FOKKER-PLANCK EQUATIONS FOR STOCHASTIC PROCESSES

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In this lecture I will discuss the various properties of the Fokker-Planck equations and the different methods, approximate and exact which are used to solve such equations.

1. Forward and Backward Equations for the Conditional Probability

Let $P(x,t|x_o,t_o)$ be the transition probability for a Markov process $x(t)$. This transition probability obeys CHAPMAN-KOLMOGOROV equation

$$P(x_3,t_3|x_1,t_1) = \int dx_2\ P(x_3,t_3|x_2,t_2)\ P(x_2,t_2|x_1,t_1),\ t_3 \geqslant t_2 \geqslant t_1 . \tag{1}$$

Using (1), a dynamical equation for $P(x,t|x_o,t_o)$ can be obtained by introducing the conditional moments $M_n = \langle (x(t+\Delta t) - x(t))^n \rangle$ and by writing (1) as

$$P(x,t+\Delta t|x_o,t_o) = \int dx_1\ P(x,t+\Delta t\ |\ x_1,t)\ P(x_1,t\ |\ x_o,t_o) \tag{2}$$

and by using the formal relation

$$P(x,t+\Delta t\ |\ x_1,t) = \sum_{n=o}^{\infty} \frac{m_n(x_1)}{n!}\left(-\frac{\partial}{\partial x}\right)^n \delta(x-x_1) \tag{3}$$

The resulting dynamical equation, which is called KRAMER-MOYAL expansion is

$$\frac{\partial P}{\partial t} = \sum_1^{\infty} \left(-\frac{\partial}{\partial x}\right)^n [D_n(x)\ P(x,t|x_o,t_o)] , \tag{4}$$

where

$$D_n(x) = \lim_{t\to o} \frac{1}{tn!} \langle (x(t)-x)^n \rangle \tag{5}$$

Similarly by using CHAPMAN-KOLMOGOROV relation in the form

$$P(x,t|x_o,t_o-\Delta t_o) = \int dx_1 P(x,t|x_1 t_o)\ P(x_1 t_o|x_o,t_o-\Delta t_o) \tag{6}$$

and by expanding $P(x,t|x_1 t_o)$ in a Taylor series around x_o, we find the KOLMOGOROV or backward equation

$$-\frac{\partial P}{\partial t_o} = \sum_1^{\infty} D_n(x_o) \left(\frac{\partial}{\partial x_o}\right)^n P(x,t|x_o,t_o). \tag{7}$$

The backward equation (7) is the adjoint of the forward equation, with all the derivatives being taken with respect to x_o and t_o. The foregoing assumes that the limit (5) exists.

Let us now examine the meaning of the various terms in (4). Suppose that $D_n = 0\ \forall\ n>1$, then (4) is a differential equation involving first order derivatives and hence

$$P(x,t_o|x_o,t_o) = \delta(x-x_o) \Rightarrow P(x,t|x_o,t_o) = \delta(x-x_o(t)),$$

$$\dot{x}_o(t) = D_1(x_o),\ x_o(t_o) = x_o. \qquad (8)$$

In such a case the motion of the system is deterministic i.e. there are no fluctuations. The term D_1 is also known as the drift term. The fluctuations in the system arise due to the nonvanishing of the coefficients like D_2 etc. For example if $D_1 = 0$, D_2 = Constant and $D_n = o\ \forall\ n > 2$, then (4) leads to

$$\langle x(t)\rangle = x_o,\quad \langle x^2(t)\rangle - x_o^2 = 2\,D_2 t, \qquad (9)$$

which corresponds to the very familiar diffusion. Using the positiveness of P and the Schwarz inequality $(\int \psi(x)\phi(x)P(x)\,dx)^2 \leqslant (\int \psi(x)P(x)dx)(\int \phi^2(x)P(x)\,d\,x)$ and by choosing $\psi = (x(t+\tau)-x(t))^n$, $\phi = (x(t+\tau)-x(t))^{n+m}$, we prove the inequality

$$[(2n+m)!\ D_{2n+m}(x,t)]^2 \leqslant 2n!\ (2n+2m)!\ D_{2n}(x,t)D_{2n+2m}(x,t) \qquad (10)$$

The inequality (10) can be shown to lead to the important result [1] - If KRAMER-MOYAL expansion terminates, then it can have at the most two terms D_1 and D_2. When KRAMER-MOYAL expansion terminates, then the resulting dynamical equation is called the FOKKER-PLANCK equation (FPE)

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[D_1(x)P] + \frac{\partial^2}{\partial x^2}[D_2(x)P], \qquad (11)$$

whose multidimensional generalization is obviously

$$\frac{\partial P}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}[D_i^{(1)}(\{x\})P] + \sum \frac{\partial^2}{\partial x_i \partial x_j}[D_{ij}^{(2)}(\{x\})P], \qquad (12)$$

with

$$D_i^{(1)} = \lim_{t\to o} \frac{1}{t}\langle x_i(t)-x_i\rangle,\quad D_{ij}^{(2)}(x) = \lim_{t\to o}\frac{1}{2t}\langle (x_i(t)-x_i)(x_j(t)-x_j)\rangle \qquad (13)$$

The diffusion matrix $D^{(2)}$ is obviously positive definite. The conservation of the probability follows form (12)

$$\frac{\partial P}{\partial t} + \vec{\nabla}.\vec{J} = 0,\quad J_i = D_i^{(1)}P - \frac{\partial}{\partial x_j}[D_{ij}^{(2)}P] \qquad (14)$$

2. Examples of FPE

The FOKKER-PLANCK equations have been traditionally used in the context of the Brownian motion. However in the meanwhile, one has discovered that the behavior of such diverse systems as lasers [2], Josephson junctions [3], chemically reacting [4] species could be described by FPE. The Brownian motion of a particle in a potential V(x) is given by the dynamical equations

$$\dot{x} = p/m,\quad \dot{p} = -\frac{\partial V}{\partial x} - \gamma p + f(t) \qquad (15)$$

where f(t) is a delta correlated Gaussian random process with zero mean

$$\langle f(t)\, f(t') \rangle = 2D\delta(t-t'), \tag{16}$$

The FPE corresponding to (15) is

$$\frac{\partial P(x,p,t)}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{p}{m}P\right)+\frac{\partial}{\partial p}\left[\left(\gamma p+\frac{\partial V}{\partial x}\right)P\right] + D\frac{\partial^2 P}{\partial p^2}\ , \tag{17}$$

which in the limit of large damping leads to the SMOLUCHOWSKI equation for $P(x,t) = \int P(x,p,t)\,dp$

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left[\frac{V'}{m\gamma}P\right] + \frac{D}{m^2\gamma^2}\frac{\partial^2 P}{\partial x^2} \tag{18}$$

which in the case of a free particle reduces to the famous EINSTEIN diffusion equation

$$\frac{\partial P}{\partial t} = \left(\frac{D}{\gamma^2 m^2}\right)\frac{\partial^2 P}{\partial x^2}\ . \tag{19}$$

The microscopic theory [2] of the single mode laser operating in the threshold region shows that the distribution P of the laser photons is given by

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial b}\left[(p-b^*b)bP\right] + 2\frac{\partial^2 P}{\partial b\,\partial b^*} + \text{c.c.}\ , \tag{20}$$

where b is the complex laser field amplitude and p is the pump parameter. A laser operating at threshold implies p = o. Very many other examples of the FPE and the contexts in which they arise will be discussed by other speakers in this school.

3. Solutions of the FPE

The FPE (12) is a partial differential equation of the parabolic type. We have to specify the boundary conditions on P(x,t) before one can solve even a one dimensional FPE. For a reflecting boundary one obviously should have the current vanish at the boundary i.e.

$$D_1 P - \frac{\partial}{\partial x}\left[D_2 P\right] = 0 \tag{21}$$

The case of an absorbing boundary is more involved. For the equation like (18), the condition at the absorbing boundary usually imposed is P = 0, though other forms have also been recently proposed [5] . In the case of natural boundaries at $\pm\infty$ one might impose $P(\pm\infty,t) = 0$. We now consider the various classes for which the FPE can be solved.

(a) $D_2 = 0$

In such a case as already discussed above, the conditional probability is exactly known provided the equations for the classical trajectories can be solved and then the probability distribution at time t will be

$$P(x,t) = \int dx_o\, P(x_o,t_o)\quad \delta(x-x_o(t)) \tag{22}$$

Equation (22) yields information on how the initial fluctuations change with time. It may be added that this type of situation has arisen in a large number of diverse problems [6] such as lasers, superradiance, chamical reactions etc.

(b) $D_i^{(1)} = \sum a_{ij} x_j$, $D_{ij}^{(2)}$ = Quadratic function of x_i

In such a case it is obvious that the equations for the moments $\langle x_i(t) \rangle$, $\langle x_i(t) x_j(t) \rangle$ etc. form a closed set of linear equations and hence can be solved by standard matrix methods. Multiplicative stochastic processes, for example a randomly modulated harmonic oscillator, lead to the above type of drift and diffusion coefficients. Recently considerable progress has been made in the field of optical resonance in fluctuating fields [7] because of this realization.

(c) $D_i^{(1)} = \sum a_{ij} x_j$, $D_{ij}^{(2)}$ = Constant

This is a very important class of FPE. The underlying stochastic process is called the ORNSTEIN-UHLENBECK process. Brownian motion by a free particle, harmonic oscillator fall under this class. The FPE is exactly soluble. The solution, which has been obtained by the method of characteristics [8] , is

$$P[x,t|x_o,o] = [(2\pi)^N \det \sigma(t)]^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}\widetilde{(x-e^{-at}x_o)}\, \sigma^{-1}(t)\, (x-e^{-at}x_o) \right\}$$

$$\sigma_{ij}(t) = \langle (x_i(t) - \langle x_i(t)\rangle)(x_j(t) - \langle x_j(t)\rangle)\rangle ,$$

$$\sigma(t) = \sigma(\infty) - e^{-at}\sigma(\infty)\, e^{-\tilde{a}\, t},$$

$$a\sigma(\infty) + \sigma(\infty)\,\tilde{a} = 2D, \quad \langle x(t)\rangle = e^{-at}\, x_o. \qquad (23)$$

The stochastic process in this case is Gaussian and markovian. The solution (23) has turned out to be of tremendous value in the study of the systems where $D^{(1)}$ is not a linear function of x_i but $D^{(2)} = g\, d^{(2)}$, where g is a infinitisimal parameter. In such a case one can calculate the behavior of the system to various orders in g. This is similar to VAN KAMPEN'S system size expansion [9] . We present the result to the lowest order in g. Write the stochastic process x(t) as

$$x(t) = x_o(t) + \sqrt{g}\; Y(t) + \ldots$$

$$\dot{x}_{oi} = D_i^{(1)}(\{x_o\}), \quad x_o(o) = x_o ,$$

$$D^{(2)}(\{x\}) = g\, d^{(2)}(\{x\}) \approx g\, d^{(2)}(\{x_o\}) ,$$

$$D_i^{(1)}(\{x\}) = D_i^{(1)}(\{x_o\}) + \sqrt{g} \sum_j Y_j \frac{\partial}{\partial x_{oj}} D_i^{(1)}(\{x_o\}) + \ldots \qquad (24)$$

Then the variables $Y_i(t)$ form an ORNSTEIN-UHLENBECK process

$$\frac{\partial P(\{y\})}{\partial t} = -\sum_i \frac{\partial}{\partial y_i}\left[\sum_j y_j \frac{\partial}{\partial x_{oj}} D_i^{(1)}(\{x_o\}) P \right] + \sum_{ij} d_{ij}^{(2)} \frac{\partial^2 P}{\partial y_i \partial y_j} \qquad (25)$$

but the drift and diffusion now depend on time t because of the time dependence of the solution of (24a). The steady state situation is much simpler and the solution given by (23) (in the limit $t \to \infty$) can be used. For the non-stationary problem, the resulting moment equations from (25) can be numerically integrated.

(d) One dimensional case-arbitrary $D^{(1)}$ and $D^{(2)}$

The one dimensional FPE can be solved by quadratures with the result

$$P(x) \sim e^{2\Phi} = \exp\left\{ \int^{x} \frac{dx}{D_2} (D_1 - D_2') \right\} \tag{26}$$

assuming that the current is zero at $\pm\infty$. The structure (26) of the steady state solution has been of great importance in the study of nonequilibriun phase transitions [10]. The time dependent problem is much harder. However it can be transformed into a Sturm-Liouville system

$$\dot{P} = \mathcal{L} P, \quad \mathcal{L} \psi_j = -\lambda_j \psi_j, \qquad \psi = e^{\Phi} \xi, \quad L \xi_j = -\lambda_j \xi_j,$$

$$L = \frac{\partial}{\partial x} D_2 \frac{\partial}{\partial x} - \left(e^{-\Phi} \frac{\partial}{\partial x} D_2 \frac{\partial}{\partial x} e^{\Phi} \right) = L^{+}. \tag{27}$$

The conditional probability can be expressed in terms of the eigenfunctions ξ_j as

$$P(x,t|x_o,o) = \sum_i e^{-\lambda_i t} \xi_i(x) \xi_i^*(x_o) e^{\Phi(x) - \Phi(x_o)} \tag{28}$$

Thus the solution to $L\xi = -\lambda\xi$ determines the stochastic behavior of the system. Since one has a Sturm-Liouville system, various perturbative methods like WKB and variational, can be used to estimate at least some of the low lying eigenvalues and eigenfunctions. It may be added that the most general exactly soluble class will correspond to the case where $L\xi = -\lambda\xi$ can be reduced to an equation for hypergeometric or confluent hypergeometric functions.

(e) Multi-dimensional FPE

In the general case of a multi-dimensional FPE, one knows very little. Even it is not possible to obtain the steady state solution. There is however a subclass where steady state can be obtained by quadratures [11] as discussed by GRAHAM and HAKEN. On imposing the detailed balance criteria i.e. invariance of the multitime joint distributions under time reversal $x_i \to \tilde{x}_i$, $t_i \to -t_i$,

$$P_n[\{x_1\}, 0, \{x_2\}, t_2 \ldots \{x_n\}, t_n] = P_n[\{\tilde{x}_1\}, 0, \{\tilde{x}_2\}, -t_2 \ldots \{\tilde{x}_n\}, -t_n] \tag{29}$$

and on using the forward and backward equations, it is possible to show that in the steady state (i) the divergence of the reversible part of the current is zero

$$\sum_i \frac{\partial}{\partial x_i} [D_i^{(1)\,rev} P_{st}] = 0, \quad D_{ij}^{(2)}(\{x\}) = \epsilon_i \epsilon_j D_{ij}^{(2)}(\{\tilde{x}\}) \tag{30}$$

and (ii) that the irreversible current is zero

$$- D_i^{(1)irr} P_{s+} + \sum_j \frac{\partial}{\partial x_j} D_{ij}^{(2)}(\{x\}) P_{st} = 0 \tag{31}$$

Thus P_{st} can be obtained by quadratures provided that the integrability conditions and (30) are satisfied. This was a major step in the field and many important FPE such as those occuring in the theory of lasers, absorptive optical bistability, optical parametric oscillators etc. were solved for the stationary distribution function. Inspite of this achievement, the time dependent problem is still a major challenge in the field basically because of our inability to solve non-hermitean partial differential equations for eigenfunctions and eigenvalues.

(f) Approximate Methods

We have already mentioned that if the Fokker-Planck equation can be transformed into a hermitean problem, then the standard variational methods can be used. The other possibility which has been extensively used [12] by Risken's group and Lambropoulos's group and others is to transform the Fokker-Planck equation into a set of difference equations which can be solved numerically by using continued fractions. For example if we transform the original Fokker-Planck equation into the form

$$\frac{\partial P}{\partial t} = L_o P + L_1 P, \quad L_o^+ = L_o \tag{32}$$

and where the eigenfunctions of L_o are known $L_o \psi_n = -\lambda_n \psi_n$, then on expanding $P = \sum_n c_n \psi_n$ we obtain from (32)

$$\dot{c}_n = -\lambda_n c_n + \sum_m (L_1)_{nm} c_m \tag{33}$$

Now depending on the structure of L_1, we will get a finite term recursion relation. A three term recursion relation is easily converted into a continued fraction. It should be borne in mind that in general one has a matrix continued fraction, the methods of solution for which are available.

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