RANDOM MATRICES IN CONDENSED MATTER PHYSICS

C.K. Majumdar
Magnetism Department
Indian Association for the Cultivation of Science
Jadavpur, Calcutta - 700032, India

The theory of ensembles of random matrices was pioneered by Wigner [1] and has been of considerable utility in the analysis of nuclear spectra [2,3,4]. There are some problems in condensed matter physics where the theory has been applied with some measure of success (see Sec. IX of [4]). It is likely that the use of random matrix ensembles in solid state physics will increase, particularly because the major ensemble types can be realized in practice in solid state problems.

1. New kind of Statistical Mechanics

It has been emphasised by Dyson [5] that the approach through random matrices constitutes a new kind of statistical mechanics characterized by the hypothesis of total ignorance. In ordinary statistical mechanics, the Hamiltonian of the system is well defined, but the system is so large that exact computation is neither feasible nor meaningful, and observation of all the details is not possible. We consider an ensemble of systems described by this fixed Hamiltonian, and assume that all states of the Hamiltonian are equally likely for the ensemble. This leads to useful characterization of the system.

In the new kind of statistical mechanics we renounce knowledge of the exact nature of the system itself. For instance, a nucleus may be pictured as a "black box" containing a large number of particles interacting strongly according to imperfectly understood laws. In the study of phonons in glass, a glass is an irregular network of several types of atoms which interact in a linear harmonic fashion with spring constants varying randomly. The problem is to define in a mathematically precise way an ensemble in which various likely Hamiltonians are equally probable.

A Hamiltonian is represented by a hermitian matrix, usually infinite. For simplicity, we restrict ourselves to large but finite matrices. Certain conservation laws are known; they are associated with some symmetry principles which are supposed to be valid for our system, whatever be its Hamiltonian. There are, for example, rotational invariance, reflection invariance and time reversal invariance. These imply certain symmetries of the matrices and special features of the matrix elements. Thus time reversal invariance implies that the matrix element can be taken to be real. Apart from these general symmetry requirements, the matrix elements are regarded as independent random variables. A common, simple assumption is that the matrix element M_{ij} is normally distributed about a mean value with some variance. Next we ask for the average spectrum of such a matrix ensemble and try to compare the calculated spectrum with some experimental result on the excited states of a nucleus, the

phonon density of states in glass, etc.

One interesting feature of this comparison is that we are often more interested in the deviations from the calculated results. If, indeed, there is perfect agreement between experiment and theory, the hypothesis of total ignorance is justified. This may be a happy situation but is devoid of any intellectual challenge. A great deal of work in nuclear physics has been done to confront the random matrix calculations with experimental data; see Reference [4].

In ordinary statistical mechanics we come across several ensembles, uniform, microcanonical, canonical, grand canonical, and so on. It has been found by Dyson [6] that matrix ensembles can be classified into three types - orthogonal, unitary and symplectic - and that all irreducible ensembles belong to one of these three types. Dyson has traced the origin of this classification to the theorem of Frobenius: over the real number field, there exist precisely three associative division algebras, namely the real numbers, the complex numbers, and the real quaternions.

It appears likely that all the three ensembles can be practically realized in solid state problems. Consider small particles of free-electron-like metals at low temperatures. The energy levels are discrete, and around 10°K with particle radius about 10^{-6} cm, the level spacing is much larger than the average thermal energy. The system of metal particles will then show "quantum size effect". Now in an actual experiment the shape and size of the small particles may not be precisely controlled. The spectrum will show fluctuations. In fact, the electronic energy levels can be thought to be the eigenvalues of a fixed Hamiltonian with random boundary conditions, which may be incorporated into a random matrix by using fictitious potentials. The appropriate ensembles are then as follows:

(i) if the number of electrons is even and there is no magnetic field, the orthogonal ensemble is applicable;

(ii) if the number of electrons is odd and there is no magnetic field, the symplectic ensemble applies; and

(iii) when the magnetic field is present and the Hamiltonian is no longer time reversal invariant, the unitary ensemble is applicable.

Brody et al have reviewed the comparison between experiment and theory in this field. There are some unsolved problems; in particular, the recent advances in the "Weyl problem" [7] can be confronted with experimental work.

We shall give below a simple derivation of the Wigner semicircular law for the asymptotic distribution of the eigenvalues of a random matrix. A paper by Wigner [8] on bordered matrices contains some other interesting results.

2. Simple Derivation of the Semicircular Law

The derivation of the semicurcular law by Edwards and Jones [9] uses nothing beyond the properties of the ordinary Gaussian integrals, which are first collected here:

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \pi^{\frac{1}{2}} a^{-\frac{1}{2}} \tag{2.1}$$

If a has a negative imaginary part, we can write

$$\int_{-\infty}^{\infty} e^{-iax^2}\,dx = \pi^{\frac{1}{2}} e^{-i\pi/4} a^{-\frac{1}{2}} \tag{2.2}$$

For a positive definite N X N matrix $A = (a_{ij})$ we have

$$\int_{-\infty}^{\infty} \exp\Big(-\sum_{i,j} a_{ij}X_iX_j\Big) \prod_i dX_i = \pi^{N/2} \det{}^{-\frac{1}{2}} A \tag{2.3}$$

$\det^{\frac{1}{2}} A$ denotes the square root of the determinant of A. Hence we also write

$$\int_{-\infty}^{\infty} \exp\Big(-i\sum_{j,k} a_{jk}X_jX_k\Big) \prod_j dX_j = \pi^{N/2} e^{-i\pi N/4} \det{}^{-\frac{1}{2}} A \tag{2.4}$$

By completing the square one proves

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2b} \pm ax}\,dx = e^{\frac{1}{2}a^2b} (2\pi b)^{\frac{1}{2}} \tag{2.5}$$

Hence we get the representation

$$e^{ia^2/2} = \frac{e^{-i\pi/4}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{\frac{i}{2}x^2 - ax}\,dx \tag{2.6}$$

The other ingradient is the elementary limit formula of calculus

$$\lim_{n\to 0} \frac{x^n - 1}{n} = \ln x. \tag{2.7}$$

Consider now the problem of calculating the average eigenvalue spectrum of a large N X N real symmetric matrix M. Each matrix element M_{ij} has a gaussian probability density function with zero mean and fixed variance σ:

$$p(M_{ij}) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-M_{ij}^2/2\sigma^2\right] \tag{2.8}$$

Let the eigenvalues of M be M_i. The density of eigenvalues is defined by

$$\nu(\lambda) = \frac{1}{N} \sum_j \delta(\lambda - M_j) \tag{2.9}$$

where $\nu(\lambda)$ is normalized to unity. Now we give λ a small negative imaginary part and use

$$\frac{1}{\lambda - i\epsilon - M_j} = P\frac{1}{\lambda - M_j} + \pi i\,\delta(\lambda - M_j) \tag{2.10}$$

to obtain

$$\nu(\lambda) = \frac{1}{\pi N} \sum_j \mathrm{Im}\ \frac{1}{\lambda - i\epsilon - M_j} \tag{2.11}$$

But

$$\det(\lambda I - M) = \prod_j (\lambda - M_j)$$

Hence

$$\ln \det (\lambda I - M) = \sum_j \ln (\lambda - M_j),$$

and

$$\frac{\partial}{\partial x} \ln \det (\lambda I - M) = \sum_j \frac{1}{\lambda - M_j} \tag{2.12}$$

Henceforth λ is supposed to have a small negative imaginary part, then (2.11) can be written as

$$\nu(\lambda) = \frac{1}{\pi N} \operatorname{Im} \frac{\partial}{\partial \lambda} \ln \det (\lambda I - M) \tag{2.13}$$

$$= -\frac{2}{\pi N} \operatorname{Im} \frac{\partial}{\partial \lambda} \ln \det^{-\frac{1}{2}} (\lambda I - M) \tag{2.14}$$

$$= -\frac{2}{\pi N} \operatorname{Im} \frac{\partial}{\partial \lambda} \lim_{n \to o} \frac{1}{n} \left[(\det^{-\frac{1}{2}} (\lambda I - M))^n - 1 \right] \tag{2.15}$$

where in the last line we use (2.7). With (2.4) we can write

$$\nu(\lambda) = -\frac{2}{\pi N} \operatorname{Im} \frac{\partial}{\partial \lambda} \lim_{n \to o} \frac{1}{n} \left\{ \left(\frac{e^{i\pi/4}}{\pi^{\frac{1}{2}}} \right)^{Nn} \right.$$

$$\left. \times \int_{-\infty}^{\infty} \prod_{\substack{i,j=i,N \\ \alpha=1,n}} dx_i^\alpha \exp\left[-i \sum_{i,j,\alpha} x_i^\alpha (\lambda \delta_{ij} - M_{ij}) x_j^\alpha \right] - 1 \right\} \tag{2.16}$$

It is here assumed that n is an integer. Equation (2.7) has no such stipulation. Assume that in the result of interest the continuation to $n \to o$ is still allowed after we get a formal answer of the integral with n an integer. This step makes the derivation nonrigorous. Such a continuation to $n \to o$ from the initial assumption of only integral n has worked in several problems in solid state physics (this is the replica trick of Edwards and Anderson). The integration is over the Nn variables x_i^α.

It is good to check that in simple cases the formal $n \to o$ limit goes through. Consider an N X N matrix with all matrix elements M_{ij} equal to M_o/N, where $M_o \sim 0(1)$. It is well known that this has one eigenvalue M_o and (N-1) eigenvalues 0. Let us check if (2.16) leads to the correct result.

Equation (2.16) becomes

$$\nu(\lambda) = -\frac{2}{\pi N} \operatorname{Im} \frac{\partial}{\partial \lambda} \lim_{n \to o} \left[\left(\frac{e^{i\pi/4}}{\pi^{\frac{1}{2}}} \right)^{Nn} \right.$$

$$\left. \times \int_{-\infty}^{\infty} \prod_{i,\alpha} dx_i^\alpha \exp\left(-i\lambda \sum_{i,\alpha} (x_i^\alpha)^2 + i \frac{M_o}{N} \sum_\alpha \left(\sum_i x_i^\alpha \right)^2 \right) \right] \tag{2.17}$$

Now we use (2.6) in the form known as the "auxiliary field identity":

$$\exp\left[i \frac{M_o}{N} \left(\sum_i x_i^\alpha \right)^2 \right] = \left[\frac{e^{-i\pi/4}}{(2\pi)^{\frac{1}{2}}} \right] \times \int_{-\infty}^{\infty} dq\, e^{iq^2/2} \exp\left[-\left(\frac{2M_o}{N} \right)^{\frac{1}{2}} q \sum_i x_i^\alpha \right] \tag{2.18}$$

The integral in (2.17) becomes

$$J_1 = \prod_\alpha \frac{e^{-i\pi/4}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dq \prod_i dx_i^\alpha \exp\left[i\lambda \sum_i (x_i^\alpha)^2 \right] \times \exp\left[-\left(\frac{2M_o}{N} \right)^{\frac{1}{2}} q \sum_i x_i^\alpha + \frac{1}{2} iq^2 \right] \tag{2.19}$$

The integral over each x_i^α is easily performed by completing the square, (2.5), and we find

$$J_1 = \prod_\alpha \frac{e^{-i\pi/4}}{(2\pi)^{\frac{1}{2}}} e^{-iN\pi/4} \left(\frac{\pi}{\lambda} \right)^{\frac{N}{2}} \int_{-\infty}^{\infty} dq \exp\left[-\frac{iq^2}{2} \left(\frac{M_o}{\lambda} - 1 \right) \right] \tag{2.20}$$

With (2.2) we get

$$J_1 = \prod_\alpha e^{-iN\pi/4} \pi^{\frac{N}{2}} \lambda^{-\frac{N-1}{2}} (\lambda - M_o)^{-\frac{1}{2}} \tag{2.21}$$

Thus (2.17) becomes

$$\nu(\lambda) = -\frac{2}{N\pi} \mathrm{Im} \frac{\partial}{\partial \lambda} \lim_{n \to o} \frac{1}{n} \left[\lambda^{-\frac{(N-1)n}{2}} (\lambda - M_o)^{-\frac{n}{2}} - 1 \right] \tag{2.22}$$

The formula allows continuation in n to fractional values and we take (2.7) as true

$$\begin{aligned} \nu(\lambda) &= -\frac{2}{N\pi} \mathrm{Im} \frac{\partial}{\partial \lambda} \ln\left[\lambda^{-\frac{N-1}{2}} (\lambda - M_o)^{-\frac{1}{2}} \right] \\ &= \frac{1}{\pi N} \mathrm{Im} \frac{\partial}{\partial \lambda} \left[(N-1) \ln \lambda + \ln(\lambda - M_o) \right] \\ &= \frac{1}{\pi N} \mathrm{Im} \left[\frac{N-1}{\lambda} + \frac{1}{\lambda - M_o} \right]. \end{aligned} \tag{2.23}$$

Recall λ has a small negative imaginary part and use (2.10):

$$\nu(\lambda) = \frac{N-1}{N} \delta(\lambda) + \frac{1}{N} \delta(\lambda - M_o) \tag{2.24}$$

We now go back to the real symmetric matrix M with $M_{ij} = M_{ji}$. From (2.8) we define J by

$$\sigma^2 = J^2/N \tag{2.25}$$

with J of order unity. The averaged density of eigenvalues $\rho(\lambda)$ is obtained by averaging $\nu(\lambda; \{M_{ij}\})$ of (2.16) over all configurations of the M_{ij} given by (2.8):

$$\rho(\lambda) = \int \nu(\lambda; \{M_{ij}\}) \prod p(M_{ij})\, d M_{ij} \tag{2.26}$$

Putting (2.8) with (2.26), we carry out all the gaussian integrations over M_{ij}:

$$\rho(\lambda) = -\frac{2}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to o}\frac{1}{n}\Big[\Big\{\Big(\frac{e^{i\pi/4}}{\pi^{\frac{1}{2}}}\Big)^{Nn}$$

$$\times\int_{-\infty}^{\infty}\prod dx_i^{\alpha}\,\exp\Big[-i\lambda\sum_i (x_i^{\alpha})^2\Big]\exp\Big[-\frac{J^2}{N}\sum_{i,j}\Big(\sum_{\alpha} x_i^{\alpha}x_j^{\alpha}\Big)^2\Big]$$

$$\times\exp\Big[\frac{J^2}{2N}\sum_i\Big(\sum_{\alpha}(x_i^{\alpha})^2\Big)^2\Big]\Big\}^{n} - 1\Big] \tag{2.27}$$

We want to retain the leading terms in N as $N\to\infty$ and the term linear in n as $n\to o$. We can simplify the calculation of (2.27) by estimating order of magnitude of the second term of the exponentials.

$$\frac{J^2}{N}\sum_{i,j}\Big(\sum_{\alpha} x_i^{\alpha}x_j^{\alpha}\Big)^2 = \frac{J^2}{N}\sum_{i,j}\sum_{\alpha\beta} x_i^{\alpha}x_j^{\alpha}x_i^{\beta}x_j^{\beta}$$

$$= \frac{J^2}{N}\sum_{\alpha}\Big(\sum_i (x_i^{\alpha})^2\Big)^2 + \frac{J^2}{N}\sum_{\alpha\neq\beta}\sum_{i\neq j} x_i^{\alpha}x_j^{\alpha}x_i^{\beta}x_j^{\beta} \tag{2.28}$$

The first term is of order Nn. The second has a zero mean, but its square is of order n. The third exponential term in (2.27) is of order n^2. Hence it is enough to keep the terms

$$\frac{J^2}{N}\sum_{\alpha}\Big(\sum_i (x_i^{\alpha})^2\Big)^2$$

Thus for large N,

$$\rho(\lambda) = -\frac{2}{\pi N}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to o}\frac{1}{n}\Big[\Big(\frac{e^{i\pi/4}}{\pi^{\frac{1}{2}}}\Big)^{Nn}$$

$$\Big\{\int_{-\infty}^{\infty}\prod dx_i\,\exp\Big[-i\lambda\sum_i x_i^2 - \frac{J^2}{N}\Big(\sum_i x_i^2\Big)^2\Big]\Big\}^{n} - 1\Big] \tag{2.29}$$

Use again an auxiliary field identity

$$e^{-\frac{J^2}{N}\left(\sum_i x_i^2\right)^2} = \Big(\frac{N}{2\pi}\Big)^{\frac{1}{2}}\frac{\lambda}{(2J^2)^{\frac{1}{2}}}\int_{-\infty}^{\infty} ds\,\exp\Big(-i\lambda s\sum_i x_i^2\Big)\,\exp\Big(-\frac{\lambda^2}{4J^2}Ns^2\Big) \tag{2.30}$$

The integral in (2.29) becomes

$$J_2 = \Big[\int_{-\infty}^{\infty} ds\prod_i dx_i\Big(\frac{N}{2\pi}\Big)^{\frac{1}{2}}\frac{\lambda}{(2J^2)^{\frac{1}{2}}}\exp\Big(-i\lambda(1+s)\sum_i x_i^2\Big)\,\exp\Big(-\frac{N\lambda^2 s^2}{4J^2}\Big)\Big]^{n} \tag{2.31}$$

The integrals over the $\{x_i\}$ are straightforward, but the convergence of this integral depends on the small negative imaginary part of λ. We could make λ real and maintain convergence by putting a small negative imaginary part to s. Hence the result is

$$J_2 = \left\{ \left(\frac{N}{2\pi}\right)^{\frac{1}{2}} \left(\frac{\pi^N}{2J^2}\right)^{\frac{1}{2}} \lambda\, e^{-\frac{N}{2}\ln\lambda} \int_{-\infty}^{\infty} ds\, \exp\left[-Ng(s)\right] \right\}^n \tag{2.32}$$

where

$$g(s) = \frac{\lambda^2 s^2}{4J^2} + \frac{1}{2}\ln\left[\, i(1+s)\,\right] \tag{2.33}$$

The negative imaginary part of s implies that the branch point in ln (1+s) lies slightly above the real axis in the upper half plane at -1. We cut the complex s-plane by a line running parallel to but above the real axis from -1 to $-\infty$. The contour of integration in (2.32) lies along the real axis. Since we are interested in the result as $N \to \infty$, we can now do a simple saddle point integration.

Now g'(s) = 0 has the roots

$$s_o^{\pm} = \frac{1}{2}\left[-1 \pm i\left(\frac{4J^2}{\lambda^2} - 1\right)^{\frac{1}{2}}\right] \tag{2.34}$$

for $|\lambda| < 2J$ at complex conjugate points $s_o^{\pm}$. For $|\lambda| > 2J$, the roots are on the real axis

$$s^{\pm} = \frac{1}{2}\left[-1 \pm \left(1 - \frac{4J^2}{\lambda^2}\right)^{\frac{1}{2}}\right] \tag{2.35}$$

In the case $|\lambda| < 2J$, the contour must be chosen such that Re g(s) is minimum at the saddle point. If the contour integration is deformed downwards to follow the line

$$s = x - \frac{i}{2}\left(\frac{4J^2}{\lambda^2} - 1\right)^{\frac{1}{2}} \tag{2.36}$$

Re g(s) has a minimum at $x = -\frac{1}{2}$ corresponding to the saddle point s_o^-. Along Re $s = -\frac{1}{2}$ we find Re g(s) has maxima at $s_o^{\pm}$ and minimum at $s = -\frac{1}{2}$. Hence the contour runs through s_o^- and, to leading order in N as $N \to \infty$, we get

$$\int_{-\infty}^{\infty} ds\; e^{-Ng(s)} \simeq e^{-Ng(s_o^-)} \tag{2.37}$$

Hence we get

$$\rho(\lambda) = -\frac{2}{N\pi}\,\mathrm{Im}\frac{\partial}{\partial\lambda}\lim_{n\to 0}\frac{1}{n}\left[\left(\frac{e^{i\pi/4}}{\pi^{\frac{1}{2}}}\right)^{Nn}\left(\frac{N}{2\pi}\right)^{\frac{n}{2}}\left(\frac{\pi}{2J^2}\right)^{\frac{Nn}{2}}\right.$$

$$\left. \times\, \lambda^n\, e^{-\frac{1}{2}Nn\ln\lambda} e^{-Nng(s_o^-)} - 1\right]$$

$$= -\frac{2}{N\pi}\,\mathrm{Im}\frac{\partial}{\partial\lambda}\ln\left\{\frac{e^{iN\pi/4}}{(2J^2)^{N/2}}\left(\frac{N}{2\pi}\right)^{\frac{1}{2}}\lambda\; e^{-\frac{N}{2}\ln\lambda - Ng(s_o^-)}\right\}$$

$$\simeq -\frac{2}{N\pi}\,\mathrm{Im}\frac{\partial}{\partial\lambda}\left\{-\frac{N}{2}\ln\lambda - Ng(s_o^-)\right\}$$

to leading order in N. Recall that λ is now real. After elementary differentiation, we get

$$\rho(\lambda) = \frac{1}{2\pi J^2}\left(4J^2 - \lambda^2\right)^{\frac{1}{2}} \quad (\text{for } |\lambda| < 2J) \tag{2.38}$$

for $|\lambda| > 2J$, the integral turns out to be real and $\rho(\lambda) = 0$.

Equation (2.38) represents the well-known semicircular law of Wigner.

If the matrix elements are distributed about a fixed mean value M_o/N, we have

$$p(M_{ij}) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(M_{ij}-(M_o/N))^2\right] \tag{2.39}$$

As before we shall define $J^2 = N\sigma^2$, where J is of order unity. The calculation is very similar and involves only rather detailed manipulations of the contour integration at the end. The details are available in the paper of Edwards and Jones [9]. We shall simply quote the result.

For a large N x N random symmetric matrix, the elements of which are independent gaussian random variables with mean M_o/N and variance J^2/N, the average density of states in the limit $N \longrightarrow \infty$ is

$$\rho_o(\lambda) = \begin{cases} (4J^2-\lambda^2)^{\frac{1}{2}}/2\pi J^2, & M_o = 0, |\lambda| < 2J \\ 0, & M_o = 0, |\lambda| > 2J \end{cases}$$

(this is (2.38)), and

$$\rho(\lambda) = \begin{cases} \rho_o(\lambda) + \frac{1}{N}\delta\left\{\lambda-(M_o+\frac{J^2}{M_o})\right\}, & |M_o| > J \\ \rho_o(\lambda), & |M_o| < J \end{cases} \tag{2.40}$$

Equation (2.40) has some relevance to the eigenvalue spectrum of a strongly coupled localized perturbation in a solid such as a substitutional impurity coupled to the phonons. For certain values of the coupling constant of the system a state may be split off from the band of extended states and contribute a delta function outside the band of continuum of states.

The other interesting question relates to "band-tailing". Equation (2.40) gives a sharp cut-off for the averaged density of states. This is only true for $N \longrightarrow \infty$. For large and finite N, the spectrum has an exponential tail of states with a finite number of eigenvalues concentrated in a region of order $N^{-1/6}$ beyond 2J [3].

3. Random Matrix in Glass

The random matrix appeared in solid state physics in connection with phonons in glass. The solution of this problem in one dimension was given by Dyson [10].

Consider a chain of N masses, each coupled to its nearest neighbours by elastic forces obeying Hooke's law. Only motion in one dimension is envisaged, so each mass is described by a single coordinate. Let the mass of the particle number j in the chain be m_j and its displacement from equilibrium position x_j. The spring constant between particles j and j+1 is k_j. Thus the equations of motion of the

system are

$$m_j \ddot{x}_j = k_j(x_{j+1}-x_j) + k_{j-1}(x_{j-1}-x_i) \tag{3.1}$$

with appropriate changes for the end masses. There is one trivial zero frequency mode with all displacements equal. The problem is to calculate the remaining (N-1) eigenmodes as N becomes large.

When the masses and the spring constants are all equal, the calculation of the frequency spectrum is elementary. In the interesting case of glass, we have several species of atoms and the positions are irregular. One could assume that the masses m_j and spring constants k_j are arranged along the chain in a random fashion.

Let us put

$$y_j = m_j^{\frac{1}{2}} x_j \tag{3.2}$$

and introduce new constants $\lambda_1, \lambda_2, \ldots, \lambda_{2N-1}$ by $\lambda_{2j-1} = k_j/m_j$, $\lambda_{2j} = k_j/m_{j+1}$. Then

$$\ddot{y}_j = (\lambda_{2j-1}\lambda_{2j})^{\frac{1}{2}} y_{j+1} + (\lambda_{2j-1}\lambda_{2j-2})^{\frac{1}{2}} y_{j-1} - (\lambda_{2j-1}+\lambda_{2j-2}) y_j \tag{3.3}$$

The coefficient matrix is now symmetric. Now we difine new variables $z_1, z_2, \ldots, z_{N-1}$ by

$$z_j = \lambda_{2j}^{\frac{1}{2}} y_{j+1} - \lambda_{2j-1}^{\frac{1}{2}} y_j , \tag{3.4}$$

so (3.3) becomes

$$\dot{y}_j = \lambda_{2j-1}^{\frac{1}{2}} z_j - \lambda_{2j-2}^{\frac{1}{2}} z_{j-1} \tag{3.5}$$

Finally, we introduce the variables $u_1, u_2, \ldots, u_{2N-1}$ by

$$u_{2j-1} = y_j, \quad u_{2j} = z_j, \quad j = 1,2,\ldots,N. \tag{3.6}$$

Then (3.4) and (3.5) are combined into a set of (2N-1) linear equations

$$\dot{u}_j = \lambda_j^{\frac{1}{2}} u_{j+1} - \lambda_{j-1}^{\frac{1}{2}} u_{j-1} \tag{3.7}$$

The eigenfrequencies ω_j of the chain are therefore the characteristic roots of the (2N-1) x (2N-1) matrix Λ whose elements are given by

$$\Lambda_{j+1,j} = -\Lambda_{j,j+1} = i\lambda_j^{\frac{1}{2}} \tag{3.8}$$

All other elements are zero. There is one zero root corresponding to the degenerate mode in which all displacements are equal. The remaining roots occur in (N-1) pairs, ω_j and $-\omega_j$.

The spectrum of eigenfrequencies is given by the function $M(\mu)$ which is defined as the proportion of the roots ω_j for which $\omega_j^2 \leqslant \mu$. As $N \longrightarrow \infty$, the function $M(\mu)$ will become a smooth differentiable function and then a density of eigen-

frequencies can be defined by $D(\mu) = dM(\mu)/d\mu$. Corresponding to given $\{\lambda_j\}$ we have to determine M and D.

There are several different ways to introduce randomization. The masses may be independent random variables, while spring constants are all equal. The masses may all be the same, while the spring constants are random. Or we may consider the λ_j's, which are combinations of masses and spring constants to be random.

Dyson studies the function

$$\Omega(x) = \lim_{N\to\infty} \frac{1}{2N-1} \sum_j \ln (1+x\omega_j^2) \tag{3.9}$$

$$= \int_0^\infty \ln (1+x\mu)\, D(\mu)\, d\mu \tag{3.10}$$

as a function of the complex variable x. That branch of the logarithm is taken which is real for real, positive x. The spectral density functions D and M are determined by the limiting values of $\Omega(x)$ on the negative real axis approached from above.

By a rather intricate analysis, Dyson shows that an alternative expression for $\Omega(x)$ is

$$\Omega(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{a=1}^{2N-1} \ln (1+\xi(a)) \tag{3.11}$$

where $\xi(a)$ is a continued fraction

$$\xi(a) = x\lambda_a/(1+x\lambda_{a+1}/(1+x\lambda_{a+2}/(\ldots \tag{3.12}$$

A simpler derivation of this equation is due to Bellman [11]. When all the λ_j's are the same, $\lambda = k/m$, ξ satisfies

$$\xi = \frac{x\lambda}{1+\frac{x\lambda}{1+\ldots}} = \frac{x\lambda}{1+\xi} \tag{3.13}$$

and $\xi \longrightarrow 0$ as $x \longrightarrow 0$. Hence

$$\xi = \frac{1}{2}\left[(1+4x\lambda)^{\frac{1}{2}}-1\right], \tag{3.14}$$

$$\Omega(x) = 2 \ln \left[\frac{1}{2} (1+4x\lambda)^{\frac{1}{2}} + \frac{1}{2} \right] \tag{3.15}$$

and

$$D(\mu) = \begin{cases} \frac{1}{\pi} (4\lambda\mu - \mu^2)^{\frac{1}{2}}, & \mu < 4\lambda \\ 0, & \mu > 4\lambda \end{cases} \tag{3.16}$$

Equation (3.16) can be checked by direct calculation. In general solvable cases are rare, but Dyson found one analytically tractable. Each of the parameters λ_j of (3.8) is an independent random variable with the probability distribution

$$G_n(\lambda) = \frac{n^n e^{-n\lambda} \lambda^{n-1}}{(n-1)!} \tag{3.17}$$

the integer n taking the values 1,2, ... The distribution has mean 1 and standard

deviation $n^{-\frac{1}{2}}$ and in the limit of large n, $G_n(\lambda)$ becomes a gaussian

$$G_n(\lambda) \sim \left(\frac{n}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{n}{2}(n-1)^2} \tag{3.18}$$

Notice the probability distribution is defined for a combination of spring constant and mass, and not of each separately.

The details of the solution are complicated. For small z, the function M(z) is given by

$$M_n(z) \sim \left[\frac{\pi^2}{6} - t_{n-1}\right] / \left[\pi^2 + \ln nz + s_{n-1} + \gamma\right] \tag{3.19}$$

with

$$s_j = \sum_{l=1}^{j} \frac{1}{l}, \qquad t_j = \sum_{l=1}^{j} \frac{1}{l^2} \tag{3.20}$$

and γ is Euler's constant = 0.5772,.... For large z,

$$M_n(z) = 1 - 2\,(\ln nz - s_{n-1} + \gamma)\, e^{-nz}\, (nz)^{2n-1} \left[(n-1)!\right]^{-2} \tag{3.21}$$

For comparison, the uniform chain result (3.16) leads to

$$\begin{aligned} M(z) &= \frac{1}{\pi} \cos^{-1}(1-\tfrac{1}{2}z), & z<4, \\ &= 1, & z>4 \end{aligned} \tag{3.22}$$

Follows a very important conclusion, which is probably valid generally : A disordered chain has a much greater proportion of very low characteristic frequencies than a uniform ordered chain.

Several points worth further study may be mentioned.

(i) Dyson's calculation is in one dimension and does not include topological disorder which is possible in three dimensions. Very little has been done in this regard.

(ii) The distribution function G_n does not correspond to any physical chain. One could consider problems of equal coupling constants but random masses. A formulation was given by H. Schmidt [12], but Schmidt's equation has been characterized by Lieb and Mattis [13] as one of the most difficult equations in mathematical physics.

(iii) Dyson's formulation, though elegant, requires analytic continuation of $\Omega(x)$ to the negative real axis through the upper half plane. This makes the formulation difficult for numerical computation. The numerical work done in this field has been reviewed by P. Dean [14] . Whether one can reformulate Dyson's method in a form suitable for computation has not been studied much. Neither has any other analytical solution, except (3.17), been discussed in the literature.

Stress Relaxation in Glass

We shall now turn to another problem in glass which is again not clearly understood. It is known that glass is a linear solid and exhibits "delayed elasticity" (see a lucid exposition by Douglas [15]). Apart from the normal instantaneous

elastic response, common soda-lime-silicate glass exhibits linear viscoelastic flow. Experimentalists have established that creep and stress relaxation in glass are non-exponential. With the strain kept constant, the stress S in glass relaxes because of delayed elasticity according to a law

$$S(t) = S_o \exp\left[-(t/\bar{\tau})^{\alpha}\right] \tag{3.23}$$

S_o is the initial stress. The mean relaxation time $\bar{\tau} \sim 4 \times 10^4$ sec and is directly proportional to viscosity, showing the same temperature dependence. The viscosity here is around 10^{14} poise . The index α changes with time; for $t \ll \bar{\tau}$, $\alpha \sim 0.3$; for $t \approx \bar{\tau}$, α is around 0.5 to 0.6; and for $t \gg \bar{\tau}$, $\alpha \longrightarrow 1$. The problem is to find an explanation for the form (3.23), the existence of large relaxation times and the values of α [16].

It is natural to start with the Navier Stokes equation for viscous flow:

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v}\right] = -\nabla p + \eta \nabla^2 \vec{v} + \tfrac{1}{3}\eta \nabla(\nabla.\vec{v}) \tag{3.24}$$

We may consider glass as an incompressible fluid. Then $\nabla.\vec{v} = 0$ and the pressure term ∇p also drops out (that is, we ignore sound propagation). As we are in the linear regime we also drop the non-linear term $(\vec{v}.\nabla)\vec{v}$. We then end up with

$$\rho \frac{\partial \vec{v}}{\partial t} = \eta \nabla^2 \vec{v} \tag{3.25}$$

which we may write as a diffusion equation

$$\frac{\partial \vec{v}}{\partial t} = D \nabla^2 \vec{v} \tag{3.26}$$

where $D = \eta/\rho$.

Equation (3.25) is a classical continuum equation. Suppose we try to write a discrete version and take into account the different atomic masses. Consider a 1-dimensional case first.

$$\rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial x^2} \tag{3.27}$$

After simple manipulations this can be replaced by

$$\frac{\partial u_i}{\partial t} = \eta\left[\frac{a_i}{m_i}(u_{i+1}-u_i) + \frac{a_{i-1}}{m_i}(u_{i-1}-u_i)\right] \tag{3.28}$$

where m_i is the i^{th} mass and a_i the distance between the i and (i+1) th masses. a_i and m_i are random variables. This equation has a form similar to Dyson's equation (3.1), with ω^2 replaced by τ^{-1}, where τ denotes a relaxation time. From the theory of random matrices we can carry over the qualitative conclusion : the disordered system will have larger proportion of longer relaxation times than ordered systems. So the delayed elasticity is expected.

The trouble is that the starting point (3.24) is not quite right. Each relaxation time τ is inversely rather than directly proportional to viscosity and the relaxation time does not increase with viscosity as observed. If we use the kinetic

theory, the diffusion coefficient $D = \eta/\rho = \frac{1}{3}c\lambda = \lambda^2/3\tau$ where λ is the mean free path, c the mean velocity and τ the time between collisions. The mean path picture breaks down when the viscosity is high and the Navier-Stokes equation becomes inapplicable.

It has been agreed [17] that the equation (3.26) being a macroscopic equation can be retained in the region of high viscosity, but the diffusion coefficient should be something like the Einstein diffusion coefficient in the Brownian motion

$$D = \frac{k_B T}{6\pi\eta a} \tag{3.29}$$

a is a characteristic average length of the network in glass. The indices α could be produced with suitable mode distribution [17]. The value of $\bar{\tau}$ comes out right,

$$\bar{\tau} \simeq \frac{\bar{\lambda}^2}{4D} \tag{3.30}$$

where $\bar{\lambda}$ is the short range order in glass. By (3.29) $\bar{\tau}$ is directly proportional to η. The problem of establishing the flow equation at large viscosities has not been satisfactorily solved. What seems to be happening is epitomized in a stochastic model of Kramers [18] on the escape of particles over potential barriers. When the viscosity is small, the rate of escape depends very little on viscosity (the Navier-Stokes situation), but when the viscosity is large the rate is inversely proportional to viscosity (the glass problem).

4. Conclusion

We have indicated several areas in solid state physics where random matrices appear. Another closely related area not touched above is that of spin waves in disordered Heisenberg magnets [19]. Analytical and numerical techniques used in the problems mentioned above and the technique of diagonalization of sparse matrices are often useful for such problems.

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