PHASE TRANSITIONS IN A SYSTEM OF ATOMS INTERACTING WITH A COHERENT FIELD

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1. INTRODUCTION

Co-operative behaviour in systems driven far from thermal equilibrium is of great current interest. For two recent reviews of this rapidly growing subject see the books by Haken[1] and by Prigogine and Nicolis[2]. As an example of such a system we consider a collection of N identical two-level atoms interacting with a coherent field and located in a volume of dimensions smaller than the wavelength. Without the external field the model was first explored by Dicke[3] as an illustration of co-operation in quantum optics. The cooperative effect manifests itself in the form of a spontaneously emitted pulse with intensity proportional to N^2 (superradiance) for suitably prepared initial atomic states. The Dicke model is an example of exactly solvable quantum mechanical model[4]. It is also of interest because of its mathematical equivalence to spatially averaged extended systems[5].

The driven Dicke model has also been the object of much recent discussions[5-7] mainly because it involves both resonance fluorescence and cooperative emission. The model admits exact steady state solutions both in the case of resonant[8-12] and off-resonant[13] coherent driving field. These exact solutions show that at resonance the system undergoes as $N \to \infty$ a non-equilibrium phase transition similar to a second order phase transition (predicted also by numerical solutions for finite N[14]). This transition exhibits itself in the form that upto a certain critical amplitude of the driving field the system behaves in pure classical manner beyond which the quantum effects become dominant. At off-resonance there is no such critical behaviour. This is discussed further in Section 2.

Next, we consider the Dicke system placed in a cavity tuned to the frequency of the atomic transition. The cavity mode is excited by an external field in resonance with the atoms and is coupled to an output mode. In this case the quantum fluctuations and the cavity feedback together lead in the limit $N \to \infty$ to a bistable behaviour[15], manifested in the relation between the output and the input fields. This is briefly outlined in Section 2.

Finally, we include interaction between atoms in the Dicke system (in a special manner)[16]. The interaction has the effect of increasing the coherence region with respect to the amplitude of the driving field. But now there is a competition between the dephasing effect of the frequency detuning and the synchronizing effect of the interaction of the atoms which leads to a first order phase transition for

certain values of the parameters involved. This is presented in Section 3.

2. THE DRIVEN DICKE MODEL

Master Equation

We consider a collection of N identical two-level atoms, each with a transition frequency ω_o, confined to a volume V of dimensions small compared to the wavelength and driven by a single mode coherent field of frequency ω:

$$\vec{E}(\vec{r},\vec{t}) = \vec{\varepsilon}_o \exp[-i(\omega t - \vec{k}_o.\vec{r})] + c.c. \quad (1)$$

The atoms decay by virtue of their coupling to the vacuum radiation field. In the electric dipole approximation the Hamiltonian reads as

$$H = \sum_{j=1}^{N} \hbar\omega_o S_{zj} + \sum_{j=1}^{N} \hbar(\Omega e^{-i\omega t} S_{+j} + \Omega^* e^{i\omega t} S_{-j}) + \sum_k \sum_{j=1}^{N} \hbar[g_k b_k (S_{+j} + S_{-j}) + h.c.] + \sum_k \hbar\omega_k b_k^\dagger b_k \quad (2)$$

where

$$g_k = -i\left(\frac{2\pi c|\vec{k}|}{\hbar V}\right)^{1/2} (\vec{d}.\vec{e}_k) e^{i\vec{k}.\vec{r}_a}; \quad \Omega = -\frac{(\vec{d}.\vec{\varepsilon}_o)}{\hbar} e^{i\vec{k}_o.\vec{r}_a}. \quad (3)$$

For simplicity, we have suppressed the polarization index. The two levels atoms are represented by the pseudo-spin operators S_{zj} and $S_{\pm j} = S_{xj} \pm iS_{yj}$ obeying the usual commutation relations. Each atom has a dipole moment $\vec{d}$ and $\vec{r}_a$ represents the centre of the atomic collection. The vacuum field modes are characterized by annihilation and creation operators b_k and b_k^+ corresponding to the wavelength $\vec{k}$, frequency ω_k and polarization $\hat{e}_k$. The summation over j in the Hamiltonian may be absorbed by defining the collective spin operators $S_\pm = \sum_j S_{\pm j}$, $S_z = \sum_j S_{zj}$ obeying the angular momentum commutation relations

$$[S_\pm, S_z] = \mp S_\pm, \quad [S_+, S_-] = 2S_z. \quad (4)$$

It is clear that the $\hat{S}^2$ commutes with the Hamiltonian, a consequence of the fact that the phase factors $e^{i\vec{k}.\vec{r}_j}$ at different atomic sites are ignored.

From Hamiltonian (1), the master equation for the reduced atomic density operator $\rho_a = \mathrm{Tr}\,\rho$ (where the trace is taken over the vacuum radiation field variables) is obtained by standard techniques[4]. In the rotating wave, Born and Markov approximations and in the frame rotating at the frequency ω this master equation reads as[4]

$$\frac{d\rho_a}{dt} = -i\Omega[S_+ + S_-, \rho_a] + i\delta_o[S_z, \rho_a] + \gamma_o([S_-, \rho_a S_+] + [S_- \rho_a, S_+]) \quad (5)$$

where $\delta_o = \omega - \omega_o$ is the frequency detuning between the coherent field and the atom 2Ω is the Rabi frequency, $2\gamma_o$ is the Einstein A coefficient ($\gamma_o = 2d^2\omega_o^3/3hc^3$).

Steady State Solution

It is readily seen that corresponding to $\hat{S}^2$ conservation in the Hamiltonian the master equation conserves $\langle S^2 \rangle$. Hence, the Dicke states $|S,m\rangle$ form a good set of basis states. These states are the simultaneous eigenstates of $\hat{S}^2$ and S_z obeying

$$\hat{S}^2|S,m\rangle = S(S+1)|S,m\rangle, \; S_z|S,m\rangle = m|S,m\rangle \; . \tag{6}$$
$$S_\pm|S,m\rangle = [(S \mp m)(S \pm m+1)]^{1/2} \; |S,m \pm 1\rangle$$

The quantum number S is the Dicke cooperation number ($0 < S < N/2$) and m is half the difference between the number of excited and unexcited atoms ($m < S$). For convenience, we assume the atoms to be in the ground state initially (just before the field is switched on). Thus $m = N/2$ and $S = N/2$ at $t = 0$ and by conservation of $\hat{S}^2$, ρ_a evolves over an (N+1)-dimensional manifold of collective atomic states $|N/2,m\rangle$ where $m = -N/2,\ldots,N/2$. For subsequent work, it is convenient to write $p = S-m$ and label the states $|S, S-p\rangle$ by $|p\rangle$.

Having fixed the space in which ρ_a evolves we proceed to derive the steady state solution. Note that the master equation does not obey the principle of detailed balance and the standard methods are ruled out. Nevertheless, at least, in the case of exact resonance ($\delta_o = 0$) it is easy to guess this solution, if we first convert the master equation into a C-number form. For this purpose, we introduce the atomic coherent states[17] $|\mu\rangle$ which are related to the Dicke states $|p\rangle$ by

$$|\mu\rangle = (1+|\mu|^2)^{-S} \, e^{\mu S_+}|2S\rangle = (1+|\mu|^2)^{-S} \sum_{p=0}^{2S} \left(\frac{2S!}{(2S-p)!p!}\right)^{1/2} \mu^{2S-p}|p\rangle \tag{7}$$

These states have the following properties

$$\langle\lambda|\mu\rangle = (1+\lambda^*\mu)^{2S}(1+|\lambda|^2)^{-S}(1+|\mu|^2)^{-S}$$
$$\frac{2S+1}{\pi}\int \frac{d^2\mu}{(1+|\mu|^2)^2}\,|\mu\rangle\langle\mu| = 1$$
$$\langle\lambda|S_+|\mu\rangle = (2S)(1+|\lambda|^2)^{-S}(1+|\mu|^2)^{-S}\lambda^*(1+\lambda^*\mu)^{2S-1} = (\langle\mu|S_-|\lambda\rangle)^* \tag{8}$$
$$\frac{2S+1}{\pi}\int \frac{d^2\lambda}{(1+|\lambda|^2)^{2S+2}}\,\lambda^{*m}\lambda^n = \left(\frac{(2S-m)!m!}{(2S)!}\right)\delta_{mn} \quad (0 < m, n < 2S).$$

These states are used to construct a diagonal representation for the matrix ρ_a. Writing

$$\rho_a(\mu^*,\mu,t) = \langle\mu|\rho_a|\mu\rangle(1+|\mu|^2)^{2S} \tag{9}$$

and using the above properties, it is easy to cast the master equation ($\delta_o=0$) in the form

$$\frac{\partial\rho_a}{\partial\tau}(\mu^*,\mu,t) = \left(\frac{\partial}{\partial\mu} - N\mu^* + \mu^{*2}\frac{\partial}{\partial\mu^*}\right)\left(\frac{\partial}{\partial\mu^*} + g\right)\rho_a(\mu^*,\mu,\tau) + C.C. \tag{10}$$

where $g = i\Omega/\gamma_o$ and $\tau = \gamma_o t$. It is now easy to verify that the steady state solution

$(\frac{\partial \rho_a}{\partial \tau} = 0)$ of Eq.(10) is given by

$$\rho_a^{ss}(\mu^*,\mu,t) = \frac{1}{D_o}\sum_{m,n=0}^{N}(g^*)^{-m}(g)^{-n}(\frac{\partial}{\partial \mu^*})^m(\frac{\partial}{\partial \mu})^n(1+|\mu|^2)^N$$

which may also be cast back in the operator form

$$\rho_a^{ss} = \frac{1}{D_o}\sum_{m,n=0}^{N}(g^*)^{-m}(g)^{-n}S_-^m S_+^n \ . \tag{11}$$

The normalization constant D_o determined by setting Tr $\rho^{ss} = 1$ is given by

$$D_o = \sum_{m=0}^{N}|g|^{-2m}H_{Nm}$$

$$H_{Nm} = \sum_{q=0}^{N-m}\frac{(N-q)!\,(q+m)!}{(N-q-m)!\,q!} \tag{12}$$

This solution is valid for all N and g. Taking the clue from this, the steady state solution for the case of finite detuning ($\delta_o \neq 0$) may be easily derived and reads as

$$\rho_a^{ss} = \frac{1}{D}\sum_{m,n=0}^{N}a_{mn}(g^*)^{-m}(g)^{-n}S_-^m S_+^n \tag{13}$$

where

$$D = \sum_{m=0}^{N}a_{mm}\,H_{Nm}|g|^{-2m}$$

$$a_{mn} = \frac{\Gamma(m+\Delta^*+1)\Gamma(n+\Delta+1)}{m!\,n!\,\Gamma(1+\Delta^*)\Gamma(1+\Delta)} \tag{14}$$

Here $\Delta = i\delta_o/\gamma_o$ and $\Gamma(z)$ is the usual Γ function of the complex argument z. Note that when $\Delta = 0$ (resonant case), $a_{mn} = 1$ for all m, n and one recovers from Eqs.(13) and (14), the steady state solution displayed in Eqs.(11) and (12).

Atomic Observables and Correlation Functions

The exact form (13) for the density operator ρ_a may now be used to obtain the expressions for the atomic operator expectation values, the atomic correlation functions and the fluctuations. These can all be derived from the operator average:

$$\langle S_+^p S_z^n S_-^q\rangle_N = \mathrm{Tr}[S_+^p S_z^n S_-^q \rho_a^{ss}]$$

$$= D^{-1}\sum_{n=\max(p,q)}^{N}(g^*)^{q-n}(g)^{p-n}a_{n-p,n-q}\sum_{m=0}^{N-n}\frac{(m+n)!\,(N-m)!}{(N-m-n)!\,m!}\,(\tfrac{1}{2}N-n-m)^r . \tag{15}$$

From this general formula, it is easy to obtain the following expressions for various atomic and radiation field observables as a function of the driving field parameter g for a given N:

$$\langle S_z\rangle = -\frac{1}{2D}\sum_{r=0}^{N}r|g|^{-2r}a_{rr}H_{Nr} = \frac{|g|}{4N}\frac{\partial}{\partial|g|}\ell n\ D \tag{16}$$

$$\langle S_+ \rangle = \frac{1}{D} \sum_{r=1}^{N} (g^*)^{-r} (g)^{-r+1} a_{r\ r-1} H_{Nr} = \langle S_- \rangle^* \tag{17}$$

$$G^{(m,n)}(0) = \langle S_+^m S_-^n \rangle = \frac{1}{D} \sum_{r=\max(m,n)}^{N} (g^*)^{n-r} (g)^{m-r} a_{r-n,r-m} H_{Nr} \tag{18}$$

We remark that $\langle S_z \rangle$, the expectation value of the atomic population inversion is also a measure of the power absorbed by the atomic system. The mean atomic polarization $\langle S_y \rangle$ and dispersion $\langle S_x \rangle$ are provided by formula (17). The functions $G^{m,n}_{(0)}$ of Eq.(18) are proportional to the correlation functions $\langle \overset{\leftarrow}{E}^m_- \vec{E}^n_+ \rangle$ of the scattered radiation. Of these, the function $G^{(1,0)} = \langle S_+ \rangle$ is proportional to the average amplitude of the scattered radiation, $G^{(1,1)}$ is proportional to the intensity and $G^{(2,2)}$ is proportional to the intensity correlation function. The correlation properties of the scattered radiation are usually determined by the normalized functions of first and second order defined as

$$g^{(1)} = G^{(1,1)} / |G^{1,0}|^2, \quad g^{(2)} = G^{(2,2)} / |G^{(1,1)}|^2 . \tag{19}$$

Note that at $g^{(1)} = 1$, the scattered radiation is fully coherent and its deviation from unity implies partial coherence. The function $g^{(2)}$ determines the probability that two photons will be simultaneously emitted by the system. The value $g^{(2)} > 1$ indicates the presence of bunching of the scattered photons, and $g^{(2)} < 1$ shows anti-bunching while $g^{(2)} = 2$ implies that the properties of the scattered light are equivalent to the properties of the equilibrium thermal radiation.

We might add that to obtain the properties of the atomic system at resonance, we have to set $a_{mn} = 1$ in the above formulae. An immediate consequence of this is that $\langle S_x \rangle \equiv 0$ at resonance.

Second Order Phase Transition

We now examine the behaviour of the atomic observables and their fluctuations when the driving field is in exact resonance ($\delta_o = 0$). We are, in particular, interested in the thermodynamic limit $N \to \infty$, $|g| \to \infty$ with $\theta = \frac{2|g|}{N}$ remaining finite. To this end we shall require the behaviour of the quantity $D_o \equiv D_o(N,\theta)$ for large N. The asymptotic expression for $D_o(N,\theta)$ for large N turns out to be

$$D_o(N,\theta) = \begin{cases} \dfrac{N\pi}{2\theta^{2N}} \dfrac{(1+\sqrt{1-\theta^2})^{2N+2} \exp(-2N\sqrt{1-\theta^2})}{\sqrt{1-\theta^2}} & (\theta < 1) \\ \dfrac{N\theta^2}{\sqrt{\theta^2-1}} \mathrm{Sin}^{-1}(1/\theta) & (\theta > 1) \end{cases} \tag{20}$$

Equations (15)-(17) may now be combined with (20) to arrive at the limiting forms of the quantities of interest. In particular, writing

$$m_\mu = \lim_{N\to\infty} \langle S_\mu\rangle/N,\quad \sigma_{\mu\nu} = [\langle S_\mu S_\nu\rangle - \langle S_\mu\rangle\langle S_\nu\rangle]/N^2 \qquad (\mu,\nu \equiv x,y,z) \tag{21}$$

we obtain

$$m_x = 0 \qquad 0 < \theta < \infty$$

$$m_y = \begin{cases} \theta/2 & 0 < \theta < 1 \\ \theta(1-f(\theta))/2 & 1 < \theta < \infty \end{cases} \tag{22}$$

$$m_z = \begin{cases} -\frac{1}{2}\sqrt{1-\theta^2} & 0 < \theta < 1 \\ 0 & 1 < \theta < \infty \end{cases}$$

$$\sigma_{zz} = \begin{cases} 0 & 0 < \theta < 1 \\ [1-\theta^2(1-f(\theta))]/4 & 1 < \theta < \infty \end{cases} \tag{23}$$

$$f(\theta) = \sqrt{\theta^2-1}/[\theta^2 \sin^{-1}(1/\theta)] \tag{24}$$

It is thus seen that in the "thermodynamic" limit the atomic observables and their fluctuations are continuous at $\theta=1$, but show a discontinuity in their derivatives (with respect to θ) at $\theta=1$. This behaviour is typical of a second order phase transition at the critical bifurcation point $\theta=1$. We show in Figs.1 and 2 the variation of $\langle S_z\rangle/N$ and the fluctuations σ_{zz} with θ for several values of N and for $N \to \infty$. Note the smooth behaviour of these quantities with θ for finite N and their approach to the asymptotic limit as N increases.

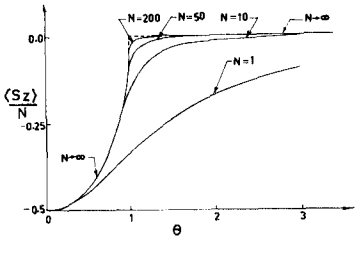

Fig.1. Steady state atomic population difference (per atom) as a function of θ for several values of N.

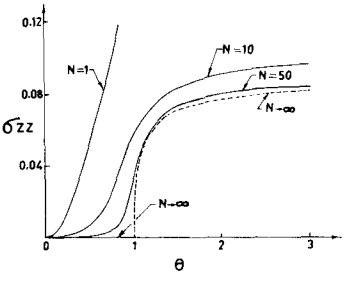

Fig.2. Fluctuations in the steady state atomic population difference as a function of θ for various N.

Further the fluorescent intensity $G^{(1,1)}(0)/(N/2)^2$ and the normalized intensity-intensity correlation function $g^{(2)}(0)$ also show the discontinuity in slope at $\theta=1$ in the limit $(N \to \infty)$. The analytical expressions are[12]

$$G^{(1,1)}(0)/(\frac{N}{2})^2 = \begin{cases} \theta^2 & \theta < 1 \\ \theta^2(1-f(\theta)) & \theta > 1 \end{cases} \tag{25}$$

$$g^{(2)}(0) = \begin{cases} 1 & \theta < 1 \\ \theta^4(1 - (1 + \frac{2}{3\theta^2})f(\theta)] & \theta > 1 \end{cases} \tag{26}$$

The behaviour of steady state intensity and the correlation function $g^{(2)}(0)$ is shown in Figs.3 and 4. It is clear from Eq.(26) that the scattered radiation is completely coherent for $\theta < 1$ whereas for $\theta > 1$ the radiation is partially coherent. Note also that $g^{(2)}(0) = 1.2$ in the limit $N \to \infty$. Curves of $g^{(2)}(0)$ also indicate photon antibunching around $\theta=1$ for finite N.

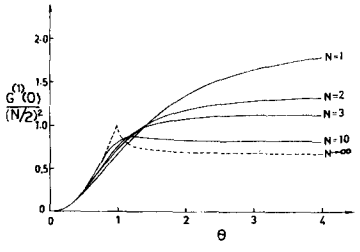

Fig.3. Behaviour of steady state fluorescent intensity with θ for several values of N.

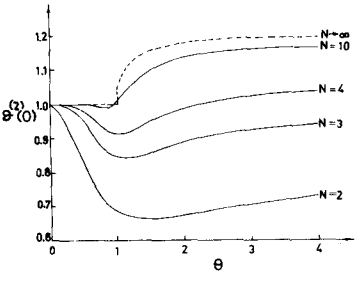

Fig.4. Steady state intensity-intensity correlation function for different values of N.

Similar analysis[6,13] can be carried out for the case when the driving field is off-resonant. Both numerical results for finite N and exact analysis in the thermodynamic limit $N \to \infty$ indicate that there is no critical point phase transition for the atomic system when the detuning parameter $\phi = 2\delta_0/\gamma_0 N \neq 0$. It is also found that as N increases, the quantum fluctuations for a given θ and ϕ go on decreasing approaching a zero value in the limit $N \to \infty$. This is in contrast with the resonance case ($\phi=0$) where $\sigma_{zz} = 0$ for $\theta < 1$ while for $\theta > 1$, σ_{zz} increases from its value zero (at $\theta=1$) approaching a value 1/12 ($\theta \to \infty$).

We might also mention that inclusion of phase fluctuations in the driving

field[18] also tends to smooth the discontinuities in the atomic observables and their correlations found in the non-fluctuating resonant case. Quantum fluctuations are also reduced due to the phase fluctuation effects.

First Order Phase Transition

There is an interesting way[15] of predicting the conventional bistable behaviour with the Dicke model. It is known that an extended system of atoms placed in a cavity predicts bistability with hysteresis, a typical first order phase transition. This behaviour arises from a competition between the cooperative and individual atom decay processes in presence of the driving field. In contrast, in the driven Dicke model for which $\hat{S}^2$ is conserved, the atomic decay is only cooperative and bistability is not predicted. However, bistable behaviour is predicted if the Dicke system is simply placed in a cavity[15] tuned to the frequency ω_o of the atomic transition. The cavity mode with which atoms interact is excited by an external field of frequency ω_o and is coupled to an output mode. In the limit of large N, the cavity field operator A may be replaced by its expectation value $\bar{A}$. The equation for the reduced atomic density operator ρ_a and that for $\bar{A}$ (in the mean field approximation) are given by[15]

$$\frac{d\rho_a}{dt} = -i\bar{A}[S_+ + S_-, \rho_a] + \gamma_o([S_-, \rho S_+] + [S_- \rho_a, S_+]) \tag{27}$$

$$\frac{d\bar{A}}{dt} = -K_A(\bar{A} - A_o) - K^2\bar{S} \tag{28}$$

where K_A is an effective decay constant describing coupling to the external field A_o and the output mode; $\bar{S} = |\langle S_+ \rangle| = |\langle S_- \rangle|$ and K^2 is a coupling constant.

Defining $x = 2\bar{A}/N\gamma_o$, $y = 2A_o/N\gamma_o$, which are proportional to the output and input fields respectively, one obtains in the steady state the equation

$$y = x + 2C|\langle S_+ \rangle|/N \tag{29}$$

where according to Eq.(22)

$$\frac{\langle S_+ \rangle}{N} = \begin{cases} x/2 & x < 1 \\ (1 - \sqrt{x^2 - 1}/[x^2 \sin^{-1}(1/x)])x/2 & x > 1 \end{cases} \tag{30}$$

in the limit $N \to \infty$ and $C = K^2/K_A\gamma_o$. A plot of x against y for different cooperation numbers C is shown in Fig.5. For $C > 0.1$ we observe the conventional bistable behaviour and hysteresis. Hysteresis arises because curves of negative slope are unstable. Note, however, that the cusps in Fig.5 are due to the second order phase transition contained in the Dicke model. The interesting point here is that the optical bistability arises from the quantum fluctuations and cavity feedback.

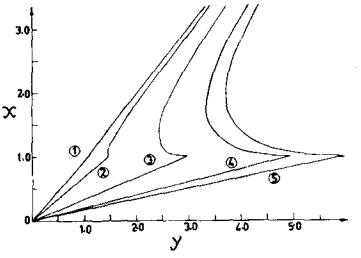

Fig.5. Bistable behaviour from Dicke system placed in a cavity. Curves (1)-(5) correspond to C=0.1,0.5, 2,4,5.

3. DRIVEN DICKE MODEL WITH INTER-ATOMIC INTERACTIONS

We now consider the driven Dicke model taking into account both the frequency detuning and the interaction between the atoms. In the special case for which the interatomic interaction operator $\hat{V}$ has the form[4,16]

$$\hat{V} = \varepsilon S_z + v S_+ S_- , \tag{31}$$

the reduced atomic density operator ρ_a satisfies the equation[4,16]

$$\frac{d\rho_a}{dt} = -i\Omega[S_+ + S_-, \rho_a] + i\delta[S_z, \rho_a] - iv[S_+ S_-, \rho_a] + \gamma_o([S_-, \rho_a S_+] + [S_- \rho_a, S_+]) \tag{32}$$

where $\delta = \delta_o - \varepsilon$. This master equation conserves $\langle \hat{S}^2 \rangle$. Incidently it describes also an extended system averaged over the space of two-level atoms[5]. This equation also admits an exact steady state solution to be presented later. We shall first discuss the semi-classical approximation.

Semi-classical Solution

The equations of motion obtained from (32) for the mean value $\langle S_\mu \rangle$, $(\mu = \pm, z)$ have the form

$$\begin{aligned}
\frac{d}{dt}\langle S_z \rangle &= -2\gamma_o \langle S_+ S_- \rangle - i\Omega[\langle S_+ \rangle - \langle S_- \rangle] \\
\frac{d}{dt}\langle S_+ \rangle &= -2i\Omega \langle S_z \rangle - i\delta \langle S_+ \rangle - 2(iv - \gamma_o)\langle S_+ S_z \rangle = \left(\frac{d}{dt}\langle S_- \rangle\right)^*
\end{aligned} \tag{33}$$

Neglecting the quantum correlations we can use the factorization $\langle S_\mu S_{\mu'} \rangle = \langle S_\mu \rangle \langle S_{\mu'} \rangle$ to rewrite equations (33) in a convenient form

$$\frac{dm_z}{d\tau} = -2m_+m_- - \frac{i\theta}{2}(m_+ - m_-)$$
$$\frac{dm_+}{d\tau} = -i\theta m_z + \frac{i\phi}{2}m_+ + 2(1-i\xi)m_+m_z \tag{34}$$

where

$$\tau = N\gamma_o t,\ m_\mu = \frac{\langle S_\mu \rangle}{N} \tag{35}$$

$$\theta = 2\Omega/N\gamma_o,\ \phi = 2\delta/N\gamma_o,\ \xi = \nu/\gamma_o$$

We note that within this factorization Eqs.(34) admit that $\langle \hat{S}^2 \rangle = \langle S_z \rangle^2 + \langle S_+ \rangle \langle S_- \rangle$ is a constant of motion. The steady state solution of Eqs.(34) is given by

$$m_\pm = \frac{(1-4m_z^2)}{2\theta}[(\phi/4m_z - \theta) \pm i] \tag{36}$$

where m_z is determined from either of the two equivalent relations

$$\theta^2 = h(m_z) = (1 - 4m_z^2)[(\frac{\phi}{4m_z} - \theta)^2 + 1] \tag{37}$$

$$\phi = 4m_z[\xi \pm (\frac{\theta}{1-4m_z^2})^{1/2}] \tag{38}$$

It is clear from Eqs.(37) and (38) that when $\xi=0$, m_z is uniquely defined function of the parameters θ and ϕ. If in addition ϕ is also set equal to zero, we recover the solution for m_z at resonance given in Eq.(21), showing a second order phase transition at the critical point $\theta=1$. On the other hand, when $\theta \neq 0$ but $\phi=0$, Eqs.(37) and (38) imply that

$$m_z = \begin{cases} -\frac{1}{2}\sqrt{1 - \frac{\theta^2}{1+\xi^2}} & \theta < \sqrt{1+\xi^2} \\ 0 & \theta > \sqrt{1+\xi^2} \end{cases} \tag{39}$$

which again shows a discontinuity in the derivative (w.r.t.θ) at $\theta = \theta_{cr} = \sqrt{1+\xi^2}$

The most interesting situation arises when both ξ and ϕ are not zero. In this case one expects from Eq.(37) that there will be a range of parameters ξ and ϕ where m_z is a multivalued function of θ. A typical behaviour of m_z is shown in Fig.6. It is seen that there is a range of θ in which each θ corresponds to three values of m_z in the physically relevant interval $[-\frac{1}{2}, 0]$. Thus one arrives at a bistable behaviour.

In a similar manner, for certain values of the parameters θ and ξ Eq.(38) predicts bistable behaviour of m_z as a function of the detuning parameter ϕ. This is shown in Fig.7. Note that in Figs.6 and 7, the curves with negative slope are unstable and a first order phase transition with hysteresis is seen.

We might mention here that the steady state bistable behaviour was predicted with the Dicke model without taking into account the atom-atom interactions. This prediction was based on the analysis of equations of motion (similar to Eqs.(33) above but with v=0) subject to a factorization ansatz for operator products. This factorization uses the semiclassical decorrelation as above but retains also the self-

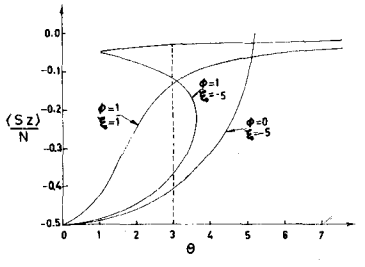

Fig.6. Plot of m_z against θ for different values of the parameters ξ and ϕ (semi-classical results)

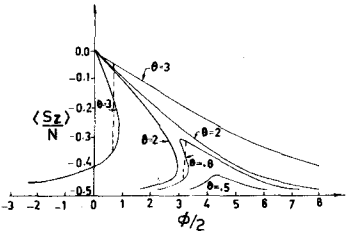

Fig.7. Plot of m_z against $\phi/2$ for several values of θ and $\xi = -5.0$

correlation of atoms so that, for example,

$$\langle S_+ S_- \rangle \simeq (1 - \frac{1}{N}) \langle S_+ \rangle \langle S_- \rangle + \langle S_z \rangle + \frac{N}{2} \tag{40}$$

This approximation thus admits independent atomic decay and leads to the usual optical bistability. However, it breaks an important property of the Dicke model, namely that $\langle \hat{S}^2 \rangle$ is no longer conserved.

Exact Quantum Mechanical Solution

The steady state solution of the master equation may be easily obtained by writing

$$\rho_a^{ss} = \sum_{m,n=0}^{N} C_{mn} S_-^m S_+^m \tag{41}$$

and determining C_{mn} by direct substitution in the steady state form of Eq.(32). The result is an expression of the same form as in Eqs.(1) with g, Δ replaced by $\tilde{g}$ and $\tilde{\Delta}$ defined as

$$\tilde{g} = (i\Omega/\gamma_o)/(1-i\xi), \quad \tilde{\Delta} = (i\delta_o/\gamma_o)/(1-i\xi) \tag{42}$$

Two limiting cases can be easily inferred from the analysis of the exact ρ_a^{ss} in the non-interacting atoms model[8]. First, it follows that for zero frequency detuning $\tilde{\Delta}=0$ the presence of interatomic interaction leads to an increase of the coherence region with respect to θ. That is, the behaviour of the atomic observables is classcal below $\theta = \theta_c = \sqrt{1+\xi^2}$ (instead of $\theta=1$ for $\theta=\delta_o=0$); while above this value quantum effects dominate. The interaction between atoms also leads to a non-zero mean-value $\langle S_x \rangle$ in contrast to the case $\xi=0$. Secondly, when $\xi=0$, but $\delta \neq 0$, the critical behaviour disappears.

More detailed analysis is required to determine the critical behaviour in the

regime of parameters ϕ and ξ where the first order phase transition is predicted by semi-classical analysis. Qualitatively one expects a sharp jump in the atomic observables and their fluctuations at the critical values $\theta=\theta_c(\phi,\xi$ fixed) from one stable branch at $\theta<\theta_c$ to another stable branch at $\theta>\theta_c$. A similar critical transition is also expected when the frequency detuning parameter is varied. This is indicated qualitatively in Figs.6 and 7 by dashed lines.

In summary, the Dicke model with atom-atom interactions predicts within $\langle\hat{S}^2\rangle$ conservation a non-equilibrium first order phase transition in the thermodynamic limit.

4. SUMMARY

The driven Dicke model exhibits interesting critical behaviour. Within the conservation of $\langle\hat{S}^2\rangle$, a mathematical condition required for the model, the model predicts a second order phase transition for resonant coherent driving field; bistable behaviour is predicted if the system is placed in a cavity in resonance with the external field as well as with the atomic transition. If interaction between the atoms is included, the system predicts a first order phase transition for certain values of the parameters related to the interaction, field amplitude and frequency detuning. At this stage it is an open question, where these critical phenomena can be observed in an experiment. Experiments with very large wavelength (low frequency) and large dipole moment characteristic of Rydberg atoms may perhaps yield this information. Because of the exactness of the model, it may also be possible to explain the critical behaviour in a more fundamental way. This issue is, however, open and needs further thought.

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