STOCHASTIC DIFFERENTIAL EQUATIONS

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Stochastic processes form a basic part of many types of modelling of phenomena in physical and mathematical sciences as well as in engineering. The random elements enter into the dynamics of the systems in very many different ways. Usually the time evolution of the system is governed by a differential equation wherein the parameters like the coefficients of the differential equation may be random. The driving forces may be fluctuating. The initial conditions may be random. Thus we are faced with the problem of integrating a stochastic differential equation which implies the knowledge of the probability density function of the solution. Alternatively we may be satisfied with some of the moments of the solution. The well known equation of Brownian motion with additive noise terms is a famous example (cf. Ramakrishnan [1]).

The breakdown of the exact parallelism between integrals of random and of deterministic functions is the key problem. The physical scientist based most of the conclusions on the Fokker-Planck equation governing the random processes and their integrals. Doob and Itô developed the properties of integrals of certain special random functions [2]. Meyer developed a comprehensive set of theorems on Martingales and stochastic integrals. Usual Riemann-sums of random functions lead to loss of uniqueness. Bartlett's book deals with stochastic integrals and stochastic equations from the view point of mean square convergence. The estimates of the probabilities of sample paths in the context of stochastic integrals were carried out in the work of Ramakrishnan et al [1]. Srinivasan and Vasudevan detailed an account of these in their book [3]. It was the work of Mcshane that provided a basis for a rigorous and unambiguous method of arriving at stochastic integrals. The work of Stratanovich in his book and also the work of Skorohod and Gikman are recommended for a deep study. Weiner's trajectories and the Feynman-Kac path integrals as used in nonrelativistic quantum mechanics are of great interest and specify methods widely in use in integrations of random equations. The list of references relating to these topics are given at the end [1-18].

Stochastic calculus

To discuss stochastic integrals and differentiation the first step is the development of a calculus to define convergence of a sequence of random variables. There are three types of convergence. Usually the mean square convergence is taken to be fundamental apart from convergence in probability and almost sure convergence [19]. With this notion mean square continuity follows.

Definition: A second order stochastic process $x(t)$, $t \in T$, has mean square derivative $\dot{x}(t)$ at t if

$$\lim_{\tau \to 0} \left[\frac{x(t+\tau) - x(t)}{\tau} \right] \longrightarrow \dot{x}(t) \tag{1}$$

exists in the mean square sense.

Higher order m.s. derivatives are defined analogously.

Differentiation in mean square criterion

A second order stochastic process $x(t)$, $t \in T$, is differentiable at t if and only if the second generalized derivative

$$\lim_{\tau,\tau' \to 0} \frac{1}{\tau\tau'} \left[\Gamma(t+\tau, s+\tau') - \Gamma(t+\tau, s) - \Gamma(t, s+\tau') + \Gamma(t,s) \right] \tag{2}$$

exists at (t,s) and is finite, where

$$\Gamma(t,s) \quad = \quad E\left[x(t)x(s)\right] \tag{3}$$

We remark that the concept of mean square differentiation leads to the notion of mean square Taylor expansion of a stochastic process and the definition of mean square analyticity.

Mean square Riemann integral

Let $x(t)$ be a second order stochastic process defined on $[a,b]$. Let $f(t,u)$ be an ordinary function defined on the same interval for t and Reimann integrable for every $u \in U$. We form the random variable

$$Y_n(u) \quad = \quad \sum_{k'=1}^{n} f(t_{k'},u)\, x(t_{k'})\,(t_{k'}-t_{k'-1}) \tag{4}$$

If for $u \in U$

$$\lim_{n \to \infty, \Delta t \to 0} Y_n(u) \quad = \quad Y(u) \tag{5}$$

exist for some sequence of subdivisions ρ_n the stochastic process $Y_n(u)$, $u \in U$, is called the mean square Riemann integral of $f(t,u)x(t)$ over the interval $[a,b]$ and it is denoted by

$$Y(u) \quad = \quad \int_a^b f(t,u)x(t)\,dt \tag{6}$$

Integration in mean square criterion

The stochastic process $Y(u)$, $u \in U$, defined by (6) exists if and only if the ordinary double Riemann integral

$$\int_a^b\!\!\int_a^b f(s,u)\,\Gamma_{xx}(t,s)\;dt\;ds \tag{7}$$

exists and is finite.

Similarly Riemann-Stieltjes integral can be defined and criteria for mean square Riemann-Stieltjes integrals to exist can be analysed.

We will see in the sequal that the definition (4) and the definition of Riemann-Stieltjes integrals will take different forms according to different prescriptions warranted by different types of modelling.

A special class of differential equations which has found important applications in control, filtering and communications theory is one where the vector stochastic process $\underline{Y}(t)$ of equation

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{Y}(t), t), \; t \in T, \; \underline{x}(t_o) = \underline{x}_o \tag{8}$$

has only white noise components. More specifically we mean equations of the form

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), t) + \underline{G}(\underline{x}(t), t) \, \underline{W}(t), \; t \in T, \; \underline{x}(t_o) = \underline{x}_o \tag{9}$$

where $W(t)$ is an m-dimensional stochastic process whose components are Gaussian white noise. $\underline{G}(\underline{x}(t),t)$ is an n x m matrix function and is independent of $\underline{W}(t), t \in T$. The popularity of this model in control and filtering applications is due to two principal reasons. The first is the mathematical simplicity - it is a natural stochastic extension of the powerful state-space approach in classical optimal theory [20-25]. Moreover as we shall see, the solution process generated by (9) is Markovian for which powerful techniques for obtaining its solution exist. The second reason is that, although white noise is a mathematical artifice it approximates closely the behaviour of a number of important noise processes in electrical and electronic systems.

Let $B(t)$, $t \geq 0$, be the Brownian motion or the Weiner process. It is Gaussian with mean 0 and covariance

$$\mu_B(t_1, t_2) = 2 \, D \min(t_1, t_2) \tag{10}$$

The formal derivative of $B(t)$, $\dot{B}(t)$ is clearly Gaussian with mean zero and its covariance is given by

$$\mu_{\dot{B}}(t_1, t_2) = 2 \, D \, \delta(t_1 - t_2) \tag{11}$$

Thus we can formally write

$$\frac{dB(t)}{dt} = W(t), \; t > 0 \tag{12}$$

With this formal representation for the white Gaussian noise $W(t)$ we can interpret (9) as formally equivalent to

$$\underline{x}(t) - \underline{x}(t_o) = \int_{t_o}^{t} f(\underline{x}(s), s) \, ds + \int_{t_o}^{t} \underline{G}(\underline{x}(s), s) \, d\underline{B}(s), \; t \in T \tag{13}$$

where $\underline{x}(s)$ is independent of the increment $d\underline{B}(t)$, $t \in T$ (cf [16,20-24] for many applications of stochastic differential equations in various fields).

The Paradox

Here we shall explain a paradox as detailed by [8]. For equations of the type (13) there is the approach of Fokker-Planck partial differential equation.

However a conceptual difficulty arises in writing the Fokker-Planck equation, since the heuristic mathematical idealization of the fluctuating force F(t) as a white noise process can lead to difficulties. The result has been that something of a controversy has appeared in the recent literature concerning the two possible ways of extending ordinary calculus to stochastic functions: The socalled Stratonovich calculus, in which the usual rules continue to apply and the socalled Itô calculus in which the rules are modified [1-17]. As remarked by Mortinsen [8] white noise is much the same kind of mathematical pathology in the theory of random processes as that the Dirac delta function is in the theory of deterministic functions.

Taking the one-dimensional version of (9) one can write

$$x(t) = f(x(t),t) + g(t)\, v(t) \tag{14}$$

where v (t) is a Gaussian noise with

$$E[v(t)] = 0, \quad E[v(t)\, v(s)] = \delta(t-s)$$

or equivalently

$$dx(t) = f(x(t),t)\, dt + g(t)\, dB(t) \tag{15}$$

or

$$x(t) = x(o) + \int_0^t f(x(t),\tau)\, d\tau + \int_0^t g(\tau)\, dB(\tau) \tag{16}$$

Since g(t) is a nonrandom function at t, both Stratanovich and Itô interpretations of these integrals agree and we take the Riemann-Stieltjes sum

$$\sum g(t_i)\,[B(t_{i+1}) - B(t_i)] \tag{17}$$

where

$$0 = t_o < t_1 < \cdots < t_n = t$$

The probability density $p(x,t|x_o,t_o)$ of the solution Markov process x(t) is obtained by the forward Fokker-Planck equation [7,8,1,2,3].

$$\frac{\partial p(x,t|x_o,t_o)}{\partial t} = -\frac{\partial}{\partial x}[f(x,t)p(x,t|x_o,0)] + \frac{1}{2}g^2(t)\frac{\partial^2 p(x,t|x_o,o)}{\partial x^2}, \quad -\infty < x < \infty,\ t > 0 \tag{18}$$

with the boundary conditions

$$\lim_{t\to 0} P(x,t|x_o,0) = \delta(x-x_o), \quad \lim_{|x|\to\infty} p(x,t|x_o,0) = 0$$

Similarly the backward Fokker-Planck equation is

$$\frac{-\partial p(x,t|x_o,t_o)}{\partial t_o} = f(x_o,t_o)\frac{\partial P(x,t|x_o,t_o)}{\partial x_o} + \frac{1}{2}g^2(t)\frac{\partial^2 (x,t|x_o,t_o)}{\partial x_o^2} \tag{19}$$

If f = 0, g = 1 the solution x(t) is a Wiener process with probability density p (x,t) given by

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} \exp(-x^2/2t) \tag{20}$$

This p(x,t) obeys the Fokker-Planck equation

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(x,t)}{\partial x^2} \tag{21}$$

Let x(t) be passed through a nonlinear device to produce a new process

$$z(t) = \sinh [x(t)] \tag{22}$$

$$\pi(z,t) = p(x,t) \left.\frac{dx}{dz}\right|_{x = \sinh^{-1}(z)} \tag{23}$$

hence

$$\pi(z,t) = \frac{(1+z^2)^{\frac{1}{2}}}{\sqrt{2\pi t}} \exp[-(\sinh^{-1} z)^2/2t] \tag{24}$$

Now going back to the F.P. equation with the change of variable $x = \sinh^{-1}(z)$ one finds the F-P-equation satisfied by $\pi(z,t)$ as

$$\frac{\partial \pi(z,t)}{\partial t} = -\frac{\partial}{\partial z}\left[\frac{z}{2}\pi(z,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left[(1+z^2)\pi(z,t)\right] \tag{25}$$

and this equation is the forward equation corresponding to the stochastic differential equation

$$dz(t) = \frac{1}{2}z(t)dt + [1+z^2(t)]^{\frac{1}{2}} dB(t) \tag{26}$$

If we simply compute dz(t) from equation (22) we get

$$dz(t) = (1+z^2)^{\frac{1}{2}} dx(t) \tag{27}$$

which means

$$dz(t) = (1+z^2)^{\frac{1}{2}} dB(t) \tag{28}$$

from equations (13,21)

The stochastic differential equations (26) and (28) differ by $\frac{1}{2}$ z(t)dt. The question is which is the correct s.d.e. for generating the process z(t). Therefore the situation is that one unambiguous way to specify a Markov process mathematically is to specify its transition density or equivalently the F-P-equation obeyed by the transition density. The divergence boils down to two different ways of associating the coefficients in the F-P-equation with the coefficients in the s.d.e. Thus we can have two ways of integrating this stochastic equation.

Using the Itô rules with the equation (26) we will obtain the F-P-equation (25). Hence integrating (26) by Itô rules one should get

$$z(t) = \sinh [x(t)] \tag{29}$$

According to Stratanovich, the equation (28) is perfectly valid and hence integration should also yield

$$z(t) = \sinh[x(t)] \tag{30}$$

Hence starting with a F-P-equation in an unambiguous way there are two possibilities of modelling the process as the solution to a s.d.e. Thus it is open to us to choose the best type of dynamical equation. However the engineer does not start with the transition density, but starts with a differential equation which he has obtained on the basis of known physical laws and then adds a white noise forcing term to get a stochastic model. If the coefficient of the noise itself is random, then there are two possible ways of interpreting the equation leading to two types of differential F-P-equations and hence two different processes. The question is which process does one really get in the physical world? Which kind of calculus one should prefer? In the following sections we will examine the situation in more detail

The Itô - calculus

If we take the Wiener process B(t) and calculate the increment in time

$$\Delta B(t) = B(t+\Delta t) - B(t) \tag{31}$$

it can be shown that

$$P_{\Delta B} d(\Delta B) = \frac{1}{\sqrt{(2\pi\Delta t)}} \exp\left(-\frac{(\Delta B)^2}{2\Delta t}\right) \tag{32}$$

The important point is that the increment ΔB is independent of B(t), the state of the process at time t. Thus B(t), the Wiener process, belongs to a special class of processes with independent increments. We know also

$$\begin{aligned} E[(\Delta B)^k] &= 0 \ ; \quad k \text{ odd} \\ &= 1.3.5 \ldots.(k-1)(\Delta t)^{k/2}; \ k \text{ even} \end{aligned} \tag{33}$$

Thus if we take a function $F[B(t+\Delta t)]$ and expand it by Taylor series and calculate the differential we can easily see that

$$E\left\{\frac{d}{dt} F[B(t)]\right\} = \lim_{\Delta t \to 0} \frac{E\{F''[B(t)]\}\Delta t + 0(\Delta t)}{\Delta t}$$

$$= \frac{1}{2} E\left\{F''[B(t)]\right\} \tag{34}$$

Since $E[(\Delta B)^2]$ is of order Δt and not $(\Delta t)^2$ using the chain rule of ordinary calculus, one will write

$$dF[B(t)] = F'(B)dB(t) \tag{35}$$

hence

$$E\{dF[B(t)]\} = E[F'(B)] \ E[dB(t)] = 0 \tag{36}$$

Clearly equation (36) is different from equation (34).

Itô showed how the rules of calculus have to modified to handle the above phenomenon. As given by Skorohod and others [cf.12]

$$d_I F[B(t)] = F'[B(t)]\, d\, B(t) + \frac{1}{2} F''[B(t)]\, dt \tag{37}$$

which is the Itô rule for differentiation in the present case. However we obtain

$$d_I \sinh B(t) = \cosh B(t) d\, B(t) + \frac{1}{2} \sinh B(t)\, dt$$
$$\text{if } F[B(t)] = \sinh[B(t)] \tag{38}$$

Hence if

$$z = \sinh[B(t)]$$
$$d_I z(t) = \frac{1}{2} z(t)\, dt + (1+z^2)^{\frac{1}{2}} d\, B(t) \tag{39}$$

which is consistent with the F-P-equation (25).

If we wish to preserve the fundamental property of calculus then we require to have

$$I\int_{t_o}^{t} d_I F[B(t)] = F[B(t)] - F[B(t_o)] \tag{40}$$

Hence we are forced to assume

$$I\int_{t_o}^{t_1} F'[B(t)]\, dB(t) = F[B(t_1)] - F[B(t_o)] - \frac{1}{2}\int_{t_o}^{t_1} F''[B(t)]\, dt \tag{41}$$

Thus

$$I\int g[B(t)]\, dB(t) = \int_{B(t_o)}^{B(t_1)} g(\xi) d\xi - \frac{1}{2}\int_{t_o}^{t_1} g'[B(t)]\, dt \tag{42}$$

Stratanovich calculus treats $\int g(\xi) d\xi$ as an ordinary integral, treating ξ as a deterministic dummy variable of integration. Hence the relation between the Itô and the Stratanovich rules of integration is given by

$$I\int_{t_o}^{t} g[B(t)] d\, B(t) = S\int_{t_o}^{t} g(B) d\, B(t) - \frac{1}{2}\int_{t_o}^{t} g'[B(t)]\, dt \tag{43}$$

where $g'[B(t)] = \left.\frac{dg(\xi)}{d\xi}\right|_{\xi=B(t)}$

Let us take $g[B(t)] = B(t)$. Then we get

$$I\int_{t_o}^{t} B(t) d\, B(t) = \frac{1}{2} B^2(t_1) - \frac{1}{2} B^2(t_o) - \frac{1}{2}(t_1-t_o) \tag{44}$$

The presence of the term $\frac{1}{2}(t_1-t_o)$ in the equation (44) can be made more plausible by the following considerations

$$E\left[I\int_{t_o}^{t} B(t) d\, B(t)\right] = \int_{t_o}^{t} E[B(t)]\ E[dB(t)] \tag{45}$$

and so

$$E\left[I\int_{t_o}^{t_1} BdB \right] = 0$$

According to (42)

$$E\left[I\int_{t_o}^{t_1} B(t)d\,B(t)\right] = \frac{1}{2}(t_1-t_o) - \frac{1}{2}(t_1-t_o) = 0 \tag{46}$$

However

$$E\left[S\int_{t_o}^{t_1} B(t)dB(t)\right] = \frac{1}{2}(t_1-t_o) \neq 0 \tag{47}$$

Thus for Stratanovich integral it cannot be true that dB(t) is independent of B(t).

The Stratanovich increment is given by

$$\Delta_s B(t) = B\left[t+\frac{\Delta t}{2}\right] - B\left[t-\frac{\Delta t}{2}\right] \tag{48}$$

This increment still has mean zero and variance Δt, but it is not independent of B(t).

For a more rigorous approach to the Itô calculus as the limit of Riemann sums, we define the integral

$$J = I\int_0^T z(t)dB(t)$$

By using n partitions, define

$$I_n = \sum z(t_{k-1})\left[B(t_k) - B(t_{k-1})\right] \tag{49}$$

which in the limit converges to a limiting random variable J. It is to be noted that $z(t_{k-1})$ is always taken at the beginning of the interval over which the increment $[B(t_k) - B(t_{k-1})]$ is taken. Hence $z(t_{k-1})$ and $\Delta B(t_k)$ are always independent and consequently we have

$$B\left[I_n\right] = \sum_{k=1}^{n} E\left[z(t_{k-1})\right] E\left[B(t_k) - B(t_{k-1})\right] = 0 \tag{50}$$

It can be proved from first principles that the Itô integral is the value given in (44) in mean square sense (cf.[6])

<u>The Stratanovich integral</u>

In this case we take the sequence of sums of the form representing the integral as

$$I_n = -\sum_{k=1}^{n} z\left(\frac{t_{k-1}+t_k}{2}\right)\left[B(t_{k-1}) - B(t_k)\right] \tag{51}$$

and let us take the generalized form of the s.d.e.

$$dx(t) = f(x(t),t) + g(x,t)dB \tag{52}$$

Explicitly we have

$$x_I(t) = x_I(0) + \int_0^t f(x,\tau)\,d\tau + I\int_0^t g(x,\tau)\,dB(\tau) \tag{53}$$

where symbol I denotes the Itô belated integrals. If the symbol S denotes the Stratanovich integration (52) yields

$$x_s(t) = x_s(o) + \int_o^t f(x,\tau)\,d\tau + S\int_o^t g(x_s(\tau),\tau)\,dB(\tau) \tag{54}$$

Because of the relationship explained earlier between the Ito and the Stratanovich integrals we have the equation

$$x_I(t) = x_I(o) + \int_o^t \left[f(x_I,\tau) - \frac{1}{2}g(x_I(\tau),\tau)\, g'(x_I(\tau),\tau)\right] d\tau$$
$$+ S\int g(x_I(\tau),\tau)\,dB(\tau) \tag{55}$$

where

$$g'(x,t) = \frac{\partial g(x,t)}{\partial x}$$

and also we have

$$x_s(t) = x_s(o) + \int_o^t \left[f(x_s,\tau) + \frac{1}{2}g(x_s,\tau)\, g'(x_s,\tau)\right] d\tau + I\int_o^t g(x_s,\tau)\,dB(\tau) \tag{56}$$

For the Itô equation (53) the F-P-equation is given by

$$\frac{\partial p_I(x,t|x_o,o)}{\partial t} = \frac{\partial}{\partial x}\left[f(x,t)p_I(x,t|x_o,o)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[g^2(x,t)p_I(x,t|x_o,o)\right] \tag{57}$$

For the Stratanovich equation the F-P-equation is given by

$$\frac{\partial p_s(x,t|x_o,o)}{\partial t} = -\frac{\partial}{\partial x}\left[f(x,t)p_s(x,t|x_o,o)\right] + \frac{1}{2}\frac{\partial}{\partial x} g(x,t)\frac{\partial}{\partial x}\left[g(x,t)p_s(x,t|x_o,o)\right] \tag{58}$$

Using the relation between the Stratanovich and the Itô integrals we find that the solutionsof equations (26) and (27) are the same, given by

$$z(t) = \sinh\left[B(t)\right] \tag{59}$$

Also we find that the F-P-equation corresponding to (28) is the same as the F-P-equation corresponding to (26) which is (25) obtained by using the usual Itô calculus.

<u>Additive and multiplicative stochastic processes</u>

In describing the physics of macroscopic systems by the deterministic time evolution of the collective variables we can find globally stable states of the system in which case the statistical nature and the dynamics can be safely neglected. However by changing some parameters large excursions from the deterministically describable values occur. We find that the fluctuations in a system play an important role in their understanding. The equilibrium phase transition is one of such class like liquid-vapour transitions etc. Another class of phenomena consists of phase transition analogous that have been found in nonequilibrium systems such as laser and nonlinear optics, hydrodynamic instabilities etc. In a variety of problems the fluctuations can be taken into account by adding a fluctuating force to the deterministic equations. The most familiar examples of these types of additive fluctuations are the vacuum fluctuations of the electromagnetic field that triggers the spontaneous emission by atoms and in the macroscopic case that of the Brownian motion.

The most important characteristic of the additive fluctuations is that they do not depend on the present state of the collective variables of the system.

In an autocatalytic chemical reaction the production of a molecule of some type is enhanced by the presence of other molecules of the same type that have been produced already. Therefore the only possible reaction channel is the autocatalytic reproduction of the molecules according to the blue print provided by the molecules of the same kind already present. As a result the fluctuations in this case do depend on the state of the system. If this dependence can be described by a function of the macroscopic variables multiplying the 'fluctuating stochastic forces' we call such process a 'multiplicative stochastic process' [18,30].

The Langevin equation for these two types of the system can be generally described by

$$\dot{x}_i(t) = L_i[\{x_j\}] + F_i \tag{60}$$

for the additive system and

$$\dot{x}_i(t) = L_i[\{x_j\}] + G_{ij}[\{x_k\}]\, F_j \tag{61}$$

for the multiplicative system with properties

$$\langle F^i_j(t+\tau)\, F^l_k(t)\rangle = Q^{(i,l)}_{jk}\,\delta(\tau) \tag{62}$$

where Q is a measure of fluctuations independent of the random variables $\{x_i\}$. $L[\{x_i\}]$ may be a linear or nonlinear function.

In the first case and the case in which $G_{ij}[\{x\}]$ is independent of x_k's the Stratanovich and the Itô calculus give the same F-P-equation. They become different if $G_{ij}[\{x_k\}]$ depend on x_k's. The F-P-equation for the multiplicative noise problem relating to the Langevin equation (61) is

$$\frac{\partial \pi(\{x_k\},t)}{\partial t} = -\frac{\partial}{\partial x_i}[\, k^{(1)}_i(\{x_k\})\pi\,] + \frac{1}{2}\,\frac{\partial^2}{\partial x_i \partial x_j}[k^{(2)}_{ij}\;(\,x_k\,)\;\pi\,] \tag{63}$$

with

$$k^{(1)}_i(\{x_k\}) = L_i[\{x_k\}] + \frac{1}{2}\frac{\partial G_{ij}}{\partial x_l}\; G_{lj}$$

$$k^{(2)}_{ij}(\{x_k\}) = G_{il}[\{x_k\}]\,G_{jl}[\{x_k\}]$$

assuming the noise has unit variance. This is the Stratanovich description and takes into account the correlations engendered by the multiplicative processes to a great extent [30].

Canonical extension

Consider the differential equation

$$dx(t) = f[x(t)]\,dt + g[x(t)]\,dB(t) \tag{64}$$

McShane [16] has developed the general calculus which includes both the Itô and the Stratanovich cases. Under this calculus the integral equation corresponding to (64) is replaced by a canonical equation which is valid for more general continuous noise processes (including Brownian motion). He has extended this when B(t) is taken to be a point process and hence one can consider the stochastic differential equations driven by jump processes also by a theory analogous to that of Itô differential equations representing Brownian motion.

The differential equation may be of the form

$$dx(t) \quad = \quad f(x,t)dt + \sum_{n=1}^{N(t)} h[x(\tau_n),U_n] \tag{65}$$

where N(t) is the number of incident points during [o,t] regardless of their mark and τ_n and U_n are the times of occurence and the random marks respectively of the nth point (it will be assumed that a counting process N(t) is almost surely left continuous). This can also be written as

$$x(t) \quad = \quad x(o) + \int_o^t f(x(\tau),\tau)d\tau \quad + \int_o^t h[x(\tau),u(\tau)] \; dN(\tau) \tag{66}$$

To arrive at a general canonical extension which can be applied to class of noise processes and also for the Brownian motion carried out by McShane, McKean and others cf. [16,17,25].

Now let us consider the differential equation

$$\dot{x} \; = \; g(x(t)), \qquad x(o) \; = \; x_o \tag{67}$$

The Lie series solution is given by

$$x(t) \quad = \quad x_o + \sum_{m=1}^{\infty} \frac{t^m}{m!} D^{m-1} [g(x_o)] \tag{68}$$

with

$$D \quad = \quad \sum_{l=1}^{n} g^l(x) \frac{\partial}{\partial x_l} \tag{69}$$

where l denotes the number of components of the process.

In all these f and g are analytic and the solutions are assumed to exist. The integral form of the stochastic differential equation may be obtained for one component system as

$$x(t) \quad = \quad x_o + \int f[x(\tau)]d\tau \; + \; \sum_{m=1}^{\infty} \frac{1}{m!} \int D^{m-1}[g(x(\tau)][dz(t)]^m \tag{70}$$

where the last integral is the Itô belated integral of McShane. If dz satisfies the conditions of the Brownian motion kicks, all the belated integrals in (70) with $m > 2$ vanish. If the s.d.e. is driven by point processes the integral form of (70) will be

$$x(t) = x_o + \int_o^t f[x(\tau)]\, d\tau + \int_o^t [e^{\alpha D} x(\tau) - x(\tau)]\, dN(\tau) \quad (71)$$

if α is the constant size of the jump (cf. [25])

Stationary solutions of Fokker-Planck- equations

Let us take the one-dimensional Fokker-Planck equation for probability density or the transition probability given by

$$\frac{\partial \pi(x,t)}{\partial t} = -\frac{\partial}{\partial x}[k(x)\pi - \frac{1}{2} D \frac{\partial \pi}{\partial x}] = -\frac{\partial}{\partial x} J(x) \quad (72)$$

if we write the current density $J(x,t)$ as

$$J(x,t) = k(x)\pi - \frac{1}{2} D \frac{\partial \pi}{\partial x}$$

This is the result of the Langevin equation with an additive suitable white noise with a deterministic force term given by $k(x)$

$$\dot{x} = k(x) + F(t) \quad (73)$$

For the n-dimensional case the Fokker-Planck equation is

$$\frac{\partial \pi}{\partial t} + \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(K_k \pi - \frac{1}{2} \sum_{l=1}^{n} D_{kl} \frac{\partial \pi}{\partial x_l}) = 0 \quad (74)$$

with the probability current given by $\vec{J} = (J_1,\ldots\ldots,J_n)$ and

$$\frac{\partial \pi}{\partial t} + \nabla.\vec{J} = 0 \quad (75)$$

This is the continuity equation for (π, J). In the stationary case

$\frac{\partial \pi}{\partial t} = 0$ and hence we have for a one dimensional process

$$\frac{1}{2} D \frac{\partial \pi}{\partial x} = K \pi \quad (76)$$

We see easily that the solution is given by stationary π^o

$$\pi^o(x) = N \exp\left(-\frac{2V(x)}{D}\right) \quad (77)$$

with

$$V(x) = -\int_{x_o}^{x} k(x')\,dx', \qquad \int \pi(x)\,dx = 1. \quad (78)$$

The normalization condition determines theconstant N. The boundary condition is $\pi(x) = 0$ as $x \to \pm\infty$. We have $V(x) = \alpha x^2/2$ if $K(x) = -\alpha x$ and $\dot{x} = -\alpha x + F(t)$. The fluctuating force pushes the particle up a slope of the potential well, but it settles down at $x = x_o = 0$ so that $\langle x \rangle = 0$ at $t = \infty$.

If $k(x) = -\alpha x - \beta x^3$

$$V(x) = \frac{\alpha x^2}{2} + \frac{\beta}{4} x^4 \tag{79}$$

This corresponds to the Langevin equation

$$\dot{x} = -\alpha x - \beta x^3 + F(t) \tag{80}$$

When $\alpha > 0$ the minimum of V(x) is at x = 0 and the stationary solution is

$$\pi(x) = N \exp\left[-\frac{2}{D}\left(\frac{\alpha x^2}{2} + \frac{\beta x^4}{4}\right)\right] \tag{81}$$

This is stable at x = 0. However for $\alpha < 0$ and $\beta > 0$ we find the following solution:

x = 0 is unstable and $x_{1,2} = \pm \left(\frac{|\alpha|}{\beta}\right)^{\frac{1}{2}}$ are stable. For the potential $V(x) = \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4$ as α goes from +ve values to -ve values the stability changes and for α negative the potential has two stable points. As α goes through 0, V(x) becomes flat near x = 0 and hence the restoring force becomes smaller and this is the socalled critical slowing down. Also we see from the equation

$$\dot{x} = -\alpha x - \beta x^3$$

that if we change $x \longrightarrow -x$ the equation remains unchanged. If α is -ve, when $x \longrightarrow -x$, $\dot{x} = \alpha x - \beta x^3$. This symmetry breaking occurs by deformation of the potential curve.

Phase transition analogy

We now consider a phase transition analogy of paramagnet becoming a ferromagnet. The ferromagnet has spontaneous magnetisation. The average magnetization $M = (N\uparrow - N\downarrow)m$ can be equated to a quantity q called an order parameter of the system. The free energy F(q,T) as a function of q and T is assumed as

$$F(q,T) = F(0,T) + qF' + \frac{q^2}{2!}F'' + \frac{q^3}{3!}f''' + \ldots.$$

with $F' = F''' = 0$ due to symmetry. Hence we take

$$F = F(o) + \frac{\alpha}{2}q^2 + \frac{\beta}{4}q^4 \tag{82}$$

The probability distribution for this thermodynamic state is

$$\pi = N \exp(-F/K_B T) \tag{83}$$

Let $\alpha = a(T-T_c)$. F_{min} occurs at $q_o = 0$ for α being positive and π is maximum. For $T < T_c$, α is negative and $q_o = \pm \left(\frac{|\alpha|}{\beta}\right)^{\frac{1}{2}}$. Entropy is continuous and specific heat is discontinuous. The equation governing q is

$$\dot{q} = -\frac{\partial F}{\partial q}$$

Hence for a model system we have

$$\dot{q} = -\alpha q - \beta q^3 + \text{fluctuating force} \tag{84}$$

The critical slowing down associated with the soft mode happens as $\alpha \rightarrow 0$. Fluctuations of q become considerable.

For the equilibrium distribution mean square fluctuation is

$$D < q^2 > = \frac{\int \frac{\partial \bar{V}}{\partial \alpha} \exp(-\bar{V})\, dq}{\int \exp(-\bar{V})\, dq}, \quad \bar{V} = \frac{2V}{D} \tag{85}$$

We expand all the quantities around q_o far from the transition point

$$D < q^2 > = -\frac{\partial}{\partial \alpha} \log\left\{ \exp(-\bar{V}(q_o)) \int \exp(-\bar{V}''(q_o)(q-q_o)^2)\, dq \right\}$$

$$= \frac{\partial \bar{V}(q_o)}{\partial \alpha} - \frac{1}{2} \frac{\partial}{\partial \alpha} \log\left(\frac{\pi}{\bar{V}''(q_o)}\right) \tag{86}$$

The second term stems from fluctuations which are large near $\alpha = \alpha_c$. The Ginsberg-Landau functional for the free energy with space-dependent q as q(x) can be written down and values of critical exponents can be arrived at (cf. Kadanoff et al Rev. Mod. Phys. 39, 395 (1967)).

Solution of F-P-equations by the path integral method

Taking the one-dimensional Langevin equation $\dot{x} = K(x) + F(t)$ we see that the Chapman-Kolmogorov equation for $\pi(x,t)$ given by

$$\pi(x, t+\Delta) = N \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2D\Delta}(x-x'-K(x')\Delta)^2\right\} \pi(x',t)\, dx' \tag{87}$$

leads to the F-P-equation for π,

$$\frac{\partial \pi}{\partial t} = \frac{\partial \pi K(x)}{\partial x} + \frac{D}{2} \frac{\partial^2 \pi}{\partial x^2} \tag{88}$$

From the Langevin equation

$$\dot{x} = K(x) + F(t)$$

we see that the jump in x process for each Δ is given by

$$(x-x'-K\Delta) = \Delta F(t) = z.$$

The amount of fluctuation jump should be z, to get x from x' in time Δ. This z is distributed as $\sim N \exp(-\frac{1}{2D\Delta}) z^2$ since it is stationary Gaussian white noise. Hence

$$\pi(x, t+\Delta) = N \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2D\Delta}\left(\xi^2 + 2\Delta\xi K(x+\xi) + \Delta^2 K(x+\xi)^2\right)\right] \pi(x+\xi, t)\, d\xi \tag{89}$$

where we take $x' = x + \xi$. Expanding K and π around x we have

$$\pi = N \int_{-\infty}^{\infty} d\xi \exp\left(-\frac{\xi^2}{2D\Delta}\right)\left[1 - \frac{K^2\Delta}{2D} - \frac{\xi^2 K'}{D} - \frac{\xi K}{D}\right]\left[\pi(x,t) + \xi\pi'(x) + \xi^2 \pi''/2 + \ldots\right] \tag{90}$$

Terms odd in ξ are dropped and keeping terms of order Δ only and choosing $N = (1/\sqrt{2D\Delta\pi})$ we obtain the Fokker-Planck equation. We may repeat the process from t_o to t by going from $t \rightarrow t+\Delta$, $t+\Delta \rightarrow t+2\Delta$, ...etc. Hence

$$\pi(x,t) \sim \iint D(x)\ \exp(-\tfrac{1}{2}\, 0\ \pi(x',o)) \tag{91}$$

with $\Delta \rightarrow 0$ and n the number of steps tending to ∞ such that $n\Delta = t$. The symbols D and 0 in (91) represent

$$D(x) = (2D\Delta\pi)^{-N/2} dx_o dx_1 \ldots dx_{n-1}$$

$$0 = \prod_i \exp\left\{-\frac{1}{2D\Delta}(x_{i+1}-x_i-\Delta K(x_i))^2\right\},\ i = 0,1,2\ldots,n-1 \tag{92}$$

This is the path integral formalism for the time dependent solution of the F-P-equation for given s.d.e. (cf. [28])

Coloured noise induced transitions

In the case of coloured noise we want to demonstrate how the transition density or F-P-equation can be obtained by a method different from the usual techniques adopted by Kitahara et al [27]. Our work (Vasudevan and Parthasarathy [2b]) is based on the method of van Kampen [18]. When the dichotomous noise in the equation goes over to white noise limit it turns out that we obtain the Stratanovich equation rather than the Itô equation. Most models are idealisations to a white noise and not a flat spectral power system. Hence the Stratanovich method leads to more physical results as West et al [28] have shown in many cases. Coloured noise has been used by Kabashima [29] for electrical systems.

Let us take a general nonlinear system of the type $\dot{x} = f(x) + g(x)\ I(t)$ where I(t) is a dichotomous Markov process which takes values $\pm$ d with probabilities P_+ and P_- described by the master equation

$$\frac{d}{dt}\begin{bmatrix} P_+ \\ P_- \end{bmatrix} = \frac{\gamma}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} P_+ \\ P_- \end{bmatrix} \tag{93}$$

$\frac{\gamma}{2}$ is the probability per unit time that a transition occurs as a jump process. If $P_o = P_+ = P_- = \frac{1}{2}$ we obtain the solution at any time t

$$\vec{P}(t) = \left(\exp\left\{\tfrac{\gamma}{2}\, t\, [I-\sigma_x]\right\}\right)\vec{P}(0) \tag{94}$$

where σ_x is the Pauli matrix. The process I(t) has a correlation function

$$\langle I(t)I(t')\rangle = d^2 \exp(-\gamma|t-t'|) \tag{95}$$

The process I(t) is also called random telegraphic signal process.

Let us now adopt the technique of van Kampen [18] to this problem by going to the solution of the stochastic differential equation and phase space density of such solutions called $\rho(x,t) = \delta(x-x_s)$ where x_s is the solution of this equation. It can be shown that $\langle \rho(x,t)\rangle$ the average over the ensemble of solutions is the probability density of x(t) at any time t, given by

$$p(x,t) = \langle \rho(x,t)\rangle \tag{96}$$

Hence the divergence equation for the Liouville density is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\dot{x}\rho) = 0 \quad \text{with} \quad \rho(x,t=0) = \delta(x-x_o) \tag{97}$$

Hence the equation for ρ , using the equation of motion is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}[\, f(x) + Ig(x)]\; \rho(x,t) \tag{98}$$

and $\rho(x,t)$ is compressible. Following the method of van Kampen [18] we arrive at the F-P-equation as

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(fp) + \Delta^2 \frac{\partial}{\partial x}\int_{-\infty}^{t} d\tau \exp\left\{-[\gamma + \frac{\partial f}{\partial x}](t-\tau)\right\}\frac{\partial}{\partial x}\, gp(x,\tau) \tag{99}$$

Now assume that I is white noise. This can be obtained as the limit of the coloured noise if we take $\Delta^2 \to \infty$, $\gamma \to \infty$ such that Δ^2/γ = finite = $\sigma^2/2$, i.e. $\frac{\Delta^2}{\gamma}\, \gamma \exp(-\gamma|t-t'|) = \infty$ if $t = t'$ and $= 0$ otherwise if $\gamma \to \infty$, $\Delta^2/\gamma \to$ finite. Hence in this limit the F-P-equation is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}[f + \tfrac{1}{2}\sigma^2 g(x)g'(x)]\; p + \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\, g^2 p \tag{100}$$

Going to the limit when $t \to \infty$ we obtain the stationary solution p_s given by

$$p_s = \left(\frac{Ng}{\Delta^2 g - f^2}\right) \exp\left\{\gamma \int dx'\; f(x')/[\Delta^2 g^2 - f^2(x')]\right\} \tag{101}$$

Hence the stable points are given by the solutions of the equation

$$f(x_m) - \left[\frac{\Delta^2}{\gamma}\right] g(x_m)g'(x_m) + \frac{2}{\gamma} f(x_m)f'(x_m) - \frac{1}{\gamma} f^2(x_m)(g'(x)/g(x_m)) = 0 \tag{102}$$

In the white noise limit we have the corresponding equation as

$$f(x_m) - \left[\frac{\Delta^2}{\gamma}\right] g(x_m)g'(x_m) = 0 \tag{103}$$

Hence coloured noise introduces two other terms in the equation for the extremal points due to the finite value of γ. Passing from coloured noise to the white noise limit we obtain the Stratanovich type of F-P-equation which is an interesting point to be noted.

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