PROJECTION OPERATOR METHODS IN LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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1. Introduction

Consider a linear stochastic differential equation

$$\frac{du}{dt} = \alpha A(t)\, u \tag{1}$$

where u is a vector , A(t) is a stochastic matrix characterised by a correlation time τ_c and α is a measure of the strength of the fluctuations. (In general, the R.H.S. of (1) may contain, in addition to A(t), a non stochastic matrix A_o. This part however can always be transformed away by going over to the interaction picture. Henceforth we shall assume that this has been done). We shall assume that the quantity $\alpha\tau_c$, called the Kubo number, is much less than unity. This serves as an expansion parameter in the perturbation expansions to be developed later.

Before going further, let us see what it means to solve a stochastic differential equation and in what ways do these equations differ from ordinary differential equations. To solve (1) we need to specify the initial conditions. Suppose that at $t=t_o$, $u(t_o)$ has a fixed value a. Now for each realisation of the stochastic process A(t), (1) is simply a deterministic equation. Its solution

$$u(t) = U(t, [A] \mid t_o)a \tag{2}$$

is a functional of A(t) i.e. it depends on all values A(t'), $t_o \leqslant t' \leqslant t$. The ensemble of all such realisations corresponding to all possible realisations of A(t) defines the stochastic process u(t) - a solution of (1) characterised by the time t_o at which the initial conditions were specified. If we were to specify the initial conditions at a time $t \neq t_o$ we would obtain a different stochastic process. We may express this alternatively by saying that the characteristic functional for u(t) depends in an essential manner on the time t_o at which the initial conditions are specified. It is in this respect, as has been emphasised by van Kampen [1], that stochastic differential equation differ from ordinary differential equations. Our main aim here is to develop systematic procedures for deriving statistical characteristics of u(t), given those of A(t). We wish to achieve this by introducing a projection operator

$$\mathcal{P}. = \langle .. \rangle \tag{3}$$

and by using two projection operator techniques.

2. Equations for Single Time Averages

2.1 An Integro-differential Equation for $\langle u(t) \rangle$ [2]

Applying $\mathcal{P}$ and $\mathcal{Q} = 1 - \mathcal{P}$ to both sides of (2) we can write it, in by now a fairly standard manner, as two coupled equations for $\mathcal{P}u$ and $\mathcal{Q}u$. Solving the equation for $\mathcal{Q}u$

we get

$$\mathcal{Q}u(t) = \mathcal{G}(t,o)\ \mathcal{Q}\,u(o) + \int_o^t d\tau\ \mathcal{G}(t,\tau)\ \mathcal{Q}A(\tau)\ \mathcal{P}u(\tau) \tag{4}$$

where

$$\mathcal{G}(t,\tau) = \overleftarrow{T}\left[\exp \alpha \int_\tau^t ds\ \mathcal{Q}\,A(s)\right] \tag{5}$$

and $\overleftarrow{T}$ prescribes chronological ordering. Substituting (4) in the equation for $\mathcal{Q}\,u$ we obtain an integro-differential equation for $\langle u(t)\rangle$

$$\frac{d}{dt}\langle u(t)\rangle = \int_o^t ds\ \ k(t,s)\ \langle u(s)\rangle + \int_o^t ds\ \mathcal{I}(t,s) \tag{6}$$

$k(t,s)$ and $\mathcal{I}(t,s)$ when expanded in powers of α have the following structure

$$k(t,s) = \sum_{m=1}^{\infty} \alpha^m k_m(t,s) \tag{7}$$

$$\mathcal{I}(t,s) = \sum_{m=1}^{\infty} \alpha^m \mathcal{I}_m(t,s) \tag{8}$$

$$k_1(t,s) = \langle A(t)\rangle \delta_+(t-s)\ ; \qquad \int_o^{\varepsilon} \delta_+(t)\ dt = 1,\ \varepsilon > o \tag{9}$$

$$k_2(t,s) = \langle A(t)\ \mathcal{Q}A(s)\rangle \tag{10}$$

$$k_{m+2}(t,s) = \int_s^t dt_1 \int_s^{t_1} dt_2 \cdots\cdots \int_s^{t_{m-1}} dt_m \langle A(t)\,\mathcal{Q}A(t_1)\ldots\mathcal{Q}A(t_m)\,\mathcal{Q}A(s)\rangle \tag{11}$$

The quantities $\langle A(t)\,\mathcal{Q}A(t_1)\cdots\cdots\ \mathcal{Q}A(t_m)\ \mathcal{Q}A(s)\rangle$ are known as totally ordered cumulants and often written as $\langle A(t)\ A(t_1)\ldots A(t_m)A(s)\rangle_t$

For $\mathcal{I}_1(t,s)$ we have

$$\mathcal{I}_1(t,s) = \langle A(t)\ \mathcal{Q}u(o)\rangle \delta_+(t-s) \tag{12}$$

and the expressions for $\mathcal{I}_m(t,s)$ $m > 1$ can be obtained from those for $k_m(t,s)$ by replacing the final operator $A(s)$ in (10) and (11) by $A(s)\ \mathcal{Q}u(o)$.

If we assume that at $t=o$, $\mathcal{Q}u(o) = o$ i.e. $u(o)$ is a non-stochastic vector (in the sense of being uncorrelated with $A(t)$) the inhomogeneous term in (6) drops out and we have

$$\frac{d}{dt}\langle u(t)\rangle = \alpha\langle A(t)\rangle + \alpha^2 \int_o^t dt_1 \langle A(t)\ A(t_1)\rangle_t \langle u(t_1)\rangle$$

$$+\alpha^3 \int_o^t dt_1 \int_o^{t_1} dt_2 \langle A(t)\ A(t_1)\ A(t_2)\rangle_t \langle u(t_2)\rangle + \cdots\cdots \tag{13}$$

This equation truncated at second order in α is known as Bourret's integral equation [3]. For the special case in which $A(t)$ involves a single dichotomic stochastic process $\alpha(t)$ i.e. $A(t) = B(t)\ \alpha(t)$ where $B(t)$ is a non stochastic matrix, then all the totally ordered cumulants higher than the second vanish and Bourret's integral equation becomes exact. In general, if $A(t)$ does not have any such special properties, then all one can hope to achieve is to calculate $\langle u(t)\rangle$ from the perturbation expansion

in (13). Terwiel [2] has estimated the integrals that appear in (13) by first proving a cluster property for the totally ordered cumulants. It turns out that (13) is not a systematic expansion in powers of $\alpha\tau_c$: the entire integro differential equation has to be rearranged in the form of a differential equation in order to find all the terms of a given order in $\alpha\tau_c$. Thus it will be desirable to derive a differential equation for $\langle u(t)\rangle$ at the very outset.

2.2 A Differential Equation for $\langle u(t)\rangle$: Time Convolutionless Projection Operator Technique [4-6] .

The projection technique above yields an integro differential equation for $\langle u(t)\rangle$ because of the convolution involved in (4). We can formally eliminate this convolution by substituting for $u(\tau)$ in (4) the following expression

$$u(\tau) = G(t,\tau)\, u(t) \tag{14}$$

where

$$G(t,\tau) \leq \vec{T}\left[\exp\left(-\alpha\int_{\tau}^{t} ds\, A(s)\right)\right] \tag{15}$$

and $\vec{T}$ prescribes anti-chronological ordering. This gives

$$\mathcal{Q}u(t) = \mathcal{G}(t,o) + \int_{o}^{t} d\tau\, \mathcal{G}(t,\tau)\, \mathcal{Q}A(\tau)\, \mathcal{P}G(t,\tau)\, (\mathcal{P}+\mathcal{Q})\, u(\tau) \tag{16}$$

Solving this for $\mathcal{Q}u(t)$ we get

$$\mathcal{Q}u(t) = \left[1 - \sum(t)\right]^{-1} \mathcal{G}(t,o) + \left[1 - \sum(t)\right]^{-1} \sum(t)\, \mathcal{P}u(t) \tag{17}$$

where

$$\sum(t) = \int_{o}^{t} d\tau\, \mathcal{G}(t,\tau)\, \mathcal{Q}A(\tau)\, \mathcal{P}G(t,\tau) \tag{18}$$

Equation (17) when substituted in the equation for $\mathcal{P}u(t)$ yields a differential equation for $\mathcal{P}u(t)$

$$\frac{d\langle u(t)\rangle}{dt} = K(t)\ \langle u(t)\rangle + I(t) \tag{19}$$

K(t) and I(t), when expanded in powers of α, are found to have the following structure

$$K(t) = \sum_{m=1}^{\infty} \alpha^m K_m(t) \tag{20}$$

$$I(t) = \sum_{m=1}^{\infty} \alpha^m I_m(t) \tag{21}$$

$$K_1(t) = \langle A(t)\rangle$$

$$K_m(t) = \int_{o}^{t} dt_1 \int_{o}^{t_1} dt_2 \cdots \int_{o}^{t_{m-2}} dt_{m-1}\, \langle A(t)A(t_1)\cdots A(t_{m-1})\rangle_p \tag{22}$$

The quantities $\langle A(t)\ldots A(t_{m-1})\rangle_p$ are known as partially ordered cumulants. These can be written down in terms of the moments using the following rules [7] .

1. Write a sequence of m dots.
2. write a zero on the first dot and any permutation of the numerals 1.2....m-1.

3. In every permutation insert a $\mathcal{P}$ between two successive numerals if $i>j$ and a $\mathcal{Q}$ if $i<j$ and place the whole expression between brackets $\langle\ldots\ldots\ldots\rangle$.

4. For each permutation with p-1 operators $\mathcal{P}$ supply a factor of $(-1)^{p-1}$.

5. Replace each numeral m by $A(t_m)$. The numeral o denotes $A(t)$. (This step can be generalised by associating different operators $A_m(t_m)$ with the numeral m and thereby define partially ordered cumulants involving different operators $A_i(t)$).

For $I_m(t)$ we have

$$I_1(t) = \langle A(t)\, \mathcal{Q}\, u(o)\rangle \tag{23}$$

$$I_m(t) = \int_o^t dt_1 \ldots \int_o^{t_{m-3}} dt_{m-2} \int_o^{t_{m-2}} dt_{m-1} \langle A(t)\ldots A(t_{m-2})\ A(t_{m-1})\, \mathcal{Q}\, u(o)\rangle_p \tag{24}$$

These expressions can simply be obtained from those for $K_m(t)$ by replacing $A(t_{m-1})$ in (22) by $A(t_{m-1})\,\mathcal{Q}\,u(o)$ and treating this as a single quantity. If we assume that $\mathcal{Q}u(o)=o$ then I(t) drops out from (19) and we have a differential equation for $\langle u(t_1)\rangle$

$$\frac{d\langle u(t)\rangle}{dt} = \Big[\alpha A(t) + \int_o^t dt_1 \langle A(t)\ A(t_1)\rangle_p + \int_o^t dt_1 \int_o^{t_1} dt_2 \langle A(t)A(t_1)A(t_2)\rangle_p + \ldots\ldots\ldots\Big]\langle u(t)\rangle \tag{25}$$

known as the cumulant expansion [8,9]. For the case in which $A(t) = B\,\alpha(t)$ where B is a constant matrix and $\alpha(t)$ is a Gaussian stochastic process then all the partially ordered cumulants, higher than the second, vanish and the expansion becomes exact in the second order approximation. From the point of view of perturbation expansions, van Kampen [9] and Roerdink [7] have proved a cluster property for the partially ordered cumulants and using this they have estimated the various terms that appear on the R.H.S. of (25). It turns out that (25) yields a systematic expansion in powers of $\alpha\tau_c$ from which $\langle u(t)\rangle$ can be calculated to a desired accuracy.

3. Equations for Higher Moments :

Equations for higher moments can be derived in exactly the same manner as above. For instance, the vector

$$t,t\ ) = u(t) \otimes u(t) = \begin{pmatrix} \underline{u}(t)\ u_1(t) \\ \underline{u}(t)\ u_2(t) \\ \vdots \end{pmatrix} \tag{26}$$

consisting of all the second moments of u(t) obeys the equation

$$\frac{d}{dt}\ g(t,t\ ) = \alpha C(t)\ g(t,t) \tag{27}$$

where

$$C(t) = A(t) \otimes I + I \otimes A(t). \tag{28}$$

Here $\otimes$ denotes direct product. (27) again has the same structure as (1) and the same methods as above apply here as well.

4. Two Time Correlation Functions :

We now wish to derive equations for the two time correlation functions

$$g(t,t') = u(t) \otimes u(t') \quad , \quad t \geqslant t' \tag{29}$$

assuming that at t=o, u(o) is a non stochastic vector. Again we have two options available

4.1 An integro differential equation for $\langle g(t,t') \rangle$:

The vector $g(t,t')$ obeys the following equation

$$\frac{d}{dt}\, g(t,t') = B(t)\, g(t,t') \tag{30}$$

$$B(t) = A(t) \otimes I \tag{31}$$

Proceeding as before and keeping in mind that the initial value for g(t,t') i.e. g(t',t') is a stochastic quantity we obtain using (6)

$$\frac{d}{dt}\langle g(t,t') \rangle = \int_{t'}^{t} ds\; k(t,s)\; \langle g(s,t') \rangle + \int_{t'}^{t} ds\; \mathcal{J}(t,s) \tag{32}$$

The structure of the terms on the R.H.S. of (32) is exactly the same as earlier with A(t) replaced by B(t) and $\mathcal{Q}u(o)$ by $\mathcal{Q}g(t',t')$. The second term in (32) involves $\mathcal{Q}g(t',t')$ and we wish to express it in terms of $\mathcal{P}g(t',t')$. This is easily done using (4) and taking into account the fact that $\mathcal{Q}g(o,o) = o$ by assumption. Hence

$$\mathcal{Q}g(t',t') = \int_{o}^{t'} d\tau\; \mathcal{G}(t,\tau)\; \mathcal{Q}C(\tau)\; \mathcal{P}g(\tau,\tau) \tag{33}$$

Here

$$\mathcal{G}(t,\tau) = \overleftarrow{T}\left[\exp \alpha \int_{\tau}^{t} ds\; \mathcal{Q}C(s)\right] \tag{34}$$

Substituting (33) in the expressions for $\mathcal{J}(t,s)$ we obtain the following integro-differential equation for $\langle g(t,t') \rangle$

$$\frac{d}{dt}\langle g(t,t') \rangle = \int_{t'}^{t} ds\; k(t,s)\; \langle g(s,t') \rangle + h(t,t') \quad . \tag{35}$$

k(t,s) has the same structure as for single time averages. The term h(t,t') has the following structure

$$h(t,t') = \sum_{m=2}^{\infty} \alpha^{m}\, h_m(t,t') \tag{36}$$

Each $h_m(t,t')$ is a sum of m-1 terms

$$h_2(t,t') = \int_{o}^{t'} dt_1 \langle B(t)\quad C(t_1) \rangle_{t} \langle g(t_1,t_1) \rangle \tag{37}$$

$$h_m(t,t') = \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{m-3}} dt_{m-2} \int_{o}^{t'} ds_1 \langle B(t)B(t_1)\cdots B(t_{m-2})C(s_1) \rangle_{t} \langle g(s_1,s_1) \rangle$$

$$+\int_{t'}^{t} dt_1 \cdots\cdots \int_{o}^{t'} ds_1 \int_{o}^{s_1} ds_2 \; \langle B(t) \cdots B(t_{m-3})\; C(s_1)\; C(s_2)\rangle_t \langle g(s_2, s_2)\rangle$$

$$+\cdots\cdots\cdots\cdots\cdots$$

$$+\int_{o}^{t'} ds_1 \int_{o}^{s_1} \cdots\cdots \int_{o}^{s_{m-2}} ds_{m-1} \langle B(t)\; C(t_1) \cdots\cdots C(s_{m-1})\rangle_t \langle g(s_{m-1}, s_{m-1})\rangle \qquad (38)$$

Equation (35) has been derived by Agarwal [10] in the context of quantum optics. The same equation trunctated at the second order has been previously derived by Morrison and Mckenna [11] .

As is clear the above equation for $\langle g(t,t')\rangle$ is coupled to the equation for the second moments $\langle g(t,t)\rangle$ and therefore becomes somewhat complicated. However simplification occurs in the case in which $A(t) = B(t)\ \alpha(t)$ where $\alpha(t)$ is a D.M.P. In this case the R.H.S. of (35) in the second order approximation becomes exact.

4.2 A Differential Equation for $\langle g(t,t')\rangle$:

If we proceed exactly as in 4.1 using the time convolutionless projection technique instead, we obtain an equation for $\langle g(t,t')\rangle$ having the following form

$$\frac{d}{dt}\langle g(t,t')\rangle = K(t,t')\langle g(t,t')\rangle + f(t,t')\ \langle g(t',t')\rangle \qquad (39)$$

where $K(t,t')$ has exactly the same form as (20) with $A(t)$ replaced by $B(t)$. This equation is not very useful. Among other things it does not even lead to exact results for the scalar Gaussian case i.e. the case in which (1) is a single variable equation with $A(t)$ Gaussian. This equation should therefore be abandoned.

An alternative differential equation for $\langle g(t,t')\rangle$ using methods not based on projection operator techniques has recently been derived somewhat heuristically in [12] and in a more systematic manner in [13] . This equation has the following form

$$\frac{d}{dt}\langle g(t,t')\rangle = M(t,t')\ \langle g(t,t')\rangle \qquad (40)$$

where $M(t,t')$ has the following structure

$$M(t,t') = \sum_{m=1}^{\infty} \alpha^m M_m(t,t') \qquad (41)$$

$$M_1(t,t') = \langle B(t)\rangle \qquad (42)$$

$$M_m(t,t') = \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots\cdots \int_{t'}^{t_{m-2}} dt_{m-1} \langle B(t)\; B(t_1) \cdots B(t_{m-1})\rangle_p$$

$$+\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{m-3}} dt_{m-2} \int_{o}^{t'} \langle ds_1\; B(t)\; B(t_1) \cdots B(t_{m-2})\; C(s_1)\rangle_p$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \int_{t'}^{t} dt_1 \int_{o}^{t'} ds_1 \cdots\cdots \int_{o}^{s_{m-3}} ds_{m-2} \langle B(t)\ B(t_1)\ C(s_1) \ldots C(s_{m-2}) \rangle_p$$

$$+ \int_{o}^{t'} ds_1 \int_{o}^{s_1} dt_2 \cdots\cdots \int_{o}^{s_{m-2}} ds_{m-1} \langle B(t)\ C(s_1) \cdots\cdots C(s_{m-1}) \rangle_p \qquad (43)$$

Equation (40) has an advantage over (35) in having a simpler structure and in not being coupled to the equations for the second moments. It becomes exact in the second order approximation in the scalar Gaussian case. Also note that the passage from the integro differential equation (35) to (40) simply consists in pulling out $g(s,t)$ and $g(s,s)$ in (35) and (38) from the integrals and replacing them by $g(t,t')$ and subsequently replacing the totally ordered cumulants that appear in the integrals, by partially ordered cumulants.

In the foregoing discussion we have developed general methods for obtaining statistical properties of solutions of linear stochastic differential equations. We have used, wherever possible, projection operator techniques to achieve this. These techniques, although by no means indespensible, prove to be convenient book keeping methods for perturbation expansions. We have seen that in some cases viz., scalar Gaussian case and the case in which $A(t)$ is a D.M.P. we obtain exact results. Mention must be made of other special cases in which exact solutions are possible. These include linear stochastic differential equations in which $A(t)$ is a (i) Kubo-Anderson process [14] (ii) Kangaroo process [14] (iii) a compound Poisson process [16,17] . Although so far we have been concerned with linear stochastic differential equations only, these methods apply to nonlinear stochastic differential equations as well. This is done by first writing a continuity equation - a linear equation corresponding to the given non linear stochastic differential equation and then making use of van Kampen's lemma [1] . In the above we have not included stochastic or non stochastic inhomogeneous terms in our starting equation (1). These can be incorporated within the framework of the formalisms presented above. For details and an excellent account of the topics discussed here we refer the reader to the works by Roerdink [7,13] .

The methods developed above have many applications in a number of areas such as resonance phenomena in quantum optics, nuclear magnetic resonance Mössbauer line shapes, wave propagation in a random medium, diffusion in a turbulent fluid etc. Some of these have been discussed at length in these proceedings [17].

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