured noise' coefficients remains unsolved.

5. Stability of Linear Second Order Systems with Narrowband Coefficients

A method of determining the regions of stability and instability of a linear second order system with ergodic narrowband coefficients is presented in this Section. The system considered is governed by the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \left(1 + F(t)\right) x = 0 \quad , \tag{12}$$

where F(t) is a sample function of an ergodic zero-mean narrowband random process. Equations of this type arise during the investigation of stability of nonlinear systems of the type

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + y + \mathcal{E}\, g(y) = f(t) \quad , \tag{13}$$

where f(t) is a sample function of an ergodic random process. If $y_o(t)$ is a solution of equation (13) and x(t) represents a perturbation of this solution, the evolution of x(t) with time is governed by an equation of the form (12). Local stability of the solution $y_o(t)$ of equation (13) depends upon the stability of the trivial solution $x(t) = 0$ of equation (12).

Since F(t) is a sample function of a narrow band process, it can be written as

$$F(t) = f_1(t)\cos 2\omega t + f_2(t)\sin 2\omega t \quad , \tag{14}$$

where 2ω is the centre frequency of F(t). The functions $f_1(t)$ and $f_2(t)$ vary slowly with time as compared to the trigonometric functions in equation (14).

Equation (12) can be converted into a system of first order equations

$$\dot{x}_1 = x_2 \quad , \tag{15}$$

$$\dot{x}_2 = -\left(1+f_1(t)\cos 2\omega t + f_2(t)\sin 2\omega t\right)x_1 - 2kx_2 \; , \tag{16}$$

where the overhead dot denotes differentiation with respect to t. The solution of equations (15) and (16) can be written as

$$x_1 = a_1(t)\cos\omega t + a_2(t)\sin\omega t \quad , \tag{17}$$

$$x_2 = \omega\left(a_2(t)\cos\omega t - a_1(t)\sin\omega t\right) \quad , \tag{18}$$

where $a_1(t)$ and $a_2(t)$ are also slowly varying functions of t. On substituting equations (17) and (18) into equations (15) and (16) and solving for $\dot{a}_1$ and $\dot{a}_2$, we get

$$\omega\dot{a}_1 = g\sin\omega t, \qquad \dot{a}_2 = -g\cos\omega t \quad , \tag{19}$$

$$g = (1-\omega^2+f_1\cos 2\omega t + f_2\sin 2\omega t)(a_1\cos\omega t + a_2\sin\omega t) + 2k\omega(a_2\cos\omega t - a_1\sin\omega t) \; . \tag{20}$$

We solve equations (19) by the method of averaging [13]. The expression for g contains the stochastic terms $a_1(t)$ and $a_2(t)$ and also the oscillatory terms $\cos\omega t$, $\sin\omega t$, etc. Since $a_1(t)$ and $a_2(t)$ are slowly varying functions of time, the change in their values during a period $(2\pi/\omega)$ is very small. Hence, equations (19) may be replaced by their time averages over a period $(2\pi/\omega)$, assuming a_1 and a_2 to be constant. If this is done, the resulting equations are

$$\omega\dot{a}_1 = \left(\frac{1}{4}f_2(t) - k\omega\right)a_1 - \frac{1}{2}\left(\omega^2-1+\frac{1}{2}f_1(t)\right)a_2 \quad , \tag{21}$$

$$\omega\dot{a}_2 = \frac{1}{2}\left(\omega^2-1-\frac{1}{2}f_1(t)\right)a_1 - \left(\frac{1}{4}f_2(t)+k\omega\right)a_2 \quad . \tag{22}$$

If the co-efficients appearing in equations (21) and (22) were constant, the solutions would be of the form

$$a_i(t) = c_i \, e^{\lambda t} \, , \; i = 1,2 \, . \tag{23}$$

In the present case, the co-efficients are slowly varying functions of time. So, we seek solutions of the form

$$a_i(t) = c_i \exp\left(\int_0^t \lambda(s) \; ds \right) . \tag{24}$$

On substituting equation (24) into equations (21) and (22), we get the following characteristic equation

$$\omega^2 \lambda^2 + 2k\omega^2\lambda - \left(\frac{1}{16} (f_1^2 + f_2^2) - \frac{1}{4} (\omega^2-1)^2 - k^2\omega^2 \right) = 0 \, , \tag{25}$$

whose roots are

$$\lambda_{1,2} = -k \pm \frac{1}{4\omega} \left(f_1^2 + f_2^2 - 4(\omega^2 - 1)^2 \right)^{\frac{1}{2}} . \tag{26}$$

Since F(t) is an ergodic process, the processes $\lambda_1(t)$ and $\lambda_2(t)$ are also ergodic. Hence, we can write

$$\lim_{t\to\infty} a_i(t) = \lim_{t\to\infty} \sum_{j=1}^{2} c_{ij} \exp\left(\int_0^t \lambda_j(s) \; ds \right)$$

$$= \sum_{j=1}^{2} c_{ij} \exp\left(E(\lambda_j)t \right) , \; i = 1,2 \, , \tag{27}$$

where E(..) denotes expectation. Hence, the solutions represented by equations (17) and (18) are stable if and only if both $E(\lambda_1)$ and $E(\lambda_2)$ have negative real parts. Let

$$(f_1^2 + f_2^2)^{\frac{1}{2}} = R \, , \tag{28}$$

and let $P_R(r)$ be the probability density function of R. Then, we have

$$\text{Re} \; E\left[\left(f_1^2 + f_2^2 - 4\,(\omega^2-1)^2 \right)^{\frac{1}{2}} \right]$$

$$= \int_{|2(\omega^2-1)|}^{\infty} \left(r^2-4\,(\;\; ^2-1)^2 \right)^{\frac{1}{2}} P_R(r) \; dr \, . \tag{29}$$

Hence, the necessary and sufficient condition for stability is

$$\int_{|2(\omega^2-1)|}^{\infty} \left(r^2-4(\omega^2-1)^2 \right)^{\frac{1}{2}} P_R(r)\,dr < 4k\omega \, . \tag{30}$$

If F(t) is a Gaussian process with mean 0 and variance σ^2, R has a Rayleigh density, i.e.

$$p_R(r) = \frac{r}{\sigma^2} \exp\left(- \frac{r^2}{2\sigma^2} \right) . \tag{31}$$

On substituting (31) and (30) and evaluating the integral, the condition for stability is obtained as

$$\sigma \exp\left(- \frac{2(\omega^2-1)^2}{\sigma^2} \right) < 4(2/\pi)^{\frac{1}{2}} \; k\omega. \tag{32}$$

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