APPLICATION OF THE CLUSTER EXPANSION TO QCD

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1. Introduction

It is the purpose of this paper to present a new theoretical frame for treating QCD in an approximate way. The basic idea is to introduce a suitable cutoff version of the QCD-Hamiltonian and to solve the emerging many-body problems for the vacuum and for the hadrons by non--perturbative many-body techniques(a variational exp S-method). The hope is to obtain in this way more realistic models for hadrons (the only parameter for massless quarks being the QCD coupling-constant which has to be adjusted by some hadron mass) with the special structure that gluonic degrees of freedom are included. The theoretical significance of the results and the validity of confinement can be checked by studying the cutoff-dependencies.

We have organized our paper as follows: In sect. 2 the general structure of the cutoff-QCD Hamiltonian and its ground state solution is discussed and compared to standard many-body theory. The basic ansatz used in the variational method is given in sect. 3. The structure of the cluster expansion used for calculating expectation values is presented for the ground state in sect. 4 and for excited states in sect. 5.

Finally, we explain in sect. 6 the principles of performing renormalization by defining a suitable cutoff-dependent QCD coupling constant and discuss the structure which is to be expected in the case of the validity of confinement.

2. General structure of the cutoff QCD many-body theory

We introduce quark (antiquark) fermion operators $q_\alpha, q_\alpha^+ (\bar{q}_\alpha, \bar{q}_\alpha^+)$ and gluon boson operators b_k, b_k^+, where α, k are abbreviations for all quantum numbers needed to specify the corresponding s.p. states (momentum, spin, flavour, colour). The $t = 0$ field operators defining the QCD Hamiltonian may be expanded in terms of these fermion and boson operators. A cutoff version of QCD is then defined by restricting this expansion for the field operators to a finite number of terms. We use a phase space cutoff as introduced in ref.[1)] which is given by the following prescription:

(a) Impose to the s.p. wave functions the boundary conditions of a finite box Ω. This makes the s.p. momentum spectrum discrete.

(b) Take only a finite number M of these s.p. states for the definition of the field operators.

The quantities Ω and M define the cutoff. As e.g. worked out by Pottinger et al.[2)], the QCD Hamiltonian has then (in the Coulomb gauge) the structure (for simplicity, we specify only the Yang-Mills-case, i.e. we disregard quarks):

$$
\begin{aligned}
H = H_2(b^+b) &+ (H_3^0(b^+b^+b^+) + H_3^1(b^+b^+b) + \text{h.c.}) \\
&+ (H_4^0(b^+b^+b^+b^+) + H_4^1(b^+b^+b^+b) + H_4^2(b^+b^+bb) + \text{h.c.}) \\
&+ \text{higher order terms} \qquad (1)
\end{aligned}
$$

($H_3^1(b^+b^+b)$ stands for an expression of the type $\sum_{0\leq k_1,k_2,k_3\leq M} H^1_{3,k_1k_2k_3}\ b^+_{k_1}, b^+_{k_2}, b_{k_3}$, etc.) Due to the cutoff, this Hamiltonian is a well-defined operator acting in the Fockspace generated from the Fockspace-vacuum $|0\rangle$ (defined by the condition $b_k|0\rangle$ for all k) by the operators b_k^+.

The many-boson-theory defined by this Hamiltonian is analogous to that of the standard non-relativistic (e.g. nuclear) case whose dynamics are described by an operator of the type

$$
H_{st} = T(a^+a) + V(a^+a^+aa) \ . \qquad (2)
$$

We mention the following structures:

(i) Both theories are defined in a Fockspace, only the statistics are different (we have fermions in the case of the nucleus).

(ii) A difference important for the interpretation of eigenstates is that the standard Hamiltonian H_{st} conserves the particle number which H does not. Thus the ground state of the fieldtheoretical Hamiltonian has to be interpreted as the "physical" vacuum, whereas that of H_{st} is the energetically lowest eigenstate of some particle number N (which has to be fixed before doing a calculation). But the mathematical analogy between the structures of the Hamiltonians (1) and (2) tells us that the vacuum "with interaction" is a state like nuclear matter (better bose-matter). This analogy with nuclear matter becomes very obvious in the case of fermions with selfinteraction, where the cutoff momentum plays the formal role of a fermi-momentum (see ref.[1]).

Eigenstates of the QCD-Hamiltonian yielding observable properties are given by "vacuum excitations": e.g. mesons correspond to p-h-excitations, glue-balls to two-particle, baryons to 3p excitations of the standard theory.

(iii) A fundamental (and for the volume-dependence of any property important) structure of the ground state ψ_o of H is that there exists a (unique) operator S

$$S = S_2(b^+b^+) + S_3(b^+b^+b^+) + \ldots \qquad (3)$$

(again, $S_2(b^+b^+) := \sum S_{2,k_1k_2}\, b^+_{k_1} b^+_{k_2}$, etc.) such that

$$\psi_o = e^S |0\rangle \qquad (4)$$

In the standard many-body case, the state $|0\rangle$ has to be replaced by a suitable slaterdeterminant of a given particle number. The only condition for the validity of this exponential form of ψ_o is that this state is not orthogonal to the Fockspace vacuum $|0\rangle$.

(iv) If perturbation theory is valid, there exists an expansion of the operator S in terms of linked (Goldstone) diagrams. A corresponding expansion of the ground state energy has important applications in standard many-body theory. In field-theory, these representations of vacuum properties play a minor role since the standard perturbation theory for Wightman functions in terms of Feynman diagrams yields at once all physical quantities relative to the vacuum.

3. Variational approach to the eigenstates of the cutoff QCD

In the case QCD, perturbation theory is assumed not to be valid and extended methods have to be applied in order to approximate the eigenvalue problem of H. We propose, herefore, to use the exponential form

$$\psi_o = e^S |0\rangle \qquad S = S_1(b^+b^+) + S_3(b^+b^+b^+) + S_4(b^+b^+b^+b^+)+\ldots$$

with a suitably truncated form for the operator S (e.g. up to 4^{th} order) as an ansatz for the ground state wave-function and to determine the coefficients defining S by the Ritz-variational principle.

In the same way, also excited states can be approximated by using an ansatz

$$\psi = De^{S'}|0\rangle$$

$$\begin{aligned} D &= D_2(b^+b^+) && \text{for a glue-ball} \\ &= D_2'(\bar{q}^+q^+) && \text{for a meson} \\ &= D_3(q^+q^+q^+) && \text{for a baryon} \end{aligned} \qquad (5)$$

Note that due to vacuum polarization effects, S' is expected to emerge from a variational calculation as different from S, the state exp S'$|0\rangle$ being then possibly interpretable as the vacuum modified through the formation of a bag.

We mention two cases where this variational method has been applied to QCD yielding nonperturbative results:

(a) The theory of Finger et al.[3] for chiral symmetry breaking. Here, only quark degrees of freedom are taken into account and the vacuum is defined as a BCS-state, i.e. $S = S_2(\bar{q}^+q^+)$. With the notations of ref.[1], this corresponds to a Hartree-Fock approximation of the vacuum state (which is then a filled Dirac--sea). The chiral-symmetry of the QCD-Hamiltonian for massless quarks yields then a self-consistent symmetry of the H.F. equations which is broken for strong enough interaction.

(b) The soliton-bag model of Friedberg and Lee[4], pursued in detail by Wilets et al.[5]: In this case the fact is used that the operator S has to be a spin-isospin-colour-scalar. It is, therefore, tempting to approximate S as being generated by an independent effective scalar quantum field. The QCD-Hamiltonian has then to be enlarged by introducing a phenomenological effective coupling of this new scalar field with itself and with the quarks and gluons. The ansatz (4) is then a coherent state which allows to treat S as a classical field. It has been shown by the above authors that the effective QCD-Hamiltonian can be chosen such that $S \neq 0$ for the vacuum and that S' for the hadron differs from S quite in the same way as the bag-model would predict.

In standard many-body theory[6,7] and in simple theories with mesonic degrees of freedom[8,9], some experience has been achieved with the ansatz (3) in a more general form (e.g. Brueckner-theory involves terms up to 4^{th} order). In view of the structure of the QCD--Hamiltonian this leads to the suggestion that the following ansatz should be necessary (whether it is sufficient has to be checked by trying out refinements in a later state of the theory):

$$S = S_2(b^+b^+) + S_3(b^+b^+b^+) + S^4(b^+b^+b^+b^+)$$

$$+ S_0^2(\bar{q}^+q^+) + S_1^3(\bar{q}^+q^+b^+) \qquad (6)$$

(here, we have included quarks for definiteness).

4. Cluster expansion of the expectation values

For a complicated ansatz for S such as in eq. (6) it is not possible to obtain rigorous expressions for the expectation value

$$E(\psi_o) = \frac{\langle\psi_o H \psi_o\rangle}{\langle\psi_o \psi_o\rangle} \qquad (7)$$

However, there exists a powerful cluster expansion[6-9] of this quantity which states that one can define diagrams, involving elements related to S, S^+ and H (fig. 1), and Feynman rules such that we have a linked cluster theorem, i.e.

$$E(\psi_o) = \sum \text{all contributions from connected diagrams} \qquad (8)$$

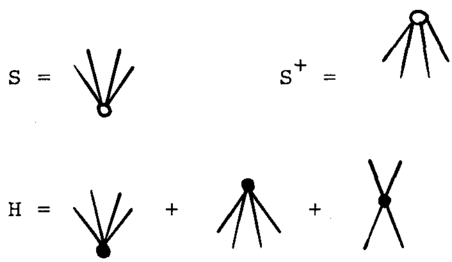

Fig. 1

Diagrammatic elements related to the operators S, S^+ and H for the case $S = S_4(b^+b^+b^+b^+)$, $H = H_4^0(a^+a^+a^+a^+) + H_4^2(a^+a^+aa) + H_4^4(aaaa)$

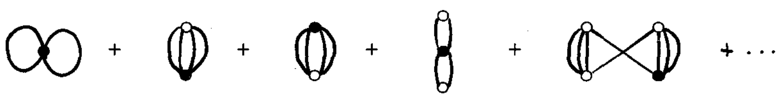

Fig. 2

Diagrams defining the linked cluster expansion of the expectation value $E(\psi_0)$ (eq. 7).

Some lowest order examples are given in fig. 2. Since (8) is a complicated infinite sum, the problem arises which technique should be used to approximate $E(\psi_0)$. Analogous to standard many-body theory[7-10], we propose as the simplest possibility the introduction of "occupation factors". This amounts to the following prescriptions:

Compute for each term of the Hamiltonian the lowest order contribution (in S).

Relate to each of these diagrams a well-defined diagram class by replacing each "free" line by a "dressed" line as depicted in an example in fig. 3 (more details are given in refs.[7,9]):

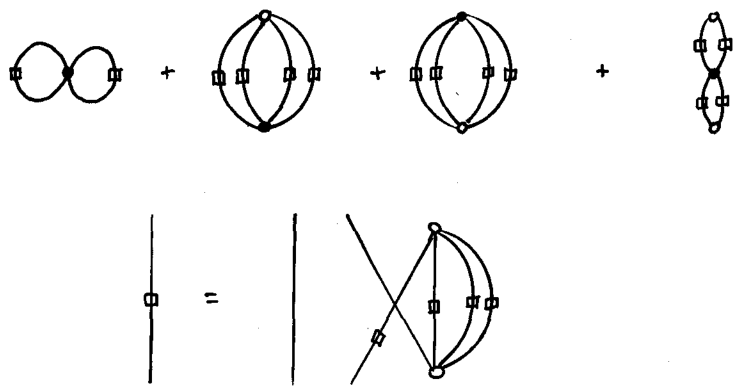

Fig. 3

Lowest order approximation of $E(\psi)$ including occupation factors.

We mention the following arguments which lead us to the hope that this "occupation factor procedure" might lead to a reliable estimate of $E(\psi_o)$. (Of course, the ultimate justification has to emerge from a numerical investigation of higher order contributions).

(a) The method is exact for $S = S_1(b^+b^+) + S_2^o(\bar{q}^+q)$

(b) It leads to the Brueckner-approximation for the standard many--body theory[7)]

(c) For the fieldtheoretical Lee-model it yields a "mesonic" Brueckner -theory consistent with renormalization[8)]

(d) The application of the method to be solvable version of the Lee model introduced in ref.[11)] leads to an approximation to $E(\psi_o)$ which is clearly valid beyond the perturbation theory limit (fig. 4).

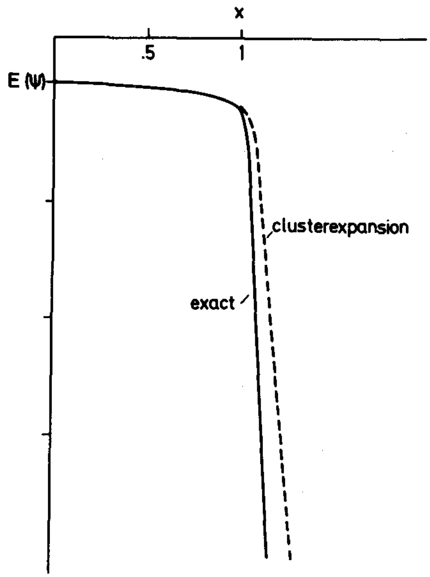

Fig. 4
Validity of the approximation given in fig. 3 for $E(\psi)$ for the solvable Lee-model. In the notation of ref.[11] we have put $N = 500$, $\alpha = 1$, $x = G\sqrt{N}$, $S = \lambda\tau_+ b^+$. Compared is the exact minimum of $E(\psi)$ ($\psi = e^S \phi$) to that of the approximate cluster expansion (fig. 3). The limit of perturbation theory is $x = 1$.

5. Cluster expansion for excitation energies

A notable general property of the cluster expansion is that it leads to an expression for excitation energies which is naturally split up into contributions from the kinetic and potential energy of the gluon in the glueball (or the quarks in the hadron) and into that of the "pressure of the vacuum on the bag": Herefore, we define

$$\psi_0' = e^{S'} |0\rangle$$

as the "vacuum modified by the bag". It is then easily proven that the energy difference $E(\psi)-E(\psi_0')$ is given by a completely linked expansion - by taking the difference all diagrams are cancelled where the operators D^+,D and H are not linked (fig. 5).

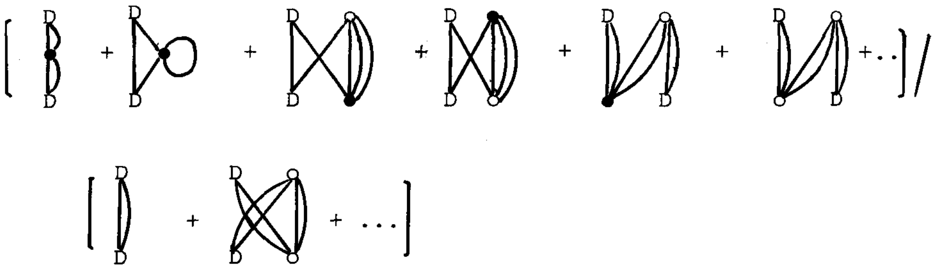

Fig. 5

Structure of the cluster expansion for the energy of a glueball ($D = D_2(b^+b^+) = $) relative to the energy of the "vacuum modified by the bag" (eq. 9) following from an analysis of the expression

$$E(\psi)-E(\psi_o') = [\langle\psi_o'|D^+HD|\psi_o'\rangle/\langle\psi_o'|\psi_o'\rangle]/[\langle\psi_o'|D^+D|\psi_o'\rangle/\langle\psi_o'|\psi_o'\rangle]$$

$$-\langle\psi_o'|H|\psi_o'\rangle/\langle\psi_o'|\psi_o'\rangle$$

The interpretation of this expansion clearly is that of "kinetic and potential energy" of the two gluons in the glueball (for $D = D_2(b^+b^+)$). Thus the physically observable glueball energy has to be written in the form $E(\psi)-E(\psi_o) = (E(\psi)-E(\psi_o'))+(E(\psi_o')-E(\psi_o))$ where the second term is interpretable as arising from the vacuum pressure as phenomenologically introduced in standard bag models.

6. Treatment of the cutoff-dependence and the confinement problem

Predictions for masses and other properties of hadrons should be made within our frame according to a "renormalization" scheme proposed in ref.[1] which is quite analogous to that of lattice QCD calculations:

For a given cutoff, the QCD coupling constant g should be adjusted to some hadronic mass (e.g. a meson or glueball mass). This fixes (for massless quarks) the QCD-Hamiltonian and allows then to predict in principle any other hadronic property. In this way, the coupling constant becomes a function of the cutoff (in fact $g = g(M)$, since g is adjusted to some energy-difference which is independent from the volume cutoff Ω). If our theoretical scheme is consistent, the prediction of hadronic properties should become independent from M if M is large enough. This is analogous to the continuum-limit in lattice QCD calculations. All this should hold for colour-singlett-quantities.

The confinement problem of QCD could be investigated within our frame by looking at the energies of colour non-zero states (the simplest case would be just gluons and quarks, defined by setting $D = b_k^+$ or $D = q_\alpha^+$). For given M and fixed coupling g(M), the mass of such a state is finite. Confinement should show up in the cutoff-dependence of these energies: When g(M) is fixed by some hadron-mass and we look at the limits for $M \to \infty$, we should obtain that the energies of colour zero-states are stable, whereas the energies of colour-nonzero states go to infinity.

We remark that all these statements about the cutoff-dependences represent expectations (for a simple case the consistency of our frame has been verified in ref.[1]) which have to be checked by working out the details of the expressions the energies $E(\psi_o)$, $E(\psi)$. Such investigations are being pursued at present in Bonn[12].

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