GAUSSIAN STOCHASTIC PROCESSES

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Introduction

Among all possible stochastic processes, Gaussian stochastic processes constitute a very important class. These occur in many areas of physics. A historically important example of a Gaussian stochastic process is that of Brownian motion. The intensity of the light emitted by a thermal source is another example of such a process. The main reason why Gaussian stochastic processes have been studied so extensively is that they are completely specified by the first two moments. This makes them particularly easy to handle.

We shall begin our discussion on Gaussian stochastic processes by studying Gaussian random variables. This, as we shall see later, will enable us to define Gaussian stochastic processes and to discuss some of their important properties.

Gaussian Random variables

Let us briefly recapitulate what a Gaussian random variable is.

A random variable X is defined by specifying

(i) the range of values x it can take and

(ii) a probability distribution over this range.

A random variable is said to be Gaussian if the range of values it can take extends from $-\infty$ to $+\infty$ and if the probability distribution over this range is a Gaussian distribution

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\langle X\rangle)^2}{2\sigma^2}\right) \tag{1}$$

If instead of a single variable we have a vector $\underline{X}$ having n components then (1) generalizes to

$$P(x_1 \ldots x_n) = \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \exp\left[-(\underline{x}-\langle\underline{X}\rangle)^T A(\underline{x}-\langle\underline{X}\rangle)\right] \tag{2}$$

where A is a positive definite symmetric matrix. The probability distribution (2) is known as a multivariate Gaussian distribution. We shall now discuss some of its properties.

(a) Let us first check that $\langle\underline{X}\rangle$ appearing on the R.H.S. of (2) is indeed the mean value of $\underline{X}$ and that the distribution is correctly normalised.

$$\langle\underline{X}\rangle = \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} d\underline{x}\,(\underline{x}) \exp\left[-\frac{1}{2}(\underline{x}-\langle\underline{X}\rangle)^T A\,(\underline{x}-\langle\underline{X}\rangle)\right]$$

$$= \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} d\underline{y}\,[\underline{y}+\langle\underline{X}\rangle] \exp\left[-\frac{1}{2}\underline{y}^T A\,\underline{y}\right]$$

Since A is a symmetric matrix, it can be diagonalized by an orthogonal matrix.

$$\Lambda = S^T A S \;;\; S^T S = 1$$

Putting $\underline{y} = S\underline{z}$ in the expression above and making use of the fact that the Jacobian of the transformation is unity we obtain

$$\langle \underline{X} \rangle = \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} d\underline{z} \left[S\,\underline{z} + \langle \underline{X} \rangle \right] \exp\left[-\frac{1}{2} \underline{z}^T \Lambda \underline{z} \right]$$

The first term, being odd in $\underline{z}$, vanishes so that

$$\langle \underline{X} \rangle = \langle \underline{X} \rangle \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} dz_1 \ldots dz_n \exp\left[-\frac{1}{2} \sum_{i=1}^{n} \Lambda_i z_i^2 \right]$$

$$= \langle \underline{X} \rangle \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \frac{(2\pi)^{n/2}}{(\Lambda_1 \ldots \Lambda_n)^{\frac{1}{2}}} = \langle \underline{X} \rangle$$

Here Λ_i are the eigenvalues of A .

This also shows that (2) is correctly normalised.

(b) We now want to show that

$$\langle (X_i - \langle X_i \rangle)\, (X_j - \langle X_j \rangle) \rangle = (A^{-1})_{ij} \tag{3}$$

$$\langle (X_i - \langle X_i \rangle)\, (X_j - \langle X_j \rangle) \rangle = \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} d\underline{y}\; (y_i y_j) \exp\left[-\frac{1}{2} \underline{y}^T A\, \underline{y} \right]$$

$$= \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} d\underline{z}\; (S\underline{z})_i (S\underline{z})_j \exp\left[-\frac{1}{2} \underline{z}^T \Lambda\, \underline{z} \right]$$

$$= \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \sum_{\alpha\beta} S_{i\alpha} S_{j\beta} \int_{-\infty}^{\infty} d\underline{z}\; z_\alpha z_\beta \exp\left[-\frac{1}{2} \sum_{i=1}^{n} \Lambda_i z_i^2 \right]$$

$$= \sum_{\alpha\beta} S_{i\alpha} S_{j\beta}\; \delta_{\alpha\beta} \frac{1}{\Lambda_\alpha} = (S \Lambda^{-1} S^T)_{ij} = (A^{-1})_{ij}$$

We thus see that a Gaussian distribution is completely determined by the mean values of the variables and the second moments.

Henceforth we shall assume for convenience that the variables X_i have zero mean i.e. $\langle \underline{X} \rangle = 0$. All the results for the case $\langle \underline{X} \rangle \neq 0$ can easily be obtained by replacing $\underline{X}$ in the following by $\underline{X} - \langle \underline{X} \rangle$.

(c) We now establish a very useful property of multivariate Gaussian distribution

$$\langle X_i f(\underline{X}) \rangle = \sum_j \langle X_i X_j \rangle \left\langle \frac{\partial f(\underline{X})}{\partial X_j} \right\rangle \tag{4}$$

where $f(\underline{X})$ is a polynomial in the X_i's.

To prove (4) we rewrite it using (3) as

$$\langle x_i f(\underline{X}) \rangle = \sum_j (A^{-1})_{ij} \langle \frac{\partial f(\underline{X})}{\partial x_j} \rangle$$

or

$$\sum_j A_{ij} \langle x_j f(\underline{X}) \rangle = \langle \frac{\partial f(\underline{X})}{\partial x_i} \rangle$$

Now

$$\sum_j A_{ij} \langle x_j f(\underline{x}) \rangle = \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} d\underline{x}\, f(\underline{x}) \sum_j A_{ij}\, x_j \exp\left[-\frac{1}{2} \sum_{l\,m} x_l A_{lm} x_m\right]$$

$$= -\frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} d\underline{x}\, f(\underline{x}) \frac{\partial}{\partial x_i} \exp\left[-\frac{1}{2} \underline{x}^T A \underline{x}\right]$$

and on integrating by parts

$$= \langle \frac{\partial f(X)}{\partial x_i} \rangle$$

Hence the proof.

(d) Repeated use of (4) enables us to show that all the even moments of a Gaussian distribution with zero mean factorize pairwise into the second moments. (The odd moments of such a distribution of course vanish as is easily seen.) Consider for instance the fourth moment $\langle x_i x_j x_k x_l \rangle$. From (4) we have

$$\langle x_i x_j x_k x_l \rangle = \sum_\alpha \langle x_i x_\alpha \rangle \langle \frac{\partial}{\partial x_\alpha} (x_j x_k x_l) \rangle$$

$$= \langle x_i x_j \rangle \langle x_k x_l \rangle + \langle x_i x_k \rangle \langle x_j x_l \rangle + \langle x_i x_l \rangle \langle x_j x_k \rangle \quad (5)$$

The R.H.S. of (5) can be written compactly as

$$= \sum_{\text{pairs}} \langle x_p x_q \rangle \langle x_r x_s \rangle$$

where the indices p,q,r,s etc. are the same as i,j,k,l and the summation extends over all different ways in which i,j,k,l can be divided into pairs.

Proceeding in a similar fashion, we have, in general

$$\langle x_i x_j x_k x_l \ldots\ldots \rangle = \sum_{\text{pairs}} \langle x_p x_q \rangle \langle x_r x_s \rangle \ldots\ldots \quad (6)$$

Thus the even moments of a Gaussian with zero mean factorise as in (6). For a moment of order 2k, there are $(2k!/2^k\, k!)$ terms on the R.H.S. of (6). Conversely one can show that if the moments of a probability distribution factorise as in (6) then the distribution is a Gaussian. It therefore follows that (6) is both necessary and sufficient for a distribution to be a Gaussian and is called the moment theorem for Gaussian distributions.

(e) A convenient way of calculating the moments of a probability function is to work out its characteristic function

$$C(\underline{h}) = \langle \exp(i\,\underline{h}.\underline{X})\rangle$$

$$= \sum_{m_i = o}^{\infty} \frac{(ih_1)^{m_1}}{m_1!} \frac{(ih_2)^{m_2}}{m_2!} \cdots\cdots \langle X_1^{m_1}\ X_2^{m_2} \cdots\cdots\rangle \tag{7}$$

Given the characteristic function, an arbitrary moment can be worked out by differentiating it an appropriate number of times w.r.t. the h_i's and then setting $\underline{h} = o$. For Gaussian distribution with zero mean $C(\underline{h})$ has a very simple form

$$C(\underline{h}) = \exp[-\tfrac{1}{2}\,\underline{h}^T A^{-1}\underline{h}] \tag{8}$$

as is easily seen:

$$C(\underline{h}) = \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int d\underline{x}\ \exp[-\tfrac{1}{2}\,\underline{x}^T A\underline{x} + \tfrac{i}{2}\,\underline{h}^T\underline{x} + \tfrac{i}{2}\,\underline{x}^T\underline{h}]$$

$$= \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int d\underline{x}\ \exp[-\tfrac{1}{2}(\underline{x}-iA^{-1}\underline{h})^T A(\underline{x}-iA^{-1}\underline{h}) - \tfrac{1}{2}\,\underline{h}^T A^{-1}\,\underline{h}\ ]$$

$$= \exp(-\tfrac{1}{2}\,\underline{h}^T A^{-1}\underline{h})\ \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int d\underline{y}\ \exp[-\tfrac{1}{2}\,\underline{y}^T A\,\underline{y}]$$

$$= \exp(-\tfrac{1}{2}\,\underline{h}^T A^{-1}\underline{h})$$

Using (3) we may write (8) as

$$C(\underline{h}) = \exp[-\tfrac{1}{2} \sum_{ij} h_i \langle X_i X_j\rangle h_j] \tag{9}$$

Similarly for a Gaussian with $\langle \underline{X}\rangle \neq 0$, $C(\underline{h})$ is found to be

$$C(\underline{h}) = \exp[-\tfrac{1}{2}\,\underline{h}^T A^{-1}\,\underline{h} + i\,\underline{h}^T\langle \underline{X}\rangle]$$

$$= \exp[i\sum_i h_i\langle X_i\rangle - \tfrac{1}{2}\sum_{ij} h_i \langle\langle X_i X_j\rangle\rangle h_j] \tag{10}$$

Another useful quantity is the logaritham of the characteristic function - the cumulant generating function:

$$K(\underline{h}) = \ln C(\underline{h})$$

$$= \sum_{m_i=o}^{\infty} \frac{(ih_1)^{m_1}}{m_1!} \frac{(ih_2)^{m_2}}{m_2!} \cdots\cdots \langle\langle X_1^{m_1}\ X_2^{m_2} \cdots\rangle\rangle \tag{11}$$

For a Gaussian we find that the cumulant generating function

$$K(\underline{h}) = i\,\underline{h}^T\langle \underline{X}\rangle - \tfrac{1}{2}\,\underline{h}^T A^{-1}\,\underline{h}$$

$$= i\sum_i h_i\langle X_i\rangle - \tfrac{1}{2}\sum_{ij} h_i \langle\langle X_i X_j\rangle\rangle h_j \tag{12}$$

is at most quadratic in the auxiliary variables h_i's and hence all cumulants higher than the second vanish.

We mention here a theorem due to Marcinkiewicz [1-3] which states that the cumulant generating function of a probability distribution can not be a polynomial in the h_i's of degree greater than 2. In other words either $K(\underline{h})$ is at best quadratic in $\underline{h}$ or contains all of powers of $\underline{h}$. This in turn implies that either all but the first two cumulants of a probability distribution vanish or there are an infinite number of non vanishing cumulants.

With this background on Gaussian random variables we now go over to defining Gaussian stochastic processes.

Gaussian Stochastic Processes

A stochastic process is a function of two variables t, the time and Ω a random variable.

$$X_{\Omega}(t) = f(t,\Omega) \tag{13}$$

We may look upon (13) in two ways:

(i) For each value ω the random variable Ω takes, $X_{\omega}(t)$ is just an ordinary function of time and is called a realisation of the stochastic process $X_{\Omega}(t)$. The stochastic process is thus an ensemble of all such realisations.

(ii) For a fixed t, $X_{\Omega}(t)$ is a sotchastic variable - being a function of a random variable X. The stochastic process $X_{\Omega}(t)$ may be regarded as a continuum of random variables one for each t.

From the second point of view it therefore logically follows that in order to define a stochastic process completely we need to specify an infinite number of joint probabilities,

$$\begin{array}{l} P_1(x,t) \\ P_2(x_2,t_2;x_1,t_1) \\ P_3(x_3,t_3;x_2,t_2;x_1,t_1) \\ \cdots\cdots\cdots\cdots \end{array}$$

which say what is the probability that $X_{\Omega}(t)$ has a value x at t, what is the probability that $X_{\Omega}(t)$ has a value x_1 at t_1 and x_2 at t_2 etc. Given this infinite (over complete) set of joint probabilities the stochastic process is completely defined.

A stochastic process is said to be Gaussian if all these joint probabilities are Gaussian.

$$P_n(x_n,t_n; \;\cdots\; ; x_2,t_2; x_1,t_1) = \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\sum_{ij} (x_i-\langle X(t_i)\rangle)\, A_{ij}\,(x_j-\langle X(t_j)\rangle)\right] \tag{14}$$

where the matrix A_{ij} is the inverse of the matrix A^{-1} with elements

$$\begin{aligned} (A^{-1})_{ij} &= \langle (X(t_i)-\langle X(t_2)\rangle)(X(t_j)-\langle X(t_j)\rangle)\rangle \\ &\equiv \langle\langle X(t_i)\; X(t_j)\rangle\rangle \end{aligned} \tag{15}$$

in analogy with (3). $\langle\langle X(t_i)\ X(t_j)\rangle\rangle$ is known as the auto correlation function.

Thus a Gaussian stochastic process is completely characterized by $\langle X(t)\rangle$ and the auto correlation function. All the formulae we had derived previously for Gaussian distributions can now be generalised to stochastic processes by replacing partial derivatives by functional derivatives, summation over i by integration over t etc. We list them below.

(a) Novikov's Theorem: For a functional $f[X]$ of the stochastic process $X(t)$ we have

$$\langle X(t)\ f[X]\rangle = \int dt'\ \langle X(t)\ X(t')\rangle \left\langle \frac{\delta f[X]}{\delta X(t')}\right\rangle \tag{16}$$

(b) Moment theorem:

For a Gaussian stochastic process with $\langle X(t)\rangle = o$, the odd moments vanish and the even moments factorise pairwise.

$$\langle X(t_i)X(t_j)\ldots\rangle = \sum_{\text{pairs}} \langle X(t_p)\ X(t_q)\rangle \langle X(t_r)\ X(t_s)\rangle \ldots\ldots \tag{17}$$

(c) The characteristic functional

$$\begin{aligned} C[h] &= \langle \exp i \int_{-\infty}^{\infty} dt\ h(t)\ X(t)\rangle \\ &= \sum_{m=o}^{\infty} \frac{i^m}{m!} \int dt_1 \ldots \int dt_m\ h(t_1)\ldots h(t_m)\ \langle X(t_1)\ldots X(t_m)\rangle \end{aligned} \tag{18}$$

for a Gaussian stochastic process with zero mean is given by

$$C[h] = \exp\left[-\frac{1}{2}\int dt_1 \int dt_2\ h(t_1)\ h(t_2)\ \langle X(t_1)\ X(t_2)\rangle\right] \tag{19}$$

and for $\langle X(t)\rangle \neq 0$ by

$$C[h] = \exp\left[\ i\int dt_1 h(t_1)\langle X(t_1)\rangle - \frac{1}{2}\int dt_1 \int dt_2 h(t_1) h(t_2) \langle\langle X(t_1)X(t_2)\rangle\rangle\right] \tag{20}$$

where

$$\langle\langle X(t_1)\ X(t_2)\rangle\rangle = \langle (X(t_1) - \langle X(t_1)\rangle)(X(t_2) - \langle X(t_2)\rangle)\rangle \tag{21}$$

(d) The cumulant generating functional

$$\begin{aligned} K[h] &= \ln C[h] \\ &= \sum_{m=o}^{\infty} \frac{i^m}{m!} \int dt_1 \ldots \int dt_m\ h(t_1)\ldots h(t_m) \langle\langle X(t_1)\ldots X(t_m)\rangle\rangle \end{aligned} \tag{22}$$

for a Gaussian distribution with $\langle X(t)\rangle = 0$ reads

$$K[h] = -\frac{1}{2}\int dt_1 \int dt_2\ h(t_1)\ h(t_2)\langle x(t_1)\ x(t_2)\rangle \tag{23}$$

and for $\langle X(t)\rangle \neq 0$ as

$$K[h] = i\int dt_1 h(t_1)\langle X(t_1)\rangle - \frac{1}{2}\int dt_1 \int dt_2\ h(t_1) h(t_2) \langle\langle X(t_1)X(t_2)\rangle\rangle \tag{23a}$$

implying that all the cumulants of a Gaussian stochastic process higher than the second vanish. The Marcinkiewicz theorem holds for stochastic processes as well.

Of special interest to physicists and mathematicians are a class of stochastic process known as Markov processes [4] . A Markov process is fully determined by a single time distribution $P_1(x,t)$ and a conditional probability defined as

$$P(x_1,t_1|x_2t_2) \equiv \frac{P_2(x_1,t_1;x_2,t_2)}{P_1(x_2,t_2)} \quad (24)$$

satisfying

(i) the Chapman-Kolmogorov equation

$$P(x_3,t_3|x_1,t_1) = \int dx_2 P(x_3,t_3|x_2,t_2)\ P(x_2,t_2|x_1,t_1) \text{ for } t_3>t_2>t_1 \quad (25)$$

and

(ii) $$P_1(x_2,t_2) = \int dx_1\ P(x_2,t_2|x_1,t_1)\ P(x_1,t_1) \quad (26)$$

Another class of stochastic processes which are of special relevance to physics are the stationary processes. A stochastic process is stationary if all the joint probabilities are invariant under a shift in time.

$$P_n(x_n,t_n+\tau;\ \ldots;x_2,t_2+\tau;\ x_1,t_1+\tau) = P_n(x_n,t_n;\ldots;x_2,t_2;x_1,t_1) \quad (27)$$

This necessarily implies that the single time probability $P_1(x,t)$ is independent of time. Equation (27) in turn implies that

$$\langle X(t_n+\tau)\ \ldots X(t_2+\tau)\ X_1(t_1+\tau)\rangle = \langle X(t_n)\ \ldots\ X(t_2)X(t_1)\rangle \quad (28)$$

Having thus defined these three important classes of stochastic processes viz. Gaussian, Markovian and stationary stochastic processes, a natural question to ask is as follows: Among all Gaussian stochastic processes what characterizes those which have the additional attributes of being stationary and Markovian? The answer to this question is provided by Doob's Theorem:

A stationary Gaussian process is Markovian only if the auto correlation function is an exponential.

$$\langle\langle X(t_1)X(t_2)\rangle\rangle \propto \exp-\gamma|t_1-t_2| \quad (29)$$

(For a multicomponent stochastic process (29) is to be replaced by its obvious generalisation

$$\langle\langle \underline{X}(t_1)\underline{X}^T(t_2)\rangle\rangle \propto \exp-\Gamma|t_1-t_2| \quad (30)$$

where Γ is a constant matrix.)

We now briefly outline the proof of this important theorem.

For a Gaussian stochastic process the joint probabilities have the form given in (14). (For simplicity we shall consider Gaussian processes with $\langle X(t)\rangle = 0$). Substituting for $P_2(x_1,t_1;x_2,t_2)$ and $P_1(x_2,t_2)$ in (24), we find that the conditional probability for a Gaussian process has the following general form

$$P(x_1,t_1|x_2,t_2) = \frac{1}{\sqrt{2\pi\sigma^2(t_1)(1-\varrho^2(t_1,t_2))}}$$

$$\exp\left[-\frac{1}{2}\frac{1}{\sigma^2(t_1)(1-\varrho^2(t_1,t_2))}\left(x_1-\frac{\varrho(t_1,t_2)\sigma(t_1)}{\sigma(t_2)}x_2\right)^2\right] \quad (31)$$

where

$$\sigma^2(t) = \langle X(t)X(t)\rangle \quad (32)$$

and

$$\varrho(t_1,t_2) = \frac{\langle X(t_1)X(t_2)\rangle}{\sigma(t_1)\sigma(t_2)} \quad (33)$$

$\varrho(t_1,t_2)$ is known as the correlation coefficient.

From (31) it follows that the conditional average of $X(t)$ at time t_1 given that it had a value x_3 at time t_3

$$\langle X(t_1)\rangle_{X(t_3)=x_3} \equiv \int dx_1\, x_1\, P(x_1,t_1|x_3,t_3) \quad (34)$$

is given by

$$\langle X(t_1)\rangle_{X(t_3)=x_3} = \frac{\varrho(t_1,t_3)\,\sigma(t_1)}{\sigma(t_3)}\, x_3 \quad (35)$$

For a Gauss-Markov process, we have, on using (35) and the Chapman Kolmogorov equation (25)

$$\begin{aligned}\langle X(t_1)\rangle_{X(t_3)=x_3} &= dx_1\, x_1\, P(x_1,t_1\ x_3,t_3)\\ &=\iint dx_1 dx_2\, x_1\, P(x_1,t_1\mid x_2,t_2)\, P(x_2,t_2\mid x_3 t_3); t_1 > t_2 > t_3\\ &= \frac{\varrho(t_1,t_2)\,\sigma(t_1)}{\sigma(t_2)}\int dx_2 x_2\, P(x_2,t_2\mid x_3,t_3)\\ &= \frac{\varrho(t_1,t_2)\,\sigma(t_1)}{\sigma(t_2)}\ \frac{\varrho(t_2,t_3)\,\sigma(t_2)}{\sigma(t_3)}\ x_3\end{aligned} \quad (36)$$

From (35) and (36) we have

$$\varrho(t_1,t_3) = \varrho(t_1,t_2)\,\varrho(t_2,t_3) \quad . \quad t_1 \geqslant t_2 \geqslant t_3 \quad . \quad (37)$$

Thus we find that a necessary condition for a Gaussian process to be Markovian is that the correlation coefficients must satisfy (36). In fact this condition turns out to be both necessary and sufficient [5].

Let us now consider a stationary Gaussian process. Stationarity implies that $\sigma(t)$ is independent of time and that $\langle X(t_1)\ X(t_2)\rangle$ and hence $\varrho(t_1,t_2)$ depends only on t_1-t_2. From (36) it follows that for such a process to be Markovian we must have

$$\rho(t_1-t_3) = \rho(t_1-t_2)\,\rho(t_2-t_3)\;. \tag{38}$$

This functional equation is satisfied only if $\rho(t_1-t_3)$ is an exponential

$$\rho(t_1-t_3) = \exp - \gamma\,(t_1-t_3)$$

i.e.

$$\langle X(t_1)\;X(t_2)\rangle \propto \exp - |t_1-t_2| \tag{39}$$

Hence the proof.

Examples of Gaussian Stochastic Processes

In conclusion we list some important Gaussian stochastic processes which one frequently encounters in physics. It contains examples of Gaussian stochastic processes which have either the Markov property or stationarity or both or neither of them.

1. Gaussian White noise : The Gaussian "stochastic process" $\xi(t)$ characterized by

$$\langle \xi(t)\rangle = 0 \tag{40}$$

$$\langle \xi(t)\quad \xi(t')\rangle = \delta(t-t') \tag{41}$$

is usually referred to as Gaussian white noise. Such a stochastic process was first introduced by Langevin in the context of Brownian motion. Gaussian white noise is not a stochastic process in a strict mathematical sense. However, in physics it is often used as a model for very rapid fluctuations.

2. Wiener Process: Wiener process W(t) is an example of a Gaussian Markovian non-stationary stochastic process and is characterised by

$$\langle W(t)\rangle = 0 \tag{42}$$

$$\langle W(t)\;W(t')\rangle = \min\,(t,t') \tag{43}$$

That it is a non stationary process is clear from (43). From (43) it also follows that $\sigma^2(t)$ and $\rho(t_1,t_2)$ for this process are given by

$$\sigma^2(t) = \langle W(t)W(t)\rangle = t \tag{44}$$

$$\rho(t_1,t_2) = \frac{\langle W(t_1)W(t_2)\rangle}{\sigma(t_1)\sigma(t_2)} = \sqrt{\frac{t_2}{t_1}}\;;\; t_1 > t_2 \tag{45}$$

The single time probability $P_1(w,t)$ is therefore given by

$$P_1(w,t) = \frac{1}{\sqrt{2\pi t}}\;\exp\left[-\frac{w^2}{2t}\right] \tag{46}$$

Substituting from (44) and (45) in (31) we find that the conditional probability $P(w_1,t_1|w_2,t_2)$ for this process is given by

$$P(w_1,t_1|w_2,t_2) = \frac{1}{\sqrt{2\pi(t_1-t_2)}}\exp\left[-\frac{1}{2}\;\frac{(w_1-w_2)^2}{(t_1-t_2)}\right] \tag{47}$$

That this process is a Markov process is easily checked by verifying that $\rho(t_1,t_2)$ satisfies (37).

We can regard Wiener process as an integral of the Gaussian white noise

$$W(t) = \int_{-\infty}^{t} dt \ \xi(t) \tag{48}$$

in the sense that (48) together with (40) and (41) reproduce (42) and (43). We can also write (48) formally as a differential equation

$$\frac{dW}{dt} = \xi(t) \tag{49}$$

3. Ornstein Uhlenbeck Process: This is an example of a Gaussian Markovian stationary stochastic process. It is characterised by

$$\langle Y(t) \rangle = 0 \tag{50}$$

$$\langle Y(t_1) Y(t_2) \rangle = \exp\left[- |t_1 - t_2| \right] \tag{51}$$

From (50) and (51) we can construct $P_1(y,t)$ and $P_1(y_1,t_1|y_2,t_2)$ just as in the case of Wiener process. These are given by

$$P_1(y,t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} y^2 \right] \tag{52}$$

$$P(y_1,t_1|y_2,t_2) = \frac{1}{\sqrt{2\pi(1-e^{-2(t_1-t_2)})}} \exp\left[- \frac{(y_1 - y_2 e^{-(t_1-t_2)})^2}{2(1-e^{-2(t_1-t_2)})} \right] \tag{53}$$

Again, as in the case of Wiener process, We may express the Ornstein Uhlenbeck process in terms of Gaussian white noise as follows

$$Y(t) = \int_{-\infty}^{t} dt' \ e^{-(t-t')} \xi(t') \tag{54}$$

in the sense that (54) reproduces (50) and (51). Equation (54) may be written as a differential equation

$$\frac{dY}{dt} = -Y + \xi(t). \tag{55}$$

4. An example of a Gaussian stochastic process which is stationary but non Markovian is easy to construct. Any Gaussian stationary process with a non exponential auto correlation function is, according to Doob's Theorem, non Markovian.

5. Finally an example of a Gaussian stochastic process which is neither Markovian nor stationary is the stochastic process defined to be the integral of Ornstein Uhlenbeck process

$$Z(t) = \int_{0}^{t} Y(t) \ dt' \tag{56}$$

For this process we can deduce using (50) and (51) that

$$\langle Z(t) \rangle = 0 \tag{57}$$

$$\langle Z(t_1)Z(t_2)\rangle = e^{-t_1}+e^{-t_2}-1-e^{-|t_1-t_2|}+2\min(t_1,t_2) \tag{58}$$

This process is not stationary as is evident from (58). It is also easy to check that (37) is not satisfied for this process and hence it is non Markovian.

We may write (55) as a differential equation

$$\frac{dZ}{dt} = Y(t) \tag{59}$$

where Y(t) obeys (54). With Z(t) and Y(t) identified with the position and velocity of a Brownian particle, (59) and (55) are the Langevin equation for a free Brownian particle. Although Z(t) is non Markovian Z(t) and Y(t) together constitute a Markov process.

The material presented above can be found in some form or another in any good text book on stochastic processes. See for instance [6] and [7].

References

1. J. Marcinkiewicz, Math. Z. 44, 612 (1939).
2. D.W. Robinson, Comm. Math. Phys. 1, 89 (1965).
3. A.K. Rajagopal and E.C.G. Sudarshan, Phys. REv. A10, 1852 (1974).
4. See Lectures by R. Vasudevan in these proceedings.
5. W. Feller, "Introduction to Probability Theory and its Applications", Vol.2 Wiley, New York (1966).
6. N.G. Van Kampen, "Stochastic Processes in Physics and Chemistry", North Holland, Amsterdam, New York, Oxford (1981).
7. C.W. Gardiner "A Hand book of Stochastic Methods for Physics, Chemistry and Natural Sciences". To be published, Springer (1983).