INSTANTONS IN THE DYNAMICAL EVOLUTION OF FOKKER-PLANCK SYSTEMS

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In this lecture we shall outline the functional integral approach and the W.K.B. method to calculate the distribution P of a single stochastic variable the evolution of which is described by a Fokker-Planck equation. We shall follow closely a series of papers by B. Caroli, C. Caroli and B. Roulet on this subject[1,2]. The spirit of the classic article on instantons pervades [3] through the CCR works. Since we shall be using the concept of instantons we shall illustrate as to what an instanton is and then proceed to develop the subject.

Instantons:

Example I

Consider the following Lagrangian

$$L = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{1}{4}(x^2-1)^2 . \tag{1}$$

The corresponding Euclidean action is obtained by setting $t \longrightarrow -i\tau$

$$S_{Eucl} = \int_{-\infty}^{\infty} d\tau\left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + \frac{1}{4}(x^2-1)^2\right]. \tag{2}$$

The Euler-Lagrange equation is

$$\frac{d^2x}{d\tau^2} = (x^3-x) . \tag{3}$$

This equation has the solution

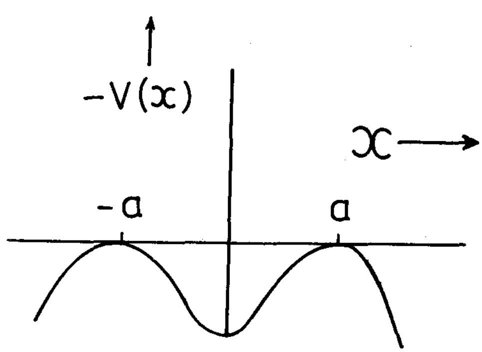
$$x(\tau) = \pm \tanh\left[\frac{\tau-\tau_0}{\sqrt{2}}\right]. \tag{4}$$

The solution with the plus sign is called the instanton solution and that with the minus sign is called the anti-instanton solution. An instanton is a soliton in the time variable.

Example II

Consider the Lagrangian

$$L(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + V(x) \tag{5}$$



where V(x) has the shape as shown in Figure 1. The Lagrangian (5) represents a particle of mass one unit moving in the potential -V as shown in the Figure 1. The classical equation of motion for the Lagrangian (5) is

$$\ddot{x}_{cl} = \left(\frac{dV}{dx}\right)_{x=x_{cl}} \quad ; \quad x_{cl}(t_i) = x_i \text{ and } x_{cl}(t) = x_1$$

At the top of the hill

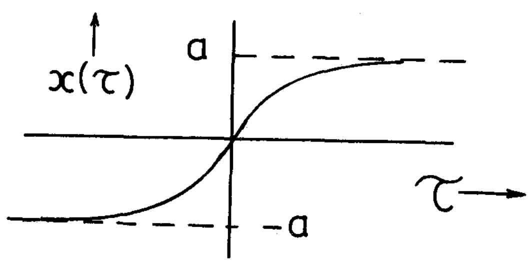
$$\frac{\dot{x}^2}{2} = V(x), \text{ (condition for zero energy)} \tag{6}$$

$$t = \int_0^x \frac{dx'}{\overline{2V(x')}} + \text{const} . \tag{7}$$

For large t the value of x approaches 'a' and therefore

$$\frac{dx}{dt} = \omega(a-x) \qquad \text{where} \qquad \omega = V''(a)$$

and $x_I(t)$ looks like as shown in Figure 2. This is the instanton solution. If instead the integration is done from +a to -a the solution $x_I(-t)$ is called the antiinstanton solution. The instanton solution looks as shown in Figure 2.



The action corresponding to this solution

$$S_o = \int dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V\right] = \int dt \left(\frac{dx}{dt}\right)^2 = \int_{-a}^{+a} dx\,\sqrt{2V} \tag{8}$$

Such solutions occur when one deals with the Fokker-Planck equation by functional methods.

Fokker-Planck Equation : A functional integral approach

Consider the probability distribution $P(x,t|x_i,t_i)$ of the stochastic variable x whose evolution is described by the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left[U'(x)P\right] + \theta\frac{\partial^2 P}{\partial x^2} \tag{9}$$

with the initial condition

$$P(x_1,t_i|x_i,t_i) = \delta(x_1-x_i) \qquad (10)$$

$U'(x) = \frac{dU}{dx}$ is a nonlinear function of x. If U is a bistable potential and $\theta \leqslant \Delta U$ where ΔU is the height of the barrier separating the wells of U then the solution of (9) satisfying the condition

$x(t_i) = x_i$ and $x(t) = x_1$ is

$$P(x_1,t|x_i,t) = \int_{x_i}^{x_1} Dx(\tau) \exp - \frac{1}{\theta} \int_{t_i}^{t} d\tau \; 0\,(\dot{x}(\tau),\, x(\tau)\,) \qquad (11)$$

where

$$0(\dot{x},x) = \frac{\dot{x}^2}{4} + V(x) + \frac{\dot{x}}{2} U'(x) \qquad (12)$$

and

$$V(x) = \frac{[U'(x)]^2}{4} - \frac{\theta}{2} U''(x) \qquad (13)$$

Noticing that $\dot{x}\, U'(x) = \frac{dU}{d\tau}$ we can write

$$P(x_1,t|x_i,t_i) = \exp\left[\frac{U(x_i)-U(x_1)}{2\theta}\right] K(x_1,t|x_i,t_i) \qquad (14)$$

where

$$K(x_1,t|x_i,t_i) = \int_{x_i}^{x_1} Dx(\tau) \exp\left[- \frac{1}{\theta} \int_{t_i}^{t} d\tau\, L(\dot{x},x)\right] \qquad (15)$$

with

$$L(x,\dot{x}) = \frac{\dot{x}^2}{4} + V(x) \;. \qquad (16)$$

This is the Lagrangian for a particle with mass $\frac{1}{2}$ moving in a potential $-V(x)$. $K(x_1,t|x_i,t_i)$ is the Feyman - Stuckelberg propogator and it has an expansion

$$K(x_1,t|x_i,t_i) = \sum_{n \geqslant 0} \phi_n(x_i)\,\phi_n(x_1) \exp\left(- \frac{(t-t_i)}{\theta}\lambda_n\right). \qquad (17)$$

$\phi_n(x)$ are the eigenfunctions of the equation

$$- \frac{\theta^2 d^2\phi_n}{dx^2} + V\phi_n = \lambda_n \phi_n \qquad (18)$$

where V is the potential derived from U.

In the functional integral approach one tries to calculate the Feynman propogator $K(x,t|x_i,t_i)$ while in the W.K.B. approach one calculates λ_n and ϕ_n which are the eigenvalues and eigenfunctions of (18) which is Schrödinger like. The approximations developed depend on the time scales of the problem. We shall illustrate as to how $K(x,t|x_i,t_i)$ is calculated using instanton methods.

Let us now evaluate $K(x,t|x_i,t_i)$ around $x_{cl}(t)$ which is the solution of the Euler-Lagrange equation corresponding to $L(x,\dot{x})$ of equation (16). We find that

$$K(x,t|x_i,t_i) = \exp\left[-\frac{1}{\theta} S_{cl}(x,t\, x_i,t_i)\right].$$

$$\times \int_{y(t_i)=0}^{y(t)=0} D(y(\tau)) \exp\left[-\frac{1}{\theta}\left\{\int d\tau \frac{\dot{y}^2(\tau)}{4} + \frac{y^2(\tau)}{2} V''(x_{cl}(\tau))\right\}\right] \quad (19)$$

where S_{cl} is the solution of the Euler-Lagrange equation of (16). Writing δS as

$$\delta S = \frac{1}{2} \int d\tau \left[y(\tau) - \frac{1}{2}\frac{d^2}{d\tau^2} + V''(x_{cl}(\tau))\right] y(\tau) \quad (20)$$

where

$$y(\tau) = x(\tau) - x_{cl}(\tau) \quad ; \quad y(\tau) = \sum_{n\geqslant 0} c_n y_n(\tau) \quad (21)$$

We can now choose solutions $y_n(\tau)$ which satisfy the equation

$$\left[-\frac{1}{2}\frac{d^2 y_n}{\tau} + V''(x_{cl})\right] y_n(\tau) = \lambda_n y_n \quad (22)$$

where

$$\int y_n(\tau)\, y_m(\tau)\, d\tau = \delta_{mn} \quad (23)$$

This gives immediately

$$K(x,t|x_i,t_i) = \frac{N}{(\pi\lambda_n)^{\frac{1}{2}}} \exp\left[-\frac{1}{\theta} S_{cl}(x,t|x_i,t_i)\right] \quad (24)$$

Here N is a nonmalisation constant which can be determined with help of the normalisation condition on P. This expression is true if λ_n are all non zero.

Now one notices thát the instanton solution $x_I(\tau)$ of figure (2) which is the solution to the potential (1) is a solution to (22) with zero eigenvalue. In such a case this mode must be separated out from the rest. In order to do this we notice

$$\lim S(x_I(\tau - t_1)) = S_o .$$

Here we have translated $\tau \rightarrow (\tau - t_1)$

$$\int_{-t}^{t/2} S(x_I(\tau - t_1)\, dt_1 = S_o \int_{-t}^{+t/2} dt_1$$

The change in $y(\tau)$ induced by a small change in the location of the centre of the instaton t , is

$$dy = \frac{dy_{cl}}{dt_1} dt_1 \quad (25)$$

Consider y_1 which is the zero eigenvalue solution of (22)

$$y_1 \propto S_o^{-\frac{1}{2}} \frac{dy_{cl}}{dt_1}$$

Therefore

$$dy = a\, S_o^{\frac{1}{2}} y, dt, \qquad (26)$$

where a is a constant $\sim \theta^{\frac{1}{2}}$. Integrating over the one instanton, we get

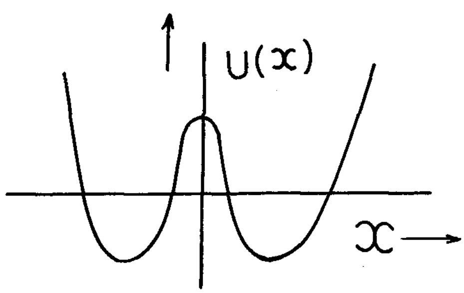
$$K(a,(t/2)|-a,(-t/2)) = \frac{N'}{(\pi'\lambda_N)^{-\frac{1}{2}}} \exp - \frac{1}{\theta} S_o . t(S_o)^{-\frac{1}{2}} \qquad (27)$$

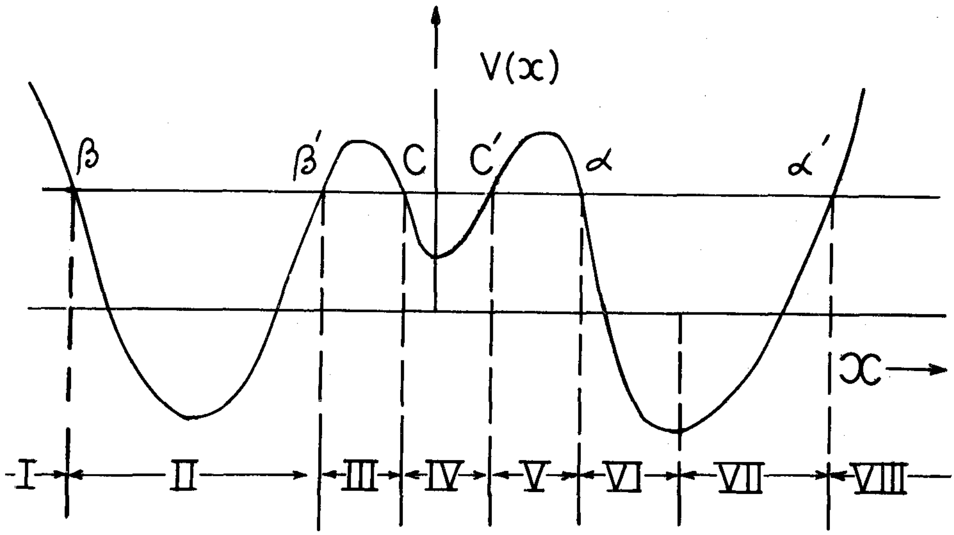
Similarly if one includes in succession an instanton and an anti-instanton one would have to evaluate integrals of the form

$\int_{-t/2}^{t/2} dt_1 \ldots \int_{-t/2}^{t_{n-1}} dt_n$ which gives a factor $\frac{t^n}{n!}(S_o)^{n/2}$ for an instanton and a factor of $(-1)^n\,(t^n/n!)(S_o)^{n/2}$ for an anti instanton. Therefore the net contribution from 'n' instantons gives a power series in t which sums up to an exponential. That is

$$K(a,(t/2)|-a,(-t/2)) \sim \frac{N'}{\pi'(\lambda_n)^{-\frac{1}{2}}} \exp -\left(\frac{1}{\theta} S_o\right) \exp\left(\frac{t}{\theta} S_o^{\frac{1}{2}}\right) \qquad (28)$$

A similar contribution will come from 'n' anti-instantons. In this way one computes K and hence $P(x,t|x_i,t_i)$. Knowing $P(x,t|x_i,t_i)$ one can compute the first passage time. Caroli et.al. calculate $K(x,t|x_i,t_i)$ for a potential which is bistable as shown in Figure 3. This give rise to V(x) which is whown in Figure 4.





In the case V(x) is bistable has been treated by Coleman [3] by W.K.B. methods and very nicely reviewed by Sakita [6]. The bistable case has also been treated by van Kampen [5].

Caroli et al choose a bistable U(x) for which V(x) is as shown in Figure (4) and solve this by W.K.B. They start from region I and construct the usual exponential W.K.B. solution and match it with the solution in region II which is the harmonic regime. This way continuing to match the solution, and finally demanding that in region VIII the exponentially increasing solution must be zero, they obtain the eigenvalue condition. For this V(x) of figure (4), this condition is an extremely complicated condition. Since usually only the first few eigenvalues of (18) are needed the problem is not too difficult.

It must be pointed out that the functional methods and the W.K.B. methods are complementary to each other and either can be used for such problems.

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