# STOCHASTIC EVOLUTION IN ISING MODELS

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## I. Introduction and Outline

Stochastic evolution models are often used to study time-dependent properties of many-body systems. Some examples are the Brownian motion of particles, tracks of nuclear particles through dense media, kinetics of nucleation in super-heated liquids etc. These models provide the basic framework for studying time-dependent phenomena in statistical mechanics, such as the approach to thermal equilibrium from an arbitrarily prepared initial state, or of non-equilibrium steady states in dissipative systems, or the appearance of co-operative long-time correlations in the neighbourhood of second-order phase-transitions. In the following, the relaxation properties of some kinetic Ising models are briefly discussed.

The plan of these lectures is as follows: In section II, we discuss briefly how the probabilistic description of evolution of many-particle systems can be reconciled with the deterministic (microscopic) mechanical evolution. In section III, the rate-equation for the Markovian evolution, and the condition of detailed balance are described. In section IV, we introduce the single-spin-flip kinetic Ising model and general crystal-growth model (of which the kinetic Ising model is a special case). The dynamical scaling hypothesis, and some of its consequences are discussed in section V. Sections VI and VII contain brief discussions of long-time relaxation in a disordered Ising model in one and higher dimensions respectively. It is shown that in the disordered Ising model with broken bonds, the relaxation of magnetization to the equilibrium-value is slower than exponential for all temperatures below the critical temperature of the model without disorder.

## II. Probabilistic Versus Deterministic Evolution

The understanding of the coexistence of thermodynamic irreversibility (as canonized in the second law of thermodynamics) with microscopic mechanical reversibility has been the central theme in non-equilibrium statistical mechanics. The fact that the time-evolution of a gas undergoing free expansion, and its time-reversed evolution

are both consistent with the laws of mechanics, clearly shows that it is not possible to 'prove' the approach to equilibrium in isolated systems without making additional assumptions about the evolution of macroscopic systems. These assumptions are necessarily extra-mechanical. They may be very plausible (e.g. the unprovable assertion that initial states corresponding to sets of measure zero in phase space are unlikely to occur in real experiments) or much less obvious ones (these also are usually preceded by the qualifier 'almost always'). These may be assumptions about the large size of the system (absence of Poincare recurrences), the tendency of the phase space trajectory of the system to diverge (the mixing property), the decay of multi-particle correlations (Boltzmann's collision-number hypothesis) or the presence of weak but uncontrollable interactions of the system with the outside (evolution in the presence of weak noise). In the following, we shall adopt the position that the macroscopic relaxation behavior of large systems is very well modelled by a probabilistic evolution law, in particular by Markovian dynamics, and shall side-step the question of deriving the master-equation from more elementary principles. (For a discussion of these issues, see [1-3]). Note that macroscopic deterministic evolution (e.g. the Navier-Stokes equations) is a special case of Markovian dynamics, and corresponds to the case when fluctuations in the macroscopic variables are small.

A simple illustrative example of a system which undergoes deterministic evolution, but may be equivalently described by a probabilistic law, is the following: Consider a particle performing a walk on the points of a linear chain. The internal state of the particle is described by an angle-variable $\theta$ ($0 < \theta < 2\pi$), which undergoes a (discrete-time) deterministic evolution according to the law

$$\theta_{t+1} = 2\theta_t \quad \text{(modulo } 2\pi\text{)}. \tag{1}$$

The particle starts at time t=0. At subsequent times t=1,2,3,..., it takes a step of unit length to the right, or to the left, according as $\cos\theta_t$ is positive or negative. If the value $\theta_{t=0}$ is known, the motion of the particle is completely determined. Assume, however, that the value $\theta_{t=0}$ is known only to a finite accuracy; the a priori probability density of $\theta_{t=0}$ being constant for $|\theta_{t=0}| < 2^{-N}\pi$, and zero elsewhere. It is easy to see that under these assumptions, the steps of the particle for $t > N$ are perfectly random and uncorrelated. As far as the motion of the particle on the chain is concerned, after the decay of initial state correlations (for $t > N$), it can be described as a simple unbiassed random walk.

In this example, the stochastic evolution is a result of the exponential growth of the initial-state uncertainty with time. Note that the stochastic characterization of the random walk does not involve the 'internal degree of freedom' $\theta_t$. In general, the number of variables in terms of which the 'mesoscopic' state of the system is characterized need not be as large as the number required for a full microscopic characterization.

## III. The Rate Equations

Consider a finite system which at any time t may exist in any one of a denumerable number of states labelled by integers 1,2,3... The system is in contact with a heat-reservoir, and the interaction causes transitions between these states, the transition rate from state m to state n being $W_{mn}$. Let $P_m(t)$ be the probability that the system exists in the state m at time t. From the general theory of Markov chains [see e.g. [4]] it follows that under very weak conditions on the transition rates $W_{mn}$'s (each state must be reachable from every other), as the time t tends to infinity, $P_m(t)$ tends to a limiting value $P_m(\infty)$ independent of the initial state. The time evolution of $P_m(t)$ is governed by the equation

$$\frac{d}{dt} P_m(t) = \sum_{n \neq m} [P_n(t) W_{nm} - P_m(t) W_{mn}] \tag{2}$$

In problems of physical interest, $P_m^{eq}$ is the well known equilibrium distribution

$$P_m^{eq} = \exp(-\beta E_m) / [\sum_n \exp(-\beta E_n)]; \tag{3}$$

where $E_m$ is the energy of the state m and $\beta$ is the inverse temperature characteristic of the heat bath. The requirement that a system at large times should tend to thermal equilibrium constrains the physically admissible transition rates $W_{mn}$'s, but it does not determine them uniquely. Many different choices of $W_{mn}$'s would be consistent with a given limiting distribution. $\{P_m^{eq}\}$ is a time-invariant probability distribution for the rate equation (2) if

$$\sum_n [W_{mn} P_m^{eq} - W_{nm} P_n^{eq}] = 0, \text{ for all } m. \tag{4}$$

These conditions are clearly satisfied, if $W_{mn}$'s satisfy the detailed balance condition

$$W_{mn} = W_{nm} \exp[-\beta E_n + \beta E_m]. \tag{5}$$

Since the limiting distribution is the unique time-invariant distribution (for finite systems), the assumptions of Markovian evolution, the detailed balance condition, and non-existence of other conserved quantities guarantee that the system will relax to thermal equilibrium with time.

Define $\bar{P}_m(t) = P_m(t)\exp(\beta E_m/2)$. (6)

In terms of $\bar{P}_m(t)$'s, eq.(2) may be rewritten as

$$\frac{d}{dt}\bar{P}_m(t) = \sum_n \bar{P}_n(t)\bar{W}_{nm} \tag{7}$$

where $\bar{W}_{mn} = W_{mn}\exp[\beta(E_n-E_m)/2]$ for $m\neq n$ (8)

and $\bar{W}_{mm} = -\sum_{n\neq m} W_{mn}$ (9)

If the transition rates $W_{mn}$ satisfy the detailed balance condition, then $\bar{W}$ is a real symmetrical matrix. Hence all its eigenvalues are real. The rate of growth of $\bar{P}_m(t)$ with time is related to the largest eigenvalue of $\bar{W}$. Since $\bar{P}_m(t)$ tends to a constant value for large t, the largest eigenvalue of $\bar{W}$ is zero.

The problem of integration of Eq.(7) is reduced to that of diagonalizing the matrix $\bar{W}$. This is quite hard, and has not been solved yet for any nontrivial model so far. Some specific models of interest are described in the next section.

## IV. Stochastic Evolution Models

Our progress in understanding irreversible phenomena in statistical mechanics is severely hampered by the lack of simple illustrative models which are exactly soluble for their nonequilibrium properties, and thus serve as testing-grounds for theories or guides to intuition. An important motivation for the study of the kinetic Ising model and the crystal-growth model discussed below is that the powerful techniques of equilibrium statistical mechanics can be used to study these models, and thus they help us bridge the gap between our understanding of these disciplines.

(a) The Kinetic Ising Model: Consider an Ising ferromagnet in which the spins $\sigma_i$ (i=1 to N), taking values ±1, are located at the sites of a d-dimensional hypercubical lattice. The Hamiltonian of the system is given by

$$H = -\sum_{ij} J_{ij}\sigma_i\sigma_j \, . \tag{10}$$

We denote a configuration of spins by $\{\sigma\}$. There are $2^N$ possible configurations. In the Glauber model [5] the evolution is assumed to be due to single-spin-flips. The probability that the $i^{th}$ spin flips between the times t and t+dt, when the configuration of spins at time t is $\{\sigma\}$ is assumed to be equal to $W_i(\{\sigma\})dt$, where

$$W_i(\{\sigma\}) = \frac{1}{2}\,[1 - \tanh \beta h_i(t)], \tag{11}$$

and

$$h_i(t) = \sum_{j \neq i} J_{ij}\sigma_j(t). \tag{12}$$

It is easy to see that the choice (11) satisfies the detailed balance condition, and hence in the long-time limit the equal-time correlation functions of this model agree with those calculated using equilibrium ensembles. Glauber studied the relaxation of a one-dimensional chain with nearest-neighbour couplings in zero external field. An exact solution for a linear chain in the presence of an external field, or for a higher dimensional lattice has not yet been found.

The single-spin-flip Glauber model is not applicable to Ising systems like binary alloys, because the assumed dynamics does not conserve magnetization (particle-number in alloys). However, a similar rate matrix W, which describes a simultaneous exchange of spins between neighbouring sites i and j is easy to write down such that it satisfies the condition of detailed balance. The evolution equations for this model are similar to the single-spin-flip case [6].

Much insight into the behavior of these models near phase transitions has been obtained by Monte-Carlo simulations. These have been comprehensively reviewed in ref.[7]. A review of the theoretical aspects may be found in Kawasaki [6].

(b) Crystal-Growth Models: Consider a model of crystal-growth shown in Fig.1. The lattice is a square lattice. At time t=0, each of the sites (x,y) is unoccupied if $x+y > 0$, and if $x+y \leqslant 0$, it is occupied by one (and only one) of two kinds of atoms A and B. The state of an occupied site (ij) is characterized by an Ising variable $\sigma_{i,j}$ which is +1 or -1 according as the site is occupied by A or B. At time t=1, each of the sites (x,y) lying on the line x+y=1 is filled by a particle A or B taken randomly from an external source of particles. We assume that sites on this new 'layer' are occupied independently of each other, and that the probability that the site (i,j) is filled by an A atom (or B) depends on the configuration of neighbouring earlier-filled

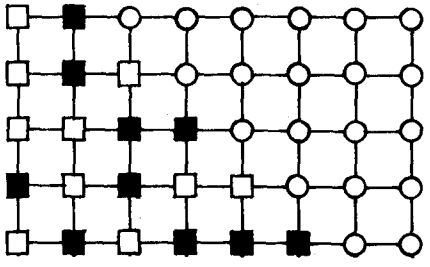

Fig.1: The crystal-growth model. The full and open squares denote sites occupied by A and B atoms respectively. Unoccupied sites are denoted by open circles.

sites (i-1,j), (i,j-1) and (i-1,j-1). Once a site is occupied, it stays occupied by the same atom at all subsequent times. At times t=2,3,..., successive layers of sites x+y=2,3,... are filled and the boundary of the 'crystal' (occupied sites) moves outwards. The model simulates the growth of mixed crystals from solution, under conditions when particle diffusion in the solid phase is negligible [8-9].

The full specification of the model requires a specification of the conditional probabilities $\text{Prob}(\sigma_{i,j}|\sigma_{i-1,j},\sigma_{i,j-1},\sigma_{i-1,j-1})$. Since $\text{Prob}(+1|\alpha,\beta,\gamma) + \text{Prob}(-1|\alpha,\beta,\gamma)$ must be 1 for all $\alpha,\beta,\gamma$, we may write

$$\text{Prob}(\sigma_{i,j}|\sigma_{i-1,j},\sigma_{i,j-1},\sigma_{i-1,j-1}) = \frac{1}{2}\,[1+\sigma_{i,j}\, f(\sigma_{i-1,j},\sigma_{i,j-1},\sigma_{i-1,j-1})] \qquad (13)$$

The model is thus defined in terms of 8 parameters, the values $f(\pm1,\pm1,\pm1)$.

We may treat t=x+y as the 'time' coordinate, and the configurations of $\sigma_{i,j}$'s on the lattice as time-histories of a discrete-time evolution kinetic Ising model on a linear chain in which the spins on odd-numbered sites are allowed to flip at odd values of the discrete time-parameter, and the even numbered ones when the time-parameter is even.

The general model involving 8 parameters is quite difficult to analyse. Of special interest is the case when the function f in Eq. (13) is independent of the spin $\sigma_{i-1,j-1}$. If we also assume mirror symmetry about the line x=y, the model can be characterized in terms of 3 parameters only. We write

$$\begin{aligned} f(+1,+1,\pm1) &= a; \\ f(+1,-1,\pm1) &= f(-1,+1,\pm1) = b; \\ f(-1,-1,\pm1) &= c. \end{aligned} \qquad (14)$$

It is easy to see that the case

$$a = 4p-2p^2-1;\ b = 2p-1;\ c = -1 \tag{15}$$

corresponds to problem of directed bond percolation with bond-concentration p. The case

$$a = b = 2p-1;\ c = -1 \tag{16}$$

corresponds to the directed site-percolation problem. These problems have attracted much attention recently [10-15]. The problem with a=b=-1 is exactly soluble and is related to the directed animals problem [16,17]. Various other special soluble cases of the general crystal-growth model have been discussed by Enting [18]. Crystal-growth models in higher dimensions, or on other lattices are easy to define, but are usually quite difficult to solve exactly.

## V. Dynamical Scaling Theory

Near phase-transitions, the spatial and temporal correlations in the Ising model became long-ranged. The singular behavior of these correlation functions is described in terms of critical exponents, which are expected to show universal characteristics, as in the case of equilibrium phase transitions. In fact, from the discussion in the previous section, it is clear that there is no essential difference between the statistical-mechanical descriptions of static and dynamical critical phenomena. Calculating time-dependent correlation functions in a d-dimensional system involves averaging over histories of configurations, which is just like a (d+1)-dimensional static calculation. Dynamical scaling theory [19] is a natural generalization of the static scaling theory used to describe equilibrium phase-transitions. Using this theory, the time-dependent critical behavior of a wide class of systems (including kinetic Ising models) can be described in terms of only 3 independent critical exponents.

Consider a d-dimensional kinetic Ising model at a temperature $T = T_c(1+\varepsilon)$, where $T_c$ is the critical temperature of the model, and $\varepsilon$ is small. We consider only the case, when there is no external magnetic field present. Inclusion of magnetic field requires only a straightforward extension of the formalism. The time-dependent two-point correlation function in equilibrium is given by

$$G(\bar{r},t,\varepsilon) \overset{\text{def}}{=\!=} < \sigma_{\bar{r}+\bar{x}}(t+t_o)\sigma_{\bar{x}}(t_o) \tag{17}$$

Note that $G(\bar{r},t,\varepsilon)$ described the time-dependent correlations <u>in</u> the

equilibrium state and hence does not depend on $\bar{x}$ or $t_o$.

According to the dynamical scaling hypothesis, the function $G(\bar{r},t,\varepsilon)$ in the limit of large spatial or temporal separations and small $\varepsilon$, equals a generalized homogenous function of $\bar{r}$, t and $\varepsilon$. The difference between $G(\bar{r},t,\varepsilon)$ and the homogeneous function is a non-universal correction-to-scaling function which is asymptotically negligible in the critical region. Since we are not going to discuss corrections to scaling here, the generalized homogeneous function will be denoted by the same symbol $G(\bar{r},t,\varepsilon)$. A function $G(\bar{r},t,\varepsilon)$ is said to be a generalized homogeneous function if there exist constants $\beta$, $\nu$ and z such that

$$G(\bar{r},t,\varepsilon) = \lambda^{+2\beta} G(\lambda^{+\nu}\bar{r},\ \lambda^{\nu z}t,\ \lambda^{-1}\varepsilon) \quad \text{for all } \lambda > 0. \tag{18}$$

A similar equation describes the behavior of more general n-point correlation function. The n-point function is scaled by $\lambda^{n\beta}$ if all distances, times and $\varepsilon$ scaled as in Eq.(18). From the scaling equation (18), the singular behavior of various physical quantities can be deduced in terms of the three critical exponents $\nu$, z and $\beta$. In the presence of external magnetic field, an exponent $\Delta$ characterizing the scaling of the external field has to be included. However, if we assume the hyperscaling relation ($d\nu = 2-\alpha$, where $\alpha$ is a known function of $\beta$, $\nu$ and $\Delta$), then the number of independent exponents is 3 again.

On putting t=0 in the above equation, it reduces to the static scaling equation. It follows that $\beta$ and $\nu$ appearing in Eq.(18) are the conventional magnetization and correlation length exponents. Consider now, the autocorrelation function $G(0,t,-\varepsilon)$. By the scaling hypothesis, it can be written in the form

$$G(0,t,-\varepsilon) = \varepsilon^{2\beta} f(t\varepsilon^{\nu z}) \tag{19}$$

where f(x) is a function of a single variable x. As x tends to infinity, f(x) tends to a constant. Since t scales as $\varepsilon^{-\nu z}$ in the above equation, the relaxation time must diverge as $\varepsilon^{-\nu z}$ for small $\varepsilon$. If we fix t, and let $\varepsilon \to 0$, then $G(0,t,-\varepsilon)$ has a smooth limit only if f(x) varies as $x^{-2\beta/\nu z}$ for small x. This then implies that

$$G(0,t,0) \sim t^{-2\beta/\nu z} \tag{20}$$

In the analysis of the decay of magnetization from an initially aligned state, we cannot use time-translational invariance. However, if we assume that an equation analogous to Eq.(19) can be written down for M(t), we get

$$M(t) \simeq \varepsilon^{\beta} g(t\varepsilon^{\nu z}), \tag{21}$$

where, since $M(t) \to \varepsilon^{\beta}$ as $t \to \infty$; $g(x)$ should tend to a constant as $x$ tends to infinity. Then arguing as before, $g(x)$ must vary as $x^{-\beta/\nu z}$ for small $x$, giving

$$M(t) \sim t^{-\beta/\nu z}, \quad \text{for } \varepsilon = 0. \tag{22}$$

This is slower than the decay of the auto-correlation function, and is due to multi-spin correlations. The relative simplicity of such results using scaling theory becomes especially valuable, as a complete solution is not available for any of the kinetic Ising models showing phase transitions. In the finite-size scaling method, similar scaling techniques are used to determine the values of the exponents $\beta$, $\nu$ and $z$ by extrapolating the behavior of infinite systems from a sequence of finite-size realizations which are solved exactly (numerically). The procedure yields very good numerical estimates of static and dynamic critical exponents [13,20].

## VI. Relaxation in Disordered Ising Chains

The Ising model in one dimension does not undergo any phase transition, and hence relaxation in Ising chains does not show all the features of critical slowing down near phase transitions. However, it is the only non-trivial model showing approach to thermal equilibrium for which the time-dependent correlation functions can be calculated without too much trouble, and hence serves as a useful test case for various approximations. In the following we study the relaxation of magnetization in a chain from an initially aligned state. The relaxation is found to be exponential in the homogeneous case. On introducing a quenched disorder in the band strengths, the relaxation is not exponential any longer, and we study its long-time behavior.

The Hamiltonian of a linear chain of Ising spins $\sigma_i$ (i=1 to N) is given by

$$H = \sum_{i=1}^{N-1} J_{i+1/2}\, \sigma_i \sigma_{i+1}. \tag{23}$$

The coupling constants $J_{i+1/2}$'s are assumed to be quenched, independent, identically distributed random variables taking values $J_1$ and $J_o$ ($0 \leq J_1 < J_o$) with probabilities (1-p) and p respectively. The time-evolution of the probability $\mathcal{P}$ of the configuration $\{\sigma_1, \sigma_2, \ldots \sigma_N\}$ in the single-spin-flip Glauber model is governed by the equation

$$\frac{d}{dt}\,\mathcal{P}(\{\sigma_1,\sigma_2\ldots\sigma_N\}) = \sum_i [W_i^+\,\mathcal{P}(\{\sigma_1\ldots-\sigma_i..\sigma_N\}) - W_i^-\,\mathcal{P}(\{\sigma_1\sigma_2\ldots\sigma_N\})]. \tag{24}$$

where $W_i^{\pm}$ are the spin-flip probabilities per unit time for the spin $\sigma_i$. The most general expression for $W_i^{\pm}(\{\sigma\})$ which depends only on $\sigma_{i-1},\sigma_i$ and $\sigma_{i+1}$, and satisfies the detailed balance condition is

$$W_i^{\pm} = \frac{1}{2}\,[1 \pm \sigma_i\,\tanh\beta h_i(t)]\,[A+B(\sigma_{i-1}+\sigma_{i+1}) + C\sigma_{i-1}\sigma_{i+1}], \tag{25}$$

where $h_i(t)$ is the effective field at site i given by

$$h_i(t) = J_{i-1/2}\sigma_{i-1}(t) + J_{i+1/2}\sigma_{i+1}(t). \tag{26}$$

In the special case A=1, B=C=0, $S_i(t)$ defined as the expectation value of $\sigma_i(t)$ evolves according to the equation

$$(1 + \frac{d}{dt})S_i(t) = C_i^-S_{i-1}(t) + C_i^+S_{i+1}(t) \tag{27}$$

where

$$C_i^{\pm} = \frac{1}{2}\,[\tanh\beta(J_{i+1/2} + J_{i-1/2}) \pm \tanh\beta(J_{i+1/2} - J_{i-1/2})] \tag{28}$$

We assume that at time t=0, all spins are +1. The average magnetization at time t is given by

$$M(t) = \frac{1}{N}\sum_{i=1}^{N} S_i(t) \tag{29}$$

Eq.(27) may be written as the matrix equation

$$\frac{d}{dt}\,|S(t)> = -\Lambda|S(t)> \tag{30}$$

where $\Lambda$ is an NxN tridiagonal matrix, independent of time. The equation may be solved formally to give

$$M(t) = \int_0^{\infty} d\lambda D(\lambda)\exp(-\lambda t) \tag{31}$$

where

$$D(\lambda)d\lambda = < \sum_{\lambda<\lambda'<\lambda+d\lambda} < S(0)|\lambda' >< \lambda'|S(0) >>_c \tag{32}$$

where $|\lambda' >$ and $< \lambda'|$ are the right-and left-eigenvectors of $\Lambda$ with eigenvalue $\lambda'$, and $<\ >_c$ denotes configuration averaging over the quenched variables $J_{i+1/2}$'s. If p=1, and $N \to \infty$, straightforward diagonalization of $\Lambda$ shows that

$$M(t) = \exp(-\lambda_o t) \tag{33}$$

with

$$\lambda_o = 1 - \tanh 2\beta J_o. \tag{34}$$

If $p \neq 1$, the arguments of Lifshitz [21] show that (details of these calculations may be found in Dhar and Barma [22]) $D(\lambda)$ is non-zero if $|\lambda-1| < \tanh 2\beta J_o$, and as $\lambda$ tends to $\lambda_o$ from above, $D(\lambda)$ varies as $\exp[-(\lambda-\lambda_o)^{-1/2}C]$, where C is a known constant. Substituting this asymptotic form of $D(\lambda)$ in Eq.(31), the behavior of M(t) for large t is easily determined. We find

$$M(t) \sim \mathrm{Exp}(-\lambda_o t - at^{1/3}), \text{ as } t \to \infty. \tag{35}$$

If the temperature is very low so that the thermal correlation length $\xi_T = \exp(2\beta J_o)$ is much larger than the percolation correlation length $\xi_p = 1/(1-p)$, then in the time domain $\xi_T\xi_p << t << \xi_T^3/\xi_p$, the magnetization is approximately given by the formula

$$M(t) \sim \exp(-\lambda_o t - bt^{1/2}). \tag{36}$$

While this is an admittedly simplified model, the existence of non-exponential relaxation in the presence of disorder is gratifying. Many experimental disordered materials show such a behavior.

## VII. Higher Dimensions

Consider now the kinetic Ising model on a d-dimensional hypercubical lattice (d > 1) with single-spin-flip dynamics given by Eq.(11). We restrict ourselves to the case of nearest-neighbour ferromagnetic couplings and $J_1=0$.

In this case, the equation of evolution for $S_i(t)$ involves multi-spin correlation functions, and a rigorous analysis of the problem is difficult. Even in the limit of high temperatures, when higher order correlations may be neglected, the resulting equations are difficult to solve in the disordered problem. However, it is easy to see that the relaxation would be non-exponential in general.

If p is less than the critical percolation probability $p_c$ on the lattice, all clusters are finite, and the equilibrium state is paramagnetic with no spontaneous magnetization. The density of clusters of size n varies as exp(-An), where A is a p-dependent constant. Because of the absence of mutual interactions, different clusters evolve independently of each other. For each finite cluster, the relaxation problem is in principle soluble, involving a diagonalization

of $2^n \times 2^n$ matrix. Let $\tau_n$ be the relaxation time (actually this is the longest of the spectrum of relaxation times) of a typical cluster of size n. [A more careful argument would take into account the fact that $\tau_n$ depends on the shape of the cluster also.] If the temperature T is greater than $T_c$, the critical temperature for the pure case (p=1), then all clusters have finite relaxation times. We may write for large n

$$\tau_n^{-1} \simeq B(T,p) + cn^{-x}. \tag{37}$$

Here we have used the fact that $\tau_n^{-1}$ is a bounded decreasing function of n. We expect B(T,p) to vanish as T tends to $T_c$ from above. In the special case p=1, $B(T,p) \simeq (T-T_c)^{\nu z}$ for T near $T_c$, where z is standard dynamical critical exponent. The extrapolation form (37) is only one of several equivalent forms showing the correct n dependence in the limit of large n. Using a Lifshitz-like argument, we would expect that x = 2/d. The average magnetization at time t is given by

$$M(t) \simeq \sum_n P_n \exp(-t/\tau_n) \tag{38}$$

For large times t, Eqs.(37) and (38) together imply that

$$M(t) \sim \exp[-B(T,p)t - Dt^{\frac{1}{1+x}}] \tag{39}$$

where D is some function of T and p.

If $T < T_c$, there will be exceptionally large clusters for which there are two metastable states of opposite magnetization with very infrequent transitions between them. For these clusters, the relaxation times are expected to be a strongly rising function of n

$$\tau_n \sim \exp[n^{x'} A'(T,p)] \tag{40}$$

Again, we expect A'(T,p) to vanish as T tends to $T_c$ from below. Also x' is equal to (1-1/d). This is because the free energy barrier between up and down magnetized states is the surface energy of a domain wall spanning the cluster. Such a wall would be of size $n^{1-1/d}$ for compact clusters having n sites.

Using Eqs.(38) and (40) we get for $T < T_c$ and $p < p_c$

$$M(t) \sim \exp[-A''(\log t)^{1/x'}], \text{ as } t \to \infty; \tag{41}$$

where A" is some constant which depends on T and p.

If $p > p_c$, there is an infinite cluster, and the possibility of a ferromagnetic-paramagnetic phase transition. However, the

analysis given above would still hold for the finite clusters, which still would relax to zero magnetization very slowly. Hence the overall relaxation of magnetization would be non-exponential. Note that for all $T < T_c$, whatever the value of p, the relaxation of magnetization is slower than exponential. This is essentially a rigorous result and depends only on the finite probability of occurrence of very large compact clusters with no holes. These clusters have large relaxation times and their contribution to the average magnetization of the sample dominates the long-time relaxation. This result is the non-equilibrium counterpart of Griffiths singularities (Griffiths [23]). For intermediate time regimes, the relaxation of noncompact clusters, as well as of the infinite cluster have to be considered.

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