PROJECTION OPERATOR TECHNIQUES IN STOCHASTIC PROCESSES

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1. Need for Projection Operator Techniques and Examples of Some Typical Projection Operators

We have already indicated in the lecture on Fokker-Planck equations, that the solution of a multidimensional system is in general not possible. Fortunately for many systems, a complete specification of the probability distribution is not quite needed if one wants to compare with experiments. Often the information about mean and the correlation function suffices. Moreover the system need not exhibit markovian behavior. In such cases we will see that the projection operator techniques are ideally suited[1]. Following examples will show the versality with which projection operator techniques can be used.

(a) Wave propagation in a Random Medium

The propagation of a wave in a random medium can be characterized by

$$[\nabla^2 + \frac{\omega^2}{c^2} n^2] U = 0 \tag{1}$$

where n is the refractive index of the random medium. The amplitude of the wave U becomes stochastic in nature as n is a random function of $\vec{r}$ as is the case of the atmospheric medium. A complete specification of the probability distribution of U is impossible due to (a) nonlinearity of (1) i.e. the random term n appears in the multiplicative form (b) our inadequate knowledge about the fluctuations[2] of n. The quantities that are usually measured correspond to the mean and correlation of U i.e. to $\langle U(\vec{r})\rangle$, $\langle U(\vec{r})U(\vec{r}')\rangle$. Let us introduce the projection operator P that takes the ensemble average over refractive index fluctuations i.e. if

$$U = \sum \int n(\vec{r}_1)\ldots n(\vec{r}_m) f(\vec{r}_1)\ldots f(\vec{r}_m) d^3r_1 \ldots d^3r_m \tag{2}$$

then

$$PU = \sum \int \langle n(\vec{r}_1)\ldots n(\vec{r}_m)\rangle f(\vec{r}_1)\ldots f(\vec{r}_m) d^3r_1 \ldots d^3r_m . \tag{3}$$

(b) Adiabatic Elimination of Fast Variables

Our systems may have too many degrees of freedom. In such a case a reduced description is desirable for then, one can hope to obtain some exact solution for example in the lecture on Fokker-Planck equations, we saw that a one dimensional Fokker-Plank equation is always soluble. More specifically consider a system composed of two parts A and B and let τ_A and τ_B be the typical time scales associated with them. If $\tau_A \ll \tau_B$, then any small perturbation on A would get back A quickly in equilibrium. Hence if we are looking for dynamics on the scale $t \gg \tau_A$, then we eliminate the degrees of freedom associated with A. This idea is very extensively used in physical systems. The simplest example being the reduction of the full Fokker-Plank equation for the Brownian motion of a particle in a potential to Smoluchowski equation in the limit of large friction . Even in the context of the laser[3], simpler equations for photon distribution can be obtained by eliminating the stochastic variables for atoms. Let $f(x_A, x_B, t)$ be the joint distribution for the coupled A and B system. The reduced distribution for the system B alone will be $f(x_B, t) = \int dx_A\, f(x_A, x_B, t)$ and hence to obtain the projection of the dynamics, we introduce P defined by

$$P.... = N_A \int dx_A ....., \quad \int N_A dx_A = 1 \ . \tag{4}$$

The choice of N_A depends on the local equilibrium of A. For the derivation of the Smoluchowski equation we can take

$$P.... = e^{-p^2/2mkT} \int dp.... \ . \tag{5}$$

(c) Systems Interacting with Random Perturbations

Usually the behavior of the system is probed by external fields such as electromagnetic fields or magnetic fields which are generally fluctuating as the sources producing such fields are fluctuating [4]. Similarly many times the internal interactions are also modelled as stochastic in nature for example the kinetic Ising Model[5] with random interaction between various sites. The dynamical equation in all such cases can be written as

$$\frac{\partial \rho}{\partial t} = (L_o + L_1)\rho \ , \tag{6}$$

where L_1 is random either in space or in time. In such a case we have to take the ensemble average of ρ over the randomness contained in L_1. Similarly we need to evaluate the ensemble average of the dipole-dipole correlation function which is important in the line shape problems. The ensemble averaged equations can be derived by using projection operator methods.

2. Ensemble Averaged Equations for One Time Expectation Values

Let us write the basic stochastic equation in the form

$$\frac{d}{dt} A = (C_o + C_1(t))A + g \tag{7}$$

which can characterize the behavior of both classical and quantum systems e.g. A may represent the expectation values of a complete set of system operators. One can regard A,g as column matrices. Assume that all the randomness is contained in $C_1(t)$. We desire to compute the ensemble average of A i.e. $\langle A \rangle$ = PA. The formal derivation of PA is by now standard. One multiplies (7) on the left by both P and (1-P) and then integrates formally the (1-P)A equation and substitutes the result in the PA equation. This procedure leads to the exact relation(1)

$$\dot{PA} = C_o PA + g + PC_1(t)PA + PC_1(t)e^{C_o t}U(t,o)(1-P)A(o) + \int_o^t d\tau\, PC_1(t)e^{C_o t}U(t,\tau)\, e^{-C_o\tau}(1-P)C_1(\tau)PA(\tau), \tag{8}$$

where

$$U(t,\tau) = T\exp\left\{\int_\tau^t dt'(1-P)e^{-C_o t'}C_1(t')e^{C_o t'}(1-P)\right\} \tag{9}$$

The third term on the right hand side of (8) represents the motion due to the coherent part of C_1 whereas the fourth term represents the effect of initial correlations. The last term involves the correlations of C_1 to all orders. For most practical applications Born approximation is sufficient. In such a case (8) reduces to

$$\dot{PA} = C_o PA + g + \int_o^t d\tau\, PC_1(t)e^{C_o(t-\tau)}C_1(\tau)PA(\tau), \tag{10}$$

where we have also assumed for simplicity that $\langle C_1\rangle = 0$, $(1-P)A(o) = 0$. Equation (10) involves the second order correlation of the randomness i.e. if

$$C_1(t) = \sum C_i F_i(t), \quad \langle F_i(t)F_j(\tau)\rangle = D_{ij}(t-\tau) \tag{11}$$

then the Laplace transform of (10) yields:

$$P\hat{A} = [Z - C_o - \sum C_i \hat{D}_{ij}(Z-C_o)C_j]^{-1}(PA(o)+\frac{g}{Z}) \tag{12}$$

where Z denotes the Laplace variable. Note that if the random forces are delta correlated $\langle F_i(t)F_j(t')\rangle = 2D_{ij}\,\delta(t-t')$, then (10) reduces to

$$\dot{PA} = C_o PA + g + \sum_{ij} C_i C_j D_{ij} PA, \tag{13}$$

which is an equation local in time. Both (12) and (13) have been extensively used in many physical systems. We like to mention two exact results[6] here — (i) If the random forces $F_i(t)$ are Gaussian and delta correlated, then (13) is <u>exact</u>, though in the derivation we made Born approximation. (ii) If $C_1(t) = C\,F(t)$ where F(t) is a telegraphical signal or a dichotomic markov process, then (12) is <u>exact</u> though <u>not</u> (13). A very important property of dichotomic markov process is

$$\langle x(t)x(t')M(t',t''..)\rangle = \langle x(t)x(t')\rangle\langle M(t',t''...)\rangle, t > t' > t''.., \tag{14}$$

It may be noted that if the random forces have very short correlation time τ_c then for times $t \gg \tau_c$, equation (10) can be made local in time. Applications of (10) - (13) will be discussed in this volume.

3. Ensemble Average of the Multitime Correlations

When a system is interacting with stochastic perturbations, then its average behavior need not be markovian and hence one can not use the fluctuation regression theorem and (8) to obtain the multitime-correlations. However it turns out convenient to obtain directly equations like (8) for the correlation matrix. Let us consider the correlation matrix defined by $R(t+\tau,t) = \langle A_i(t+\tau)B(t)\rangle$, which satisfies equation analogous to (7) :

$$\frac{d}{d\tau} R(t+\tau,t) = (C_o + C_1(t+\tau))\, R(t+\tau,t) + \langle g\, B(t)\rangle . \tag{15}$$

The ensemble average of R over the fluctuations of C_1 can be obtained as before but now the inhomogeneous term $\langle gB(t)\rangle$ will make an additional contribution since $(1-P)\ \langle gB(t)\rangle \neq o$. The final result being [6]

$$\begin{aligned}\frac{d}{d\tau} PR(t+\tau,t) = {} & C_o PR + Pg\langle B(t)\rangle + PC_1(t+\tau)\ e^{C_o\tau}\, V(\tau,o)\,(1-P)R(t,t) \\ & + \int_o^{\tau} d\alpha\ PC_1(t+\tau)\,C\, e^{C_o\tau}\, V(\tau,\alpha) \Big\{ e^{-C_o\alpha}(1-P)g\langle B(t)\rangle \\ & + (1-P)\ e^{-C_o\alpha}\, C_1(t+\tau)PR(t+\alpha,t)\Big\} ,\end{aligned} \tag{16}$$

$$V(\tau,\alpha) = T \exp\Big\{\int_{\alpha}^{\tau} dt'(1-P)\ e^{-C_o t'}\ C_1(t+t')\ e^{C_o t'}\ (1-P)\Big\}. \tag{17}$$

Note that the initial condition will involve $R(t,t)$ which is a single time expectation value and which can be obtained from the result of Sec II. The result in Born approximation is relatively simple

$$\begin{aligned}\frac{d}{d\tau} PR(t+\tau,t) = {} & C_o PR + Pg\langle B(t)\rangle + I_1 + I_2 \\ & + \int_o^{\tau} d\alpha\ PC_1(t+\tau)\ e^{C_o(\tau-\alpha)}\ C_1(t+\alpha)PR(t+\alpha,t),\end{aligned} \tag{18}$$

$$I_1(t+\tau,t) = \int_o^t d\alpha\ PC_1(t+\tau)\ e^{C_o\tau}\, M\, e^{C_o(t-\alpha)}\ C_1(\alpha)\ PA(\alpha)\ ,$$

$$I_2(t+\tau,t) = \int_o^{\tau} d\alpha \int_o^t d\beta\ PC_1(t+\tau)\ e^{C_o(\tau-\alpha)}\ N\ e^{C_o(t-\beta)}\ C_1(\beta)PA(\beta), \tag{19}$$

where the matrices M and N enter through $(1-P)R(t,t) = M(1-P)A,\ (1-P)g\langle B(t)\rangle = N(1-P)A$. Note that the structure of (18) is very similar to (10) except for the new feature that inhomogeneous terms appear. It is because of such terms that the fluctuation-regression theorem is no longer valid and the system's behavior is nonmarkovian. If the random forces are delta correlated, then $I_1 = I_2 = 0$ and the markovian property is recovered. Again it must be noted that the result (18) is <u>exact</u>[6] if $C_1(t)$ has a single stochastic variable of random telegraphical signal type.

We close this chapter with a simple example which will illustrate how the dynamical behavior of the system can change depending on the stochastic nature of the randomness. Consider the stochastic equation

$$\dot{\psi} = i\varsigma X(t)\psi\ . \tag{20}$$

If $X(t)$ is a delta-correlated Gaussian random process $\langle X(t)X(t')\rangle = X_o^2\ \delta(t-t')$ then (10) and (18) lead to exact results:

$$\langle \dot{\psi} \rangle = \frac{-g^2 X_o^2}{2} \langle \psi \rangle, \frac{d}{d\tau} \langle \psi(t+\tau)\ \psi^*(t) \rangle = \frac{-g^2 X_o^2}{2} \langle \psi(t+\tau)\ \psi^*(t) \rangle \qquad (21)$$

On the other hand for a telegraphic signal with jump rate Γ we get from (10) and (18)

$$\langle \dot{\psi} \rangle = -g^2 x_o^2 \int_o^t d\tau\, e^{-2\Gamma(t-\tau)} \langle \psi(\tau) \rangle \quad ,$$

$$\frac{d}{d\tau} \langle \psi(t+\tau)\, \psi^*(t) \rangle = -g^2 x_o^2 \int_o^{\tau} dt'\ e^{-2\Gamma(\tau-t')} \langle \psi(t+t') \psi^*(t) \rangle$$
$$+2g^2 x_o^2 \int_o^t dt'\ e^{-2\Gamma(t+\tau-t')} \langle \psi(t') \psi^*(t') \rangle \qquad (22)$$

In this case the memory effects are very important. The differences in the two models are obvious from (21) and (22). It may also be added that (22) allows for the possibility of going from a completely coherent to incoherent situation by changing Γ from a very small value to a very large value.

References

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