WAVE PROPAGATION IN RANDOM MEDIA

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In this lecture we will apply some of the techniques developed in this volume to the study of the wave propagation in a random medium. This is a subject on which enormous body of the literature [1] exists and hence we will cover only certain basic stochastic aspects of the system. In the specific context of this talk, wave propagation refers to the propagation of the electromagnetic waves through a medium whose refractive index is a random function. However one could very well also consider the propagation of acoustic waves or even Schrödinger equation in a random potential, the latter being covered by other speakers at this school.

Let us consider the steady state phenomena and let $\epsilon(\vec{r})$ be the spatially inhomogeneous dielectric function of the medium. The Maxwell equations in such a case lead to an equation for the electric field $\vec{E}$ alone

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} - k_m^2(\vec{r})\, \vec{E} = 0, \quad k_m = (\omega/c)\sqrt{\epsilon(\vec{r})} \tag{1}$$

which can also be written as

$$\nabla^2 \vec{E} + k^2 \epsilon(\vec{r})\, \vec{E} + \vec{\nabla}(\vec{E}\cdot\vec{\nabla} \ln \epsilon(\vec{r})) = 0, \quad k = \omega/c \tag{2}$$

Since $\epsilon(\vec{r})$ is a random function whose statistical properties are supposedly known (2) represents a very complex type of Langevin equation. No exact solutions for such an equation appear to be known even if $n(\vec{r})$ is a Gaussian random process and hence one needs to use approximate methods in order to obtain information onthe characteristics of $\vec{E}$ for such a medium.

(1) Weak Refractive Index Fluctuations

We first consider the case when the refractive index fluctuations are weak i.e. $n(\vec{r}) = 1+n_1(r)$; $n_1(r) \ll 1$. We can then integrate equation (1) in a perturbative manner. Let us introduce the Green's dyadic defined by

$$\vec{\nabla} \times \vec{\nabla} \times \overleftrightarrow{G}(\vec{r},\vec{r}\,',\omega) - k^2 \overleftrightarrow{G}(\vec{r},\vec{r}\,',\omega) = 4\pi \overleftrightarrow{I}\, \delta(\vec{r}-\vec{r}\,'). \tag{3}$$

Assuming outgoing boundary conditions at infinity, the solution of (3) is

$$\overleftrightarrow{G}(\vec{r},\vec{r}\,',\omega) = \left(\overleftrightarrow{I} + \frac{\vec{\nabla}\vec{\nabla}}{k^2}\right) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \tag{4}$$

This represents the free space Green's function. Using (4) and $\epsilon(r) = 1+\epsilon_1(r)$, one can convert (1) into an integral equation

$$\vec{E}(\vec{r},\omega) = \vec{E}^{(i)}(\vec{r},\omega) + \frac{k^2}{4\pi}\int d^3r'\, \overleftrightarrow{G}(\vec{r},\vec{r}\,',\omega)\cdot \epsilon_1(\vec{r}\,')\vec{E}(\vec{r}\,',\omega) \tag{5}$$

where $\vec{E}^{(i)}(\vec{r},\omega)$ represents the incident field on the medium. Note that $\epsilon_1(\vec{r}) \sim 2n_1(\vec{r})$. The scattered field can now be evaluated to different orders in ϵ_1. In Born approximation, the scattered field will be

$$\vec{E}_{Sc}(\vec{r},\omega) = \frac{k^2}{4\pi}\int d^3r'\ \vec{G}(\vec{r},\vec{r}',\omega).\ \epsilon_1(\vec{r}').\vec{E}^{(i)}(\vec{r}',\omega). \tag{6}$$

If the fluctuations in ϵ_1 are Gaussian, then the scattered field will also be Gaussian provided that the input field $\vec{E}^{(i)}$ is non-stochastic in nature. It may be of interest to note that $\vec{E}_{Sc}$ will not be Gaussian even if both ϵ_1 and $\vec{E}^{(i)}$ are Gaussian random processes. The expression for the scattered field can be simplified in the far zone limit whence

$$\vec{G}\ (\vec{r},\vec{r}',\omega) \longrightarrow (\vec{I}-\vec{n}\ \vec{n})\ \frac{e^{ikr-i\vec{n}.\vec{r}'k}}{r}\ , \tag{7}$$

where $\vec{n}$ is the unit vector in the direction $\vec{r}$. The scattering amplitude then becomes

$$\vec{E}_{Sc} \sim -\frac{k^2}{4\pi}(\frac{e^{ikr}}{r})\int d^3r'\ \epsilon_1(\vec{r}')e^{-i\vec{n}.\vec{r}'k}\ \vec{n}\ x\ (\vec{n}\ x\ \ \vec{E}^{(i)}(\vec{r}')) \tag{8}$$

which on assuming $\vec{E}^{(i)}(\vec{r}) = \vec{\xi}\ e^{i\ \vec{K}.\vec{r}}$ leads to the following expression for the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\ \pi^2}\int d^3r_1\int d^3r_2\ \epsilon_1(\vec{r}_1)\ \epsilon_1^*(\vec{r}_2)e^{i(\vec{K}-\vec{n}k).(\vec{r}_1-\vec{r}_2)}\sin^2\theta\ , \tag{9}$$

where θ is the angle between $\vec{n}$ and $\vec{\xi}$. Expression (9) is the basic formula for the electromagnetic scattering. For our medium ϵ_1 is a random quantity and hence we have to average (9) over the stochastic fluctuations of $\epsilon_1(\vec{r})$. Let us write the correlation function of n as

$$\Gamma(\vec{r}_1,\vec{r}_2) = \langle n_1(\vec{r}_1)n_1^*\ (\vec{r}_2)\rangle = F\left\{\frac{\vec{r}_1+\vec{r}_2}{2}\right\}\ g\ (\vec{r}_1-\vec{r}_2) \tag{10}$$

For a homogeneous medium F is unity . Then the average cross section can be written as

$$\frac{d\sigma}{d\Omega} = \frac{k^4\sin^2\theta}{4\pi^2}\int_V d^3\ R\ \ F(\vec{R})\int_V g(\vec{r})e^{i(\vec{K}-k\vec{n}).\vec{r}}d^3r, \tag{11}$$

which is the basic formula relating the scattered flux to the refractive index fluctuations of the random medium. Further progress can be made depending on the structure of F and g. Often Kolmogorov spectrum is used in connection with the atmospheric turbulence

$$\langle(n_1(\vec{r}_1+\vec{r}) - n_1(\vec{r}_1))^2\rangle = D(r) \sim r^{2/3},\ L_o \gg r \gg l_o \qquad \sim r^2\ ,\quad r \ll l_o \tag{12}$$

where l_o and L_o are, respectively, the inner and outer scales of turbulence. This form was generalized by Kármán to

$$1 - \frac{2^{2/3}}{\Gamma(\frac{1}{3})} \left(\frac{r}{L_o}\right)^{1/3} K_{1/3} \left(\frac{r}{L_o}\right).$$

More generally we can write

$$g(\vec{r}) = \frac{\bar{n}_1^2}{2^{\nu-1}\Gamma(\nu)} \left(\frac{r}{L_o}\right)^{\nu} K_{\nu} \left(\frac{r}{L_o}\right), \qquad \nu \sim 1/3, \tag{13}$$

whose cosine Fourier transform $g(\vec{k})$ is

$$g(\vec{k}) = \frac{\Gamma(\nu+3/2)}{\pi^{3/2}\Gamma(\nu)} \bar{n}_1^2 L_o^3/(1+k^2L_o^2)^{\nu+3/2} \tag{14}$$

On substituting (14) in (11) and assuming F = 1 we find the result

$$\frac{d\sigma}{d\Omega} = \frac{2k^4 \sin^2\theta \;\; V \;\; \Gamma(\nu+3/2) \; \bar{n}_1^2 \; L_o^3}{\pi^{\frac{1}{2}} \;\; \Gamma(\nu) \;\; [1+4L_o^2 k^2 \sin^2\frac{\theta}{2}]^{\nu+3/2}}, \tag{15}$$

which is quite a general expression for the scattering from a random medium. Notice the important frequency dependence coming from the denominator. The refractive index fluctuations are characterized by the parameter $\bar{n}_1^2/L_o^{2\nu}$.

2. Strong Fluctuations : Non Perturbative Methods

When the refractive index fluctuations are strong, then one has to use methods which yield the scattered fields to all orders in the refractive index. This obviously is difficult and it turns out that a good deal of progress can be made by incorporating several physical assumptions. We assume that the wavelength is small compared to the typical scale of inhomogeneities and hence we can use the idea of small angle scattering in the forward direction and ignore any back scattering. We will also, for simplicity, ignore the depolarization effects so that the electromagnetic problem can be treated as a scalar problem. Writing further $E = ue^{ikz}$ and making the slowly varying approximation, we find that the original equation (2) reduces to the well known parabolic equation

$$2ik\frac{\partial u}{\partial z} + \nabla_t^2 u + k^2 \epsilon_1 u = 0; \tag{16}$$

where z is the direction of propagation. This is a stochastic differential equation of the type treated earlier [2]. As noted there, exact results for the moments and the correlation functions of u can be obtained by assuming that ϵ_1 is a Gaussian delta correlated random process

$$\langle \epsilon_1(\vec{r}) \;\; \epsilon_1(\vec{r}') \rangle = \delta(z-z') \;\; \Gamma(|\vec{\rho} - \vec{\rho}'|), \tag{17}$$

where $\vec{\rho}$ is the coordinate transverse to z. This assumption has been more or less universally used by every one in the field and very many different methods like the ones based on Novikov's theorem [3], diagrams [4] have been developed to obtain the equations for the mean and the correlations. Here we will show how the results of [2] can be used to derive in a systematic manner the various averaged equations. For this purpose we recall the result from [2]: If the basic stochastic differential equation is written in the form (z now playing the role of time)

$$\frac{\partial \psi}{\partial z} = [L_o + L_1 (z, \cdot\cdot)] \psi \tag{18}$$

and if

$$L_1 = \sum F_\alpha (z, \cdot\cdot) L_{1\alpha} , \tag{19}$$

where F_α's are Gaussian delta correlated process with diffusion coefficient $2 D_{\alpha\beta}$, then the exact equation for the ensemble average of ψ is

$$\frac{\partial \langle\psi\rangle}{\partial z} = (L_o + \sum_{\alpha\beta} D_{\alpha\beta} L_{1\alpha} L_{1\beta}) \langle\psi\rangle . \tag{20}$$

we can now directly apply (20) to (16) by using identification $L_o = \frac{-1}{2ik} \nabla_t^2$, $L_1 = -k^2 \epsilon_1 (z, \vec{\rho})/2ik$ with the result

$$2ik \frac{\partial \langle u\rangle}{\partial z} + \nabla_t^2 \langle u \rangle + \frac{ik^3}{4} \Gamma(o) \langle u \rangle = 0, \tag{21}$$

which has the solution

$$\langle u(\vec{\rho}, z)\rangle = u_o (\vec{\rho}, z) \exp(-\alpha z), \quad \alpha = \frac{k^2 \Gamma(o)}{2} , \tag{22}$$

where u_o is the field in free space and which for a Gaussian beam has the structure

$$u_o = \frac{1}{(1+i\beta z)} \exp\left\{-\frac{k\beta}{2} \frac{\rho^2}{(1+i\beta z)}\right\}. \tag{23}$$

Thus α can be identified with the absorption coefficient and is related to the spectral density g(k) of fluctuations for $\Gamma(\rho)$ can be written as

$$\Gamma(\rho) = (2\pi)^2 \int_o^\infty kdk\ g(k)\ J_o (k\rho), \tag{24}$$

where we can, for example, choose for g(k), the Kolmogorov form (14).

We next show how the equations for the mutual coherence function $K(\vec{\rho}_1, \vec{\rho}_2, z)$

$$K (\vec{\rho}_1, \vec{\rho}_2, z) = \langle u(\vec{\rho}_1, z)\ u^* (\vec{\rho}_2, z)\rangle \tag{25}$$

can be obtained. Using the parabolic equation (16), we find the equation for $u(\vec{\rho}_1, z)\ u^*(\vec{\rho}_2, z) \equiv u_1 u_2^*$

$$2ik \frac{\partial}{\partial z}(u_1 u_2^*) + (\nabla_{t1}^2 - \nabla_{t2}^2) u_1 u_2^* + k^2 \left\{\epsilon_1(z, \vec{\rho}_1) - \epsilon_1^*(z, \vec{\rho}_2)\right\} u_1 u_2^* = 0. \tag{26}$$

The application of (20) to (26) is straight forward if we identify $\psi = u_1 u_2^*$, $L_o = (-\nabla_{t1}^2 + \nabla_{t2}^2)/2ik$, $L_1 = -k^2 \{\epsilon_1(z, \vec{\rho}_1) - \epsilon_1^*(z, \vec{\rho}_2)\}/2ik$ with the result

$$2ik \frac{\partial K}{\partial z} + (\nabla_{t1}^2 - \nabla_{t2}^2) K + \frac{ik^3}{2}\left[\Gamma(o) - \Gamma(\vec{\rho}_1 - \vec{\rho}_2)\right] K = 0. \tag{27}$$

Note that the mutual coherence function gives information regarding the correlations in a given plane and hence (27) essentially shows how the correlations change in propagation from one plane to another plane. Thus (27) will yield the generalization of the Van Zittert-Zernike theorem [5] for propagation in a random medium. Exact solution to (27) is known from the work of Tatarskii. Introducing the sum and difference coordinates $\vec{\rho} = \vec{\rho}_1 - \vec{\rho}_2$, $\vec{R} = (\vec{\rho}_1 + \vec{\rho}_1)/2$ (27) becomes

$$2ik \frac{\partial K}{\partial z} + 2 \vec{\nabla}_\rho \cdot \vec{\nabla}_R K + \frac{ik^3}{2} (\Gamma(o) - \Gamma(\vec{\rho})) K = 0. \tag{28}$$

Equation (28) can be solved by first taking the Fourier transform with respect to the variable $\vec{R}$. This results in a first order equation in $\vec{\rho}$ and z and hence can be solved by the method of characteristics. The result being

$$K(\vec{R}, \vec{\rho}, z) = \int d^2K \, m \, (\vec{K}, \vec{\rho} - \frac{\vec{\rho} z}{k}, 0) \exp\left\{ i\vec{K}.\vec{R} - H(\vec{K}, \vec{\rho}, z)\right\},$$

$$H(\vec{K}, \vec{\rho}, z) = \frac{k^2}{4} \int_0 [\Gamma(o) - \Gamma(\vec{\rho} - \frac{\vec{K}}{k} z')] dz' \tag{29}$$

where m is the Fourier transform of K in the plane z = 0

$$m(\vec{k}, \vec{\rho}, o) = \frac{1}{(2\pi)^2} \int K(\vec{R}, \vec{\rho}, 0) \, e^{-i\vec{k}.\vec{R}} d^2R. \tag{30}$$

For the Gaussian beam case the function m has a simple form, however the function H and hence K has to be evaluated numerically for the case of Kolmogorov spectrum.

The closed form of equations for the higher oreder moments can also be obtained using (20). However such equations acquire increasingly complex structure, for example the intensity correlation

$$K_2 = \langle I(\vec{\rho}_1, z) I(\vec{\rho}_2, z)\rangle, \quad I = uu^* \tag{31}$$

obeys the equation

$$2ik \frac{\partial K_2}{\partial z} + (\nabla_{\rho_1}^2 + \nabla_{\rho_2}^2 - \nabla_{\rho_1'}^2 - \nabla_{\rho_2'}^2) K_2$$

$$+ \frac{ik^3}{2}\left[2\Gamma(o) + \Gamma(\vec{\rho}_2 - \vec{\rho}_1) + \Gamma(\vec{\rho}_2 - \vec{\rho}_1') - \Gamma(\vec{\rho}_1 - \vec{\rho}_1')\right.$$

$$\left. - \Gamma(\vec{\rho}_1 - \vec{\rho}_2') - \Gamma(\vec{\rho}_2 - \vec{\rho}_1') - \Gamma(\vec{\rho}_2 - \vec{\rho}_2')\right] K_2 = 0 \tag{32}$$

No exact solutions to (32) appear to be known; approximate solutions can be found for example in [6].

Finally we would like to add that if ϵ_1 were to correspond to a random telegraphic signal, then certain exact results can be obtained following the general formulation given in [2]. Similarly if the fluctuations of ϵ_1 had a finite correlation length, in the direction of propagation, then we have to use more general formula than (20) such as those given in [2] .

References

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