THE QUARK-QUARK INTERACTION IN THE NONRELATIVISTIC AND IN THE LATTICE APPROXIMATION

F. Palumbo
INFN - Laboratori Nazionali di Frascati, P.O. Box 13 - 00044 Frascati, Italy

Abstract

The Galilean Hamiltonian for QCD is presented and illustrated. In a small quantization box such Hamiltonian exhibits a confining potential which, however, is a finite size effect, becoming purely Coulombic for infinite quantization volume. This does not necessarily imply absence of confinement in Galilean QCD, and a possible mechanism for confinement is discussed, suggested by the exact solution of the infrared sector in Galilean QED.

Finally the derivation of classical Galilean gauge theories and their quantization are sketched.

1. - Introduction

Nonrelativistic approximations of the so far well understood relativistic theories provide an accurate description of low-energy phenomena, in particular of bound states. A nonrelativistic approximation to QCD would therefore seem useful in the study of confinement, unless this is a purely relativistic feature of the theory. In fact not all of the low-energy properties of relativistic theories must necessarily be retained by their nonrelativistic approximations, and radiative corrections in QED are usually given as an example. This example is of particular relevance, because just the divergences which are related to radiative corrections in QED are thought to be responsible for quark confinement in QCD. But radiative corrections are related to gauge invariance. Therefore the question: Is it possible to maintain gauge invariance in the nonrelativistic approximation? A positive answer would imply the possibility of formulating a nonrelativistic version of QED which should give rise to radiative corrections.

Before going on, however, we must specify the meaning of the world nonrelativistic. Here it will be used as a synonym for Galilean. When the velocity of light c becomes infinite, the Poincaré group contracts into the Galilean group. We therefore require that the limit $c \to \infty$ of gauge theories be gauge-invariant Galilean theories. We will see that such theories exist and that in Galilean QED there are indeed radiative corrections which are the $c \to \infty$ limit of the radiative corrections of relativistic QED.

The fact that Galilean QED retains the infrared features of relativistic QED gives some confidence that the same could be true for QCD.

Galilean QCD has been formulated and it has been shown that the q-q interaction is not confining in the usual sense. Since this is a nonperturbative result, it is natural to compare it to lattice calculations which give indication of a linear potential. I will therefore present Galilean QCD from this point of view, which is the reason for the title of my talk, although I will not discuss lattice calculations.

In Galilean QCD the q-q interaction depends on the size of the quantization volume L^3. Calculations must be done at finite L and the limit $L \to \infty$ must be taken at the end. Now while for infinite volume the q-q interaction is purely Coulombic, for small volume it has a confining term (a finite size effect) which is of the order of magnitude of that resulting from lattice calculations.

The absence of a confining potential does not necessarily imply, however, absence of confinement, and a possible mechanism for confinement will be discussed.

I will first show and illustrate the Hamiltonian in Galilean QCD. From this Hamiltonian I will derive the effective q-q interaction to be compared to that obtained in lattice calculations. After that I will present Galilean QED whose infrared sector has been solved exactly and I will compared it to relativistic QED. The solution of the infrared sector in QED suggests the mentioned mechanism for confinement in the absence of a confining potential.

Finally I will sketch the derivation of classical Galilean gauge theories and their quantization.

2. - Galilean QCD and the effective q-q interaction in a small quantization volume.

I will present the Galilean QCD Hamiltonian[1] for two quarks in the fundamental representation of SU(2). This Hamiltonian is separated into two terms

$$H = H_o + V, \tag{1}$$

the potential having the form

$$V = \frac{1}{8}\,\alpha_s\,\frac{1}{|x|}\,\sigma_a(1)\left\{\delta^{ab} - \left[1 - \cos g\,L^{-3/2}\,\vec{q}\cdot\vec{x}\right]\left[\delta^{ab} - \frac{v^a v^b}{v^2}\right] + \varepsilon^{abc}\frac{v^c}{v}\sin g\,L^{-3/2}\,\vec{q}\cdot\vec{x}\right\}\sigma_b(2)\,. \tag{2}$$

In Eq. (2) $\vec{x}$ is the separation distance between the quarks, σ_a the Pauli matrices in color space, $\vec{q}$ and v^a two variables originating from the gluon field, g the coupling constant, $\alpha_s = g^2/4\pi$, L^3 the volume of the quantization box, and we have put $\hbar = c = 1$. For $L \to \infty$ V becomes purely Coulombic, but as we will see, the prescription is to calculate at finite L and take the limit $L \to \infty$ in the final results.

H_o is given by

$$H_o = -\frac{1}{2m_q}(\Delta_1+\Delta_2) + 2\,m_q + \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{1}{2}\,\frac{1}{q^2}(\vec{v}\wedge\vec{\pi})^2 +$$
$$+\,\omega\, q_k \frac{v^a}{v}\;\frac{1}{\sqrt{2\,m_q}}\; i\,(\alpha_a(1)\;\nabla_{1k} + \sigma_a(2)\Delta_{2k}) \tag{3}$$

where m_q is the quark mass, p_k and π^a are the momenta conjugate to q_k and v^a respectively and

$$\omega^2 = \frac{g^2}{2\,m_q L^3}. \tag{4}$$

The above equations hold for quantization with periodic boundary conditions (p.b.c.), which seem appropriate to QCD. I have no time to discuss this point for which I refer to the literature[2].

The physical states are subject to the constraint

$$\left[(\vec{v}\wedge\vec{\pi})_a + \frac{1}{2}\,\sigma_a(1) + \frac{1}{2}\,\sigma_a(2)\right]\Psi = 0. \tag{5}$$

Note that H and therefore Ψ do not depend on v but only on $\frac{v^a}{v}$.

Evaluation of cross sections at finite L is not trivial, and we will show in the next section such an evaluation in the abelian case as an illustration and in order to suggest a possible mechanism for confinement.

In order to compare to lattice calculations, however, we must keep L finite. If we take two quarks in color singlet state, according to Eq. (5) their wave function Ψ does not depend on v^a/v, so that we can define an effective Hamiltonian as the average of H over v^a/v

$$H_{eff} = -\frac{1}{2\,m_q}(\Delta_1 + \Delta_2) + 2\,m_q + \frac{1}{2}\,\omega^2 q^2 - \frac{3}{8}\,\alpha_s\frac{1}{|x|} + \frac{1}{4}\,\alpha_s\frac{1}{|x|}(1-\cos g\, L^{-3/2}\,\vec{q}.\vec{x}), \tag{6}$$

and the effective potential energy is

$$W = -\frac{3}{8}\,\alpha_s\frac{1}{|x|} + \frac{1}{2}p^2 + \frac{1}{2}\,\omega^2 q^2 - \frac{3}{2}\,\omega + \frac{1}{4}\,\alpha_s\frac{1}{|x|}(1-\cos g\, L^{-3/2}\,\vec{q}.\vec{x})\,. \tag{7}$$

This potential energy still depends on $\vec{q}$. At fixed $\vec{x}$ the lowest eigenvalue of W as an operator acting on $\vec{q}$ is the effective q-q potential energy. The evaluation of this eigenvalue has already been done for finite quark mass[3], but in order to compare to lattice calculations we need it for infinite quark mass. Denoting by W_o this eigenvalue we have[4]

$$W_o = -\frac{3}{8}\,\alpha_s\frac{1}{|x|} + \sigma|x|\,, \tag{9}$$

$$\sigma = \frac{1}{2}\,\sqrt{\pi}\,\alpha_s\,\frac{1}{L^2}\sqrt{\frac{L}{|x|}}\;. \tag{10}$$

The present results comes from an asymptotic expansion valid for large values of the parameter

$$K = \frac{1}{2\pi}\left(\frac{L}{x}\right)^3. \tag{11}$$

For $L \sim 1$ fm, $\alpha_s \sim 1$, $x = 0.2$ L, $\sigma \sim 400$ MeV/fm.

Thus we see that although the true potential is not confining in the usual sense ($\sigma \longrightarrow 0$ for $L \longrightarrow \infty$), for small L it appears to have a confining term. This term is spurious and such a spurious contribution to the string tension should also be present in lattice calculations with p.b.c., due to the contribution of Galilean configurations of the gauge field.

3. - Galilean QED

The e^+e^- Hamiltonian in Galilean QED is[2)]

$$H_o = -\frac{1}{2\,m_e}(\Delta_1+\Delta_2) - \frac{1}{2}\,\alpha\,\frac{1}{|x|} + \frac{1}{2}p^2 + \frac{1}{2}\,\omega^2 q^2 + \omega\, q_k\,\frac{1}{\sqrt{2\,m_e}}\,i\,(\nabla_{1k} - \nabla_{2k}), \qquad (12)$$

with

$$\omega^2 \doteq \frac{2\,e^2}{m_e L^3}. \qquad (13)$$

Eq. (12) holds for p.b.c.. Although phenomenology definitely requires that quantization be done by requiring that the fields vanish on the surface of the quantization box in QED, I will consider here Galilean QED with p.b.c. for comparison with Galilean QCD and with relativistic QED.

If we let $L \longrightarrow \infty$ in Eq. (12) the coupling between the "photon" and the electrons disappears. But we know that we must perform this limit only on the final results.

In order to solve the infrared sector[2)] it is convenient to introduce creation and destruction operators

$$a_k^+ = (2\,\omega)^{-1/2}\,(p_k + i\,\omega\, q_k), \qquad (14)$$

and rewrite H as

$$H = -\frac{1}{2\,m_e}(\Delta_1 + \Delta_2) + \frac{3}{2}\,\omega + \omega\, a_k^+ a_k + \frac{1}{2}\,\sqrt{\omega}\,i\,(a_k^+ - a_k)\, I_k - \frac{1}{2}\,\alpha\,\frac{1}{|x|}, \qquad (15)$$

where

$$I_k = -\frac{i}{\sqrt{m_e}}\,(\nabla_{1k} - \nabla_{2k}). \qquad (16)$$

If we denote the purely electronic states by $|\, i\rangle$, $|\, f\rangle$, ... the eigenstates of H can be written

$$|\, i, n_1, n_2, n_3\,\rangle = \prod_{k=1}^{3}\frac{1}{\sqrt{n_k!}}\left[a_k^+ + \frac{i}{\sqrt{2\omega}}\, I_k(i)\right]^{n_k} e^{\frac{i}{\sqrt{2\omega}}(a_k^+ + a_k)\, I_k(i)}\,|\, i\,\rangle, \qquad (17)$$

where $I_k(i)$ is the eigenstate of the operator I_k in the state $|\, i\rangle$. Let $|\, i\rangle$ be the initial state and $|\, f\rangle$ the final state, $\Delta I_k = I_k(f) - I_k(i)$. Assuming a proper choice of axes, $\Delta I_k = \delta_{k3}\,\Delta I$, and the Smatrix elements have the expression

$$|\langle i,0,0,0,|S|\, f, n_1, n_2, n_3 \rangle|^2 = \delta_{on_1}\,\delta_{on_2}\,|\langle i\,|S(n_3\,\omega)|\,f\rangle|^2 \cdot \frac{1}{n_3!}\left[\frac{(\Delta I)^2}{2}\right]^{n_3} e^{-\frac{(\Delta I)^2}{2}}, \tag{18}$$

which is the probability for a classical source to radiate n_3 photons of energy ω.

The limit $L \to \infty$ can now be performed in Eq. (18) showing that

$$\lim_{L\to\infty} |\langle i, 0,0,0\,|\,S\,|\,f, n_1, n_2, n_3\rangle|^2 = 0. \tag{19}$$

In an actual experiment, however, there will be a finite energy resolution ΔE in a final state. The theoretical cross section to be compared with experiment is

$$\begin{aligned}\sigma \sim \lim_{L\to\infty} \sum_{n_3=0}^{\infty} |\langle i,0,0,0\,|S|\,f, n_1, n_2, n_3\rangle|^2 \theta(\Delta E - n_3\,\omega) \\ = |\langle i\,|\,S\left[\tfrac{1}{2}(\Delta I)^2\right]|\,f\rangle|^2 \,\theta\left[\Delta E - \tfrac{1}{2}(\Delta I)^2\right].\end{aligned} \tag{20}$$

Now we must compare this formula to the corresponding one in relativistic QED. The Bloch-Nordsieck theorem with p.b.c. gives[5]

$$\sigma \sim \left[\frac{\Delta E - \frac{1}{2}(\Delta I)^2}{E}\right]^{\beta} \theta\left[\Delta E - \frac{1}{2}(\Delta I)^2\right], \tag{21}$$

where E is some experimental energy and β vanishes for infinite light velocity so that the Galilean limit is reproduced.

I have derived here Eq. (20) for two reasons. First to show that it agrees with the limit of the relativistic formula for the radiative corrections, which gives confidence in Galilean gauge theories. The second reason is that it suggests a new mechanism for confinement.

The radiative correction of Eq. (20) arises from the overlapping of the soft photon state relative to the initial state to the soft photon state relative to the final state of the electrons, the photon state being a function of the current. This is because $\vec{q}$ is coupled only to the current in the electronic Hamiltonian.

In Galilean QCD the gluon appears not only with the quark color current, but also in the quark-quark potential, so that the gluon state will depend on the structure of the q-q wavefunction. Confinement would result if the gluon state relative to a bound state were orthogonal to the gluon state relative to a free state of quarks in the $L \to \infty$ limit. In such a case, in fact, the disintegration cross-section would vanish.

4. Classical gauge theories and their quantization

In this final part I will show how classical Galilean gauge theories have been formulated[6] and quantized[1]. I will confine myself to pure gauge theories, without coupling to matter fields, for the sake of brevity. I will maintain $\hbar=1$ but I have obviously to reintroduce the light velocity c.

The relativistic gauge field Lagrangian density is

$$\mathscr{L}_G = \frac{1}{2}(\mathscr{F}^a_{oi})^2 - \frac{1}{4}(\mathscr{F}^a_{ij})^2 , \tag{22}$$

where

$$\begin{aligned}\mathscr{F}^a_{oi} &= \frac{1}{c}\ \partial_t \mathscr{A}^a_i - \partial_i \mathscr{A}^a_o + \frac{g}{c}\ f^{abc} \mathscr{A}^b_o \mathscr{A}^c_j \\ \mathscr{F}^a_{ij} &= \partial_i \mathscr{A}^a_j - \partial_j \mathscr{A}^a_i + \frac{g}{c}\ f^{abc} \mathscr{A}^b_i \mathscr{A}^c_j ,\end{aligned} \tag{23}$$

f^{abc} being the structure constants of the color group.

Now if we let $c \to \infty$ other things remaining constant, $\mathscr{F}^a_{ij} \to \partial_i \mathscr{A}^a_j - \partial_j \mathscr{A}^a_i$, $\mathscr{A}^a_{oi} \to -\partial_i \mathscr{A}^a_o$ and we loose gauge invariance, without reaching Galilean invariance (see below). We therefore rescale the gauge fields according to

$$\begin{aligned}\mathscr{A}^a_o &= - V^a \\ \mathscr{A}^a_i &= c\, A^a_i ,\end{aligned} \tag{24}$$

so that

$$\begin{aligned}\mathscr{F}^a_{oi} &\overset{def}{=} F^a_{oi} = \partial_t A^a_i + \partial_i V^a - g\, f^{abc}\, V^b A^c_i \\ \mathscr{F}^a_{ij} &\overset{def}{=} F^a_{ij} = c(\partial_i A^a_j - \partial_j A^a_i + g\, f^{abc} A^b_i A^c_j) ,\end{aligned} \tag{25}$$

$$\mathscr{L}_G = +\frac{1}{2}(F^a_{oi})^2 - \frac{1}{4}\ c^2 (F^a_{ij})^2 . \tag{26}$$

Now gauge invariance is maintained and Galilean invariance is acquired in the $c \to \infty$ limit

$$\mathscr{L}_G \xrightarrow[c \to \infty]{} \frac{1}{2}\ (F^a_{oi})^2 , \tag{27}$$

with the constraint

$$F^a_{ij} = 0 . \tag{28}$$

In fact, under Galilei transformations of parameters v_k

$$\begin{aligned}&A^a_i \to A^a_i ,\ V^a \to V^a + v_k A_k \\ &F^a_{ij} \to F^a_{ij} ,\ F^a_{oi} \to F^a_{oi} + v_k F^a_{ki} ,\end{aligned} \tag{29}$$

so that the constraint (28) is Galilei-invariant and under such constraint the Lagrangian (27) is Galilei invariant.

In order to quantize it is convenient to rewrite $\mathscr{L}_G$ in first order formulation, including the constraint by means of Lagrange multipliers Λ^a_{ij}

$$\mathscr{L}_G = F^a_{oi}\, \partial_t\, A^a_i - \frac{1}{2} F^a_{oi}\ F^a_{oi} - V^a\, \mathscr{D}^{ab}_i\, F^b_{oi} - \Lambda^a_{ij}\, F^a_{ij}\,, \tag{30}$$

where

$$\mathscr{D}^{ab}_i = \partial_i\, \delta^{ab} - g\, f^{abc}\, A^c_i \tag{31}$$

The dynamical varaibles are A^a_i with canonical momenta

$$F^a_{oi} = \frac{\partial \mathscr{L}_G}{\partial \partial_t A^a_i}\ , \tag{32}$$

while V^a are Lagrange multipliers for the constraints

$$\phi^a = \mathscr{D}^{ab}_k\ F^b_{ok}\,, \tag{33}$$

which are common to the relativistic case.

The Hamiltonian is

$$H = \frac{1}{2} F^a_{oi}\, F^a_{oi}\,, \tag{34}$$

apart from a linear superposition of first class constraints.

We apply Dirac's theory of canonical quantization of constrained systems, and find that the only secondary constraints are

$$\chi^a_i = \varepsilon_{ijk}\ \mathscr{D}^{ab}_j\ F^b_{ok} = 0\ . \tag{35}$$

Among the constraints F^a_{ij}, ϕ^a, χ^a_i, only the ϕ^a are first calss, i.e. they commute among themselves, with the other constraints and with the Hamiltonian. According to Dirac's theory first class constraints can either be supplemented by gauge fixing, or can be used as conditions to be satisfied by physical states. We will use an intermediate procedure.

We will expand fields and constraints in Fourier series, and will single out the zero-momentum component of ϕ^a

$$\phi^a_o = g\, L^{-3/2} \int d^3x\, \phi^a = g\, L^{-3/2}\, Q^a_G\,, \tag{36}$$

for its physical meaning, which is the gauge field color charge Q^a_G, and we will use it as constraint on physical states. We will supplement the other Fourier components of ϕ^a by the gauge fixing

$$A^a_{3\,\vec{n}} = 0, \; n_3 \neq 0$$

$$A^a_{2\,\vec{n}} = 0, \; n_3 = 0, \; n_2 \neq 0 \qquad (37)$$

$$A^a_{1\,\vec{n}} = 0, \; n_3 = n_2 = 0, \; n_1 \neq 0 \, .$$

The solution to Eqs. (28), (33), (35) and (37) for the color group SU(2) is

$$A^a_i = \frac{1}{L^{3/2}} q_i \frac{v^a}{v}$$

$$E^a_i = \frac{1}{L^{3/2}} \left(p_i \frac{v^a}{v} + \frac{q_i \ell^a_\perp}{q^2}\right) \qquad (38)$$

with

$$\ell^a_\perp = (\pi^a - v^a \frac{1}{v^2} v^b \pi^b) \, v, \qquad (39)$$

and q_k, p_k, v^a, π^b satisfying canonical poisson brackets.

Introducing Eqs. (38) into Eq. (39) we obtaine the pure gauge part of the Hamiltonian (3).

References

1) F. Palumbo, to be published.
2) F. Palumbo, Phys. Letters B132, 165 (1983).
3) F. Palumbo, in "Mesons, Isobars, Quarks and Nuclear Excitations" Erice 6-18 Aprile 1983.
4) G. De Franceschi and F. Palumbo, in preparation.
5) F. Palumbo and G. Pancheri, in preparation.
6) F. Palumbo, Nuclear Phys. B182, 261 (1981).