GOLDSTONE MODES IN NON-EQUILIBRIUM PHASE TRANSITIONS

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Spontaneous symmetry breaking is a very important concept that pervades almost every branch of physics. Of course success of this concept in one branch stimulates one to apply it to another. When one does discover that this indeed works in a new area it reinforces our faith in the concept itself. The spectacular success of this concept in high-energy physics, condensed matter physics and even cybernetics makes spontaneous symmetry breaking a universal fundamental concept. The success of this concept also reinforces our belief that there is an underlying unity among the various branches of physics.

The history of the subject itself is quite interesting. T. Holstein and H. Primakoff [1] in 1940 introduced a method to diagonalize the Hamiltonian in the exchange interaction model of a ferromagnet by introducing second quantized creation and annihilation operators a_ℓ^+ and a_ℓ for magnons. They wrote the spin operators in terms of the magnon creation annihilation operators and the spins satisfy the usual SU(2) algebra. They then approximated the Hamiltonian by neglecting all terms that were not bilinear in a_ℓ and a_ℓ^+. This meant that in this approximation spin operators no longer satisfied the SU(2) commutation relation but E(2). They found a magnon mode whose frequency went to zero as $k \rightarrow o$. Holstein and Primakoff did not worry too much about the fact that rotational invariance present in the Hamiltonian is absent for the ground state. Nor did they link the zero frequency magnon mode to the asymmetric ground state. Nor did they bother about checking the conservation of the current that follows from rotational invariance.

The next person to choose an asymmetric ground state is N.N. Bogoliubov in his theory for liquid helium [2]. He chose a ground state that did not conserve the particle number and also obtained a phonon mode whose frequency went to zero as $k \rightarrow o$. But he again did not link this phonon mode to the asymmetric ground state.

The theory of superconductivity of Bardeen, Cooper and Schrieffer [3] is probably the forerunner to this theorem. In their epoch making paper, probably one of the greatest contributions to physics in the twentieth century, Bardeen, Cooper and Schrieffer chose a ground-state wave function which was not gauge invariant as their trial wave function. They diagonalized their Hamiltonian with this wave function, and showed that almost all properties of superconductors follow from their theory. The lack of gauge invariance of the ground state worried physicists for a while for it was thought that this would mean violation of charge conservation locally. P.W. Anderson and Y. Nambu [4] investigated the B.C.S. theory and showed that all is right with the B.C.S. theory. If one went beyond the B.C.S. approximation the current so calculated is conserved. Nevertheless there appears a phonon mode $\omega(k) \longrightarrow o$ as

$k \to o$. Also they showed that when the Coloumb interactions were taken into account this mode disappeared and became the plasma mode.

Later Nambu extended this idea to elementary particle physics in what is now known as the Jona Lasino-Nambu model. In fact it is Nambu who realized significance of the mechanism of spontaneous symmetry breaking. Later J. Goldstone [6] using a simple model illustrated this concept. It was finally proved as a theorem by S. Weinberg, A. Salam and J. Goldstone [7].

The Goldstone theorem states that if a Lagrangian is invariant under a continuous transformation while the ground-state is not, as a consequence of the conservation of the current that follows from the symmetry there exists a massless boson in the theory. [In the context of condensed matter physics there exists a mode $\omega(k) \to o$ as $k \to o$.] [8]

It is now known that in a large class of driven systemsin optics, chemical systems etc. "nonequilibrium phase transitions (N.P.T.) are observed between competing steady states. The similarities between N.P.T. and its equilibrium cousin has been well studied and a new area called synergetics has now emerged [9]. In what follows we shall show [10] that the Goldstone theorem in equilibrium phase-transition in condensed matter physics can be now extended to N.P.T. The analog of this theorem in N.P.T. is as follows:

"Whenever the order parameter is invariant under a continuous transformation while the steady state is not, there appears, in the limit the drive $J \to o$, a mode whose complex frequency goes to zero as $k \to o$." We shall now demonstrate this result by considering a few examples.

Consider the following order parameter equation:

$$\left[\frac{\partial}{\partial t} + \omega(\nabla^2) \right] \phi_\alpha = - \frac{\partial V}{\partial \phi_\alpha} + J_\alpha \tag{1}$$

Here J_α is a coherent external driving field and V is the potential which is a functional of the ϕ's. V transforms like a scalar under SU(N). $\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_N \end{bmatrix}$ is an n component field which transforms like the fundamental representation of SU(N).

Let $- \frac{\partial V}{\partial \phi_\alpha} = F(\phi^+\phi)\ \phi_\alpha$

Then equation (1) becomes

$$\left[\frac{\partial}{\partial t} + \omega(\nabla^2) \right] \phi_\alpha = F(\phi^+\phi)\ \phi_\alpha + J_\alpha \tag{2}$$

We now look for a steady state solution of equation (2) of the form

$$\begin{aligned} \langle \phi_\alpha \rangle &= \lambda \qquad \text{for} \qquad \alpha = 1, \ldots . m \\ \langle \phi_\alpha \rangle &= 0 \qquad \text{for} \qquad \alpha = m+1, \ldots . n \end{aligned} \tag{3}$$

Since the steady state is space-time independent it is obvious that as $J \to o$

$$F(m|\lambda|^2)\lambda = 0 \implies F(m|\lambda|^2) = 0 \tag{4}$$

since $\lambda \neq 0$ for the asymmetric steady state. Let us now look for the eigenmodes of the problem. To this effect we write

$$\phi_\alpha = \lambda + \psi_\alpha \qquad \alpha = 1 \ldots\ldots m$$

$$\phi_\alpha = \chi_\alpha \qquad \alpha = m+1 \ldots\ldots n \tag{5}$$

The linearisation of the steady state equations lead to

$$\left[\frac{\partial}{\partial t} + \omega(\nabla^2)\right]\psi_\alpha = F(m|\lambda|^2)\psi_\alpha + J_\alpha + \sum_\beta \left(\frac{\partial F}{\partial \phi_\beta}\right)_{\phi_\beta = \lambda} \psi_\beta \tag{6}$$

$$\left[\frac{\partial}{\partial t} + \omega(\nabla^2)\right]\chi_\alpha = F(m|\lambda|^2)\chi_\alpha + \ldots\ldots \tag{7}$$

Equation (7) with (4) shows that in the limit $J \to o$ the frequency of the oscillation associated with χ field goes to zero as $k \to o$. In this limit the frequency associated with ϕ is

$$\omega_\phi \xrightarrow[k \to o \; J \to o]{} 2\lambda^2 mF'$$

where F' denotes the derivative of F w.r.t. its argument. Thus we have demonstrated the theorem, using a linearized approximation.

One might, instead of a local interaction, consider some non-local interaction which is short range. So let us consider the following order parameter equation

$$\left[\frac{\partial}{\partial t} + \omega(\nabla^2)\right]\psi = F(|\psi|^2)\psi - i\psi(x)\int d^3x' \, |\psi(x')|^2 \Omega(x-x') \tag{8}$$

$F(|\psi|^2)$ is a real function of ψ.

We now look for a symmetry breaking steady state

$$\psi = \psi_o e^{-iE_o t} \qquad E_o = \int d^3x \, \Omega(x) \, |\psi_o|^2$$

$$F(|\psi_o|^2) = 0 \tag{9}$$

To obtain the excitation frequencies we can linearize ψ around the steady state solution

$$\psi = (\psi_o + \delta\psi) \, e^{-iE_o t} \tag{10}$$

From (8) and (9) we obtain

$$\left[\frac{\partial}{\partial \tau} + \omega(\nabla^2)\right]\delta\psi = |\psi_o|^2\left[F'\delta\psi - i\int d^3x' \, \delta\psi(x')\Omega(x-x')\right] + |\psi_o|^2\left[F'\delta\psi^* - i\int d^3x \, \Omega(x-x')\delta\psi^*(x')\right] \tag{11}$$

A simple Fourier analysis of (11) and its conjugate shows that the system admits Goldstone modes provided the interaction is short range. i.e.

$$\underset{k \to 0}{\mathrm{Lt}} \quad [\mathrm{Im}\,\omega(-k^2)]\,\Omega_k \quad = \quad 0$$

Note that generally $\mathrm{Im}\,\omega(-k^2) \longrightarrow k^2$ if it is non-zero.

We have thus shown that the generalized Goldstone theorem is valid in N.P.T. An exact proof of this has been now obtained by us and will be published elsewhere. We also have found that the Higgs mechanism goes through [11]. Finally it should be borne in mind that the general belief is that in two dimensions the Goldstone theorem is not true. However Nakanishi [12] has shown that one can construct a massless field in two dimensions if one uses indefinite metric. Since in our case we have complex energies probably Nakanishi's results can be extended here.

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