NUCLEAR SHELLS FROM A QUARK-THEORETICAL VIEWPOINT

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Part I: A quark shell model of the nucleus

1. Introduction

Modern particle theory considers all hadrons as being made out of quarks which interact via gluon exchange (QCD). In particular, the nucleon is a 3-quark system of considerable size. This size is so large that the basic postulates of the nuclear shell model, assuming approximately point-like nucleons which move quasi-freely in a common average potential, become highly questionable. In fact the measured rms charge radius of the proton (0.9 fm) indicates that the nucleon radius is 1.2 fm (for a uniform charge distribution) whereas the average nucleon-nucleon distance in a nucleus is 1.8 fm. Thus we have to expect that the quark wave functions of the individual nucleons largely overlap and that the quarks develop a tendency to spread out over the whole nucleus. This effect seems to be observable in the recent μ-Fe scattering experiment which is discussed by Rith in this volume. Under these circumstances the question arises whether the overall nuclear structure might be considered from the outset as a system of quarks whose lowest energy states have the characteristic properties predicted by the standard shell model. This report reviews a first joint attempt[1] in this direction. Part I concentrates on the physical aspects, part II on the mathematical techniques which we need.

2. Application of the MIT model to nuclei

The main ingredients of our new picture of the nucleus are taken from the MIT bag model of hadrons[2]. This model is believed to constitute an intermediate step towards a true QCD calculation of hadronic properties and we just enlarge its application by treating now the nucleus as a system of 3N non-strange quarks. Hence the quarks are described by the Dirac equation

$$(|x|<R) \qquad (\vec{\alpha}\vec{p}+\beta m)\psi = \varepsilon\psi \ ,$$

together with the boundary condition

$$(|x| = R) \quad \vec{\alpha}\cdot\vec{x}/|x|\psi = i\beta\psi \ ;$$

i.e. the quarks move in a spherical, infinite potential well with a radius R, which we have to identify with the nuclear radius. It follows that quark states form shells of good angular momentum and parity and that we can label these states by quantum numbers (n,j,m,τ,i) representing radial excitations (n), total angular momentum (j), angular 3-momentum (m), isospin (τ) and colour (i). If we order these states with respect to increasing energy ε we find a sequence of shells with the same quantum numbers as in the Mayer-Jensen shell model, namely $1s\ \frac{1}{2}$, $1p\ \frac{3}{2}$, $1p\ \frac{1}{2}$, $1d\ \frac{5}{2}$ etc. Note that the spin-orbit splitting is already enforced here by the Dirac equation without any additional parameter.

Nuclear states are now formed by putting 3N quarks in these shells. In addition, we have to fulfill the requirement that the total wave function has colour zero. This requirement is trivially satisfied when a shell is completely filled. Remarkably enough one obtains in this simple way already the analogues of the magic nuclei within the quark model.

The major difficulty arises when a shell is not completely filled. We come to this problem later and discuss first the total energy E (the nuclear mass) which is up to now equal to the sum of the occupied single-particle energies ε. These energies depend on the quark mass m and the bag radius R. In particular, E is radius-dependent, too. In the MIT-model the hadronic system is stabilized by a universal outer pressure B which is supposed to be an effect of the degenerate gluonic vacuum. The equilibrium equation reads:

$$-\frac{\partial E}{\partial R} = 4\pi BR^2 \ ,$$

and can be used to determine the nuclear radius R, once the values of m and B are fixed. Choosing m = 270 [MeV] and $B^{1/4}$ = 127 [MeV], ($B^{1/4}$ = 100 [MeV] is the MIT value), we find the remarkable result that the nuclear radius computed in this way satisfies the well-known law $R = 1.3\ N^{1/3}$ [fm] valid all over the periodic table. Thus in our model the gluonic pressure B is directly related to the saturation properties of nuclei.

It should be mentioned that the last equation is slightly too complicated for practical computations. Using the approximate relation

$$-\frac{\partial E}{\partial R} = \frac{1}{R} E$$

(strictly valid for massless quarks), it can be converted into the simpler form

$$E = 4\pi B \cdot R^3$$

which states that the energy and the volume of nuclei are strictly proportional.

3. Non-filled quark shells

We now turn to the discussion of non-filled quark shells. For simplicity we consider the case of a core $|0\rangle$ of completely filled quark shells and M quarks in a shell with fixed quantum numbers n and j. It is convenient to lump state labels together, $(n,j,m,\tau,i) \to (\alpha,i)$, and to introduce the corresponding creation and annihilation operators $a^+_{i\alpha}$, $a_{i\alpha}$. Clearly the vector space of all M-quark states described above is then given by the vectors:

$$\prod_{r=1}^{M} a^+_{i_r\alpha_r} |0\rangle \ .$$

Most of these states are of no physical interest because they carry colour. The vector space of colour-free states alone is spanned by vectors of the form:

$$\prod_{r=1}^{N} B^+(\alpha_r,\beta_r,\gamma_r) |0\rangle \ ,$$

where M = 3N and (using from now on the convention of summing over repeated indices):

$$B^+(\alpha,\beta,\gamma) = \varepsilon_{ikl} a^+_{i\alpha} a^+_{k\beta} a^+_{l\gamma} \ .$$

The proof of this statement has already been given by H. Weyl[3] who applied it in his discussion of spinless electron systems (Russell--Saunders coupling). We denote the space of colour-free states from now on by V_N and proceed by characterizing it in more detail by certain group-theoretical properties. Let $d_{\alpha\beta}$ $(\alpha,\beta = 1,\ldots,n=2(2j+1))$,

be an antihermitean trace-free matrix, i.e.

$$d_{\alpha\beta} = -\bar{d}_{\beta\alpha} \ .$$

Such matrices form the Lie-algebra of SU(n) and the assignment

$$d_{\alpha\beta} \to \hat{d} = d_{\alpha\beta} a^{+}_{i\alpha} a_{i\beta}$$

yields a representation of SU(n) in the 3N-quark space. This representation commutes with colour rotations and, therefore, leaves V_N invariant. Hence we can restrict it to V_N, and this restriction turns out to be irreducible[3]. Fortunately, all such irreducible representations are classified, either by Young diagrams[4] or by maximal weights[5]. Our representation belongs to the Young diagram with 3 columns and N rows, or, equivalently to the maximal weight $3f_N$[5].

From this maximal weight we can compute[5] the dimension of V_N (see part II). It turns out that

$$\dim V_N = 2\binom{n}{N}\binom{n+1}{N}\binom{n+2}{N}/(N+1)^2(N+2) \ .$$

A moment's thought will show us that this leads us into trouble. In fact, the situation with 3N quarks in one shell should correspond to N nucleons in a shell with the same quantum numbers. Hence the dimension of V_N should be equal to $\binom{n}{N}$ which is in general much smaller than the number given before. We come to the conclusion that the nuclear states we are looking for, can at best be a subspace of V_N, which must contain in addition other hadronic states. Such new states are, of course, expected because the nucleon itself has a lot of excited states arising from different quark configurations, and our description of the nucleus will include such excitations, too. The simplest new excitation will, for example, occur when a nucleon is changed into a Δ. Note, however, that in our model such a change is described by an excitation of the whole nucleus.

A Δ-excitation is energetically separated from the normal nuclear states by 300 MeV, but up to now, it has in our model the same energy as the ground state. We conclude that we have to include an additional part into our Hamiltonian, and the natural choice is a quark-quark interaction. From the discussion above it is clear what this force should do: It should yield as the lowest energy states a subspace of V_N which can be identified with the space of N nucleons in a shell of

angular momentum j; this requirement fixes in particular the total spin and isospin content of this subspace completely. Largely separated in energy (300 MeV) from these states we should then find new excited states, which we can identify with Δ-excitations.

4. The quark-quark interaction

Unfortunately, we are working on a low energy problem far away from asymptotic freedom; hence QCD does not provide us with a reliable expression for our quark-quark interaction. We may assume, however, that it is short-ranged because long-range effects are already built into the bag boundary conditions. Since we are interested in an effective force P, which acts only between quarks in one shell the short-range character leads us to the assumption that our force will be of the pairing type, i.e. it will act only between two quarks coupled to angular momentum zero. It is well-known that the pairing part is always dominant when a force is short-ranged; thus our assumption yields at least a reasonable approximation. But then SU(3)--invariance and the Pauli-principle allow only two possibilities: Either the quarks are in a colour antitriplett, isospin zero state, or they form a colour sextett, isospin one state. The latter possibility can be disregarded because such a force would favour energetically high isospin states. We are left with the first possibility which can be written down in the form:

$$P = -gA_i^+ A_i/4 \quad ;$$

$$A_i = \varepsilon_{ikl} g_{\alpha\beta} a_{k\alpha} a_{l\beta} \quad .$$

In this formula ε_{ikl} represents the totally skew Kronecker symbol and

$$g_{\alpha\beta} = (-1)^{j_\alpha - m_\alpha - \tau_\alpha + \frac{1}{2}} \cdot \delta_{m_\alpha, -m_\beta} \delta_{\tau_\alpha, -\tau_\beta}$$

is essentially the Clebsch-Gordon coefficient achieving the coupling to spin and isospin zero. g is a coupling constant which we discuss later.

Our total Hamiltonian reads as follows:

$$H = \varepsilon N_{op} + P + E_{core} \ .$$

E_{core} is the energy contribution of the filled core which can be treated as a pure number in this context; ε and N_{op} are the energy and the number operator for the non-filled shell. This Hamiltonian commutes with colour rotations and, therefore, leaves the physical subspace V_N (the colourfree states) invariant. We have to diagonalize it in this subspace in order to determine the new mass values which will be shifted by the interaction P. The diagonalization can be done exactly with the help of group-theoretical methods as will be shown now.

First note that $g_{\alpha\beta}$ can be interpreted as symmetric bilinear metric. Consider the antihermitean trace-free matrices $d_{\alpha\beta}$ which leave in addition the metric $g_{\alpha\beta}$ invariant, i.e. the equation

$$d_{\alpha\gamma}g_{\gamma\beta} + d_{\beta\gamma}g_{\alpha\gamma} = 0$$

holds. These matrices form a Lie subalgebra of SU(n), which is naturally isomorphic to the Lie algebra of SO(n). The representation of SU(n) described in section 3 does not have the property to commute with H, but when we restrict it to the subalgebra SO(n) we can easily check that all the generators $\hat{d} = d_{\alpha\beta}a^{+}_{i\alpha}\alpha_{i\beta}$ commute with P provided that the last equation holds.

5. The diagonalization of the Hamiltonian

The invariance of our Hamiltonian under the group SO(n) can be made even more explicit; in fact P can be expressed entirely by the Casimir operators C_1 and C_2 of the Lie algebras SU(n) and SO(n), respectively. More precisely, the following formula holds (see part II):

$$P = -g[(n-2)C_2 - nC_1 + \frac{1}{2N} N_{op}^2 - \frac{1}{2} N_{op}] \ .$$

(We use the same normalization of the Casimir operators as in ref. 5)).

In an irreducible representation the values of these Casimir operators are constant all over the representation space and completely determined by the maximal weights[5]. Since we know already that SU(n) acts irreducibly in V_N and has the maximal weight $3f_N$ in this space, we can already compute the value of C_1. To compute the values of C_2 we note that our irreducible representation of SU(n) in V_N will yield a reducible representation of SO(n) (the invariance group of our Hamiltonian) which splits into irreducible components. We can determine the maximal weights of these irreducible subspaces and hence compute the values of the Casimir operators C_2. Alternatively we may solve our problem by using Young diagram techniques[4]. The first method is described in full detail in part II; here for the moment we just quote the result which is remarkably simple: Set k = min(N,n-N); then the space V_N splits into k+1 components which are irreducible under SO(n). Moreover, we can label these subspaces by l = 0,1,...,k such that the eigenvalue of H in each subspace takes the form

$$E_l = E_{core} + 3N\cdot\varepsilon - g[N(n+3-N) - l(n+3-l)] \ .$$

The lowest eigenvalue occurs for l = 0. The states in the corresponding subspace should be the normal nuclear shell model states corresponding to N nucleons in a shell with angular momentum. The first excitation occurs for l = 1; those states should describe Δ-excitations of the nucleus. (We will prove this in the next section). Hence $E_1 - E_o$ must equal the Δ-N mass difference which fixes the value of g to be g = 120/(2j+1) [MeV].

The parameters of our model are now completely determined and we can compute nuclear masses. We do this by starting with 1s $\frac{1}{2}$ quarks and $E_{core} = 0$. Having filled the 1s $\frac{1}{2}$ shell we begin to fill the p-shells with the 1s $\frac{1}{2}$ core energy taken from the preceding computation. This process can obviously be continued as far as we wish, but we stop at Ca^{40}(N = 40) for physical reasons because afterwards the Coulomb energy begins to destroy the isospin invariance which, up to now, is a strict symmetry of our model. The numerical values for the nuclear masses then turn out to satisfy approximately the relation E = μN where μ is the nucleon mass, as it should be. The deviations from this formula are for an individual nucleus, on the average ± 1.5 percent and become only slightly higher when a shell is completely filled. This result can be combined with the relation $E = 4\pi BR^3$ and leads to the radius law $R = 1.3\ N^{1/3}$ [fm] anticipated in section 2.

6. The structure of the SO(n)-irreducible subspaces

As we have stated in the previous section, we have to split the SU(n)-irreducible space V_N into SO(n)-irreducible components. This problem can be solved in the following way: Recall that V_N is spanned by vectors of the form

$$\psi = \prod_{r=1}^{N} B^+(\alpha_r,\beta_r,\gamma_r)\,|0\rangle \ ,$$

where

$$B^+(\alpha,\beta,\gamma) = \varepsilon_{ikl} a^+_{i\alpha} a^+_{k\beta} a^+_{l\gamma} \ .$$

Next note that any contraction of the indices carried by the vector ψ defines a linear map in V_N which commutes with the SO(n)-action. This invariance property of the contraction operator has already been used by H. Weyl[6] in order to determine all irreducible representations of SO(n). We will see that a similar technique works also here. Let us define the operator

$$C^+_\alpha = B^+(\alpha,\beta,\gamma)\cdot g_{\beta\gamma} \ ,$$

and let F_l denote the vector space spanned by all vectors φ_l of the form

$$\varphi_l = \prod_{r=1}^{l} B^+(\alpha_r,\beta_r,\gamma_r)\cdot \prod_{s=1}^{N-l} C^+_{\alpha_s}\,|0\rangle \ .$$

In particular F_N equals V_N and $F_0 \subset F_1 \subset \ldots \subset F_{N-1} \subset F_N$. Moreover, due to the SO(n)-invariance of the contraction operator, each space F_l is SO(n)-invariant. Next we define the new spaces G_l by $G_0 = F_0$ and

$$G_l \oplus F_{l-1} = F_l \ , \qquad (l = 1,\ldots,N) \ ;$$

i.e. G_l is the orthogonal complement of F_{l-1} in F_l. Clearly the new spaces G_l are also SO(n)-invariant and, by construction,

$$V_N = \bigoplus_{l=0}^{N} G_l \ .$$

Hence we have achieved an orthogonal decomposition of V_N into SO(n)--invariant subspaces. (In part II we will show that $G_l = 0$ if $N > n/2$ and $l > n-N$).

In order to show that G_1 has indeed the properties stated in the last section, we must establish SO(n)-irreducibility and then compute the maximal weight. The lengthy details will be described in part II. Here we want to discuss the states in G_o and G_1, i.e. the ground-states and the first excited states of our model. Evidently G_o is spanned by the vectors

$$\varphi_o = \prod_{r=1}^{N} C^+_{\alpha_r} |O\rangle \ .$$

φ_o is antisymmetric in the indices α_r because the operators C_{α_r} anti-commute. Obviously, such states are in one-to-one correspondence with Slaterdeterminants build from nucleonic states in a shell with the same angular momentum and parity as our quarks. Thus we have shown that the quark-quark interaction P has indeed energetically singled out a nucleonic subspace with the correct physical properties and hence fulfills all the requirements of section 3. The fact that the states in this subspace are still degenerate can be accepted as a reasonable approximation and should not disturb us; in reality the corresponding energy splittings are of the order of a few MeV, and hence indeed small in comparison with the first genuine excitation in our model at 300 MeV.

These first excitations form the subspace G_1 and can be described as follows: Let $d_{\alpha\beta\gamma}$ be a symmetric tensor with the property that contractions with the metric $g_{\alpha\beta}$ vanish. Assign to any such $d_{\alpha\beta\gamma}$ the operator

$$D^+_d = B^+(\alpha,\beta,\gamma) d_{\alpha\beta\gamma} \ .$$

From its very definition, it then is obvious that G_1 is spanned by vectors of the form

$$\varphi_1 = D^+_d \cdot \prod_{r=1}^{N-1} C^+_{\alpha_r} |O\rangle \ .$$

We have seen before that $C^+_{\alpha_r}$ is the creation operator of a nucleon with the quantum numbers α_r. On the other hand, a straightforward comparison with the standard quark model calculations of baryons shows that the operator D^+_d creates an excited nucleon, i.e. a Δ-particle. (The known physical quantum numbers arise when d is appropriately specialized).

It has now been rigorously established that the ground states and the first excited states of our model have the right properties. It remains to analyse the higher excitations. The most obvious interpretation would be to relate them to states with more Δ-particles on even higher resonances of the nucleon, but we have only made an incomplete analysis so far. An interesting point which we do not want to omit here, is the phenomenon that, e.g., a naive study of the Δ-excitations of the nucleus treating the Δ as an elementary particle, yields much more states than our quark model. This effect is due to Pauli principle alone and we illustrate it here with 6 quarks in the 1s $\frac{1}{2}$ shell. The two-nucleon states have in our model the spin and isospin quantum numbers (1,0) and (0,1), respectively; they stand for a rough description of the deuteron and the nucleon-nucleon antibound state and are degenerate as they should. The first excitation consists of a (2,1) and a (1,2) system. Note that N-Δ system with Δ elementary yields in addition states with the quantum numbers (2,2) and (1,1). The next higher excitations which then exhaust all possibilities of our model have the quantum numbers (3,0) and (0,3), whereas two elementary Δ-particles also give the combinations (3,2), (1,0), (1,2), (0,1), (2,3), (2,1). Hence, the use of the quark model reduces the number of states tremendously, and we speculate that this effect should be directly observable.

Conclusion

Our model description of nuclear states in terms of quarks can be summarised as a sequence of correlations which are built into the multi-quark wave functions step by step. The first correlation arises from the requirement of vanishing colour which enforces a grouping of quark creation operators into triplett clusters which are individually colourfree. This is achieved by contracting the colour indices with the ε-symbol. The second correlation then is introduced by coupling two creation operators in each cluster to angular momentum and isospin zero and amounts to contraction with the metric tensor $g_{\alpha\beta}$. The main result of our work consists in the proof that such a configuration is energetically favoured by our pairing force. Bearing in mind that this force can at best be regarded as an approximation to the full effective quark-quark interaction, it is to be expected that additional

correlations must occur when a more realistic expression is used. Nevertheless we believe that our model will be of practical interest also for such a more refined theory, provided that the pairing part of the force remains dominant. In this case one could diagonalize the pairing part exactly by our methods and treat the rest perturbatively.

That more correlations are actually needed can indeed be checked by computing certain nuclear properties like magnetic moments, weak decay constants, etc. In particular, we have found that magnetic moments become too large for high angular momentum states. This behaviour is, in fact, due to the specific couplings of angular momenta in a nucleonic cluster which we use in our model, and it can be cured by admixing different configurations probably due to a more complicated quark force.(As long as the quark-quark interaction remains theoretically undetermined such calculations can be used to gain some phenomenological insight into its physical properties).

The basically new feature of our quark model of the nucleus consists in a radical reinterpretation of the nuclear saturation properties. These are usually determined by the following procedure (see R. Machleidt's contribution to this volume): First an expression for the nuclear force is derived partly from theoretical models (boson exchange) but still containing phenomenological ingredients (form factors). In the next step a certain approximation of the many-body problem, mostly based on Brueckner's theory, is applied to compute the density and energy of nuclear matter. Such a program is notoriously difficult and never gave completely satisfactory values for both of these data. Instead we apply now the MIT model prescription and obtain nuclear saturation directly as the result of the universal gluonic pressure on the walls of the nuclear bag.

Our model is obviously not invariant under translations. Hence our total energies must be corrected by subtracting the energy ΔE of the center of mass motion. We have no reliable method to compute ΔE but we can assume that ΔE will be proportional to the inverse of the quark number. Hence we make the phenomenological ansatz $\Delta E = \gamma/3N$, and treat the constant γ as an additional fit parameter. We find $\gamma = 370$ [MeV]. This purely numerical correction has already been incorporated into the results which we have described before.

Part II: The group theory of the model

1. Introduction

This part describes our mathematical techniques. They are based on the representation theory of compact Lie algebras. In section 2 we summarize the main theorems which we need and which we have all taken from ref. 5). In sections 3 and 4 we specialize these theorems to the groups SU(n) and SO(n) and we prove the basic mathematical properties of our model.

2. Compact Lie algebras

Let A be a real Lie algebra of finite dimension k with Lie product $[,]$. The adjoint representation $ad: A \to L(A)$, (the algebra of linear transformations of A), is defined by the formula

$$ad(x)y = [x,y], \quad (x,y \in A) \ . \tag{1}$$

Setting

$$\langle x,y \rangle_A = -\mathrm{tr}\ ad(x)ad(y) \ , \tag{2}$$

one obtains a bilinear symmetric form $\langle , \rangle_A$ on A, called the Killing form. A is called compact, if $\langle , \rangle_A$ is positive definite. From now on we assume that A has this property. (In particular, this implies that A is semi-simple).

Let H be an abelian subalgebra of A. H is called a Cartan algebra of A, if its dimension r is maximal which means that any abelian subalgebra of A has a dimension less than or equal to r. We fix H and denote by H^* its dual, i.e. the space of linear functions on H. Let $\{C_i\}$ be a basis of H. Hence $f \in H^*$ is completely specified by the values $f(C_i)$. Let γ_{ij} denote the inverse of the matrix $\langle C_i, C_j \rangle_A$. For arbitrary $f,g \in H^*$ we may set

$$\langle f,g \rangle_{H^*} = \sum_{i,j} f(C_i)g(C_j)\gamma_{ij} \tag{3}$$

to obtain a positive definite scalar product $\langle , \rangle_{H^*}$ on H^*. With these definitions we are now able to recall the basic theorems on compact Lie algebras.

Theorem I (structure of compact Lie algebras): There is a unique set ϕ of linear functions $\omega \in H^*$ with associated vectors A_ω, B_ω which form an orthonormal basis of $H^\perp$, (the space orthogonal to H in A), and fulfill the equations

$$[h, A_\omega] = \omega(h) B_\omega \quad ,$$

$$[h, B_\omega] = -\omega(h) A_\omega \quad , \tag{4}$$

for all $h \in H$. Moreover, ϕ contains a subset $\{\omega_i\}$ which forms a basis of H^* and has the property that each $\omega \in \phi$ can be written in the form

$$\omega = \sum_{i=1}^{r} k_i \omega_i \tag{5}$$

with integers $k_i \geq 0$.

$\omega \in \phi$ is called a positive root of A, ϕ is the positive root space of A and $\{\omega_i\}$ is called a root basis.

Theorem I is needed in order to classify the unitary representations of compact Lie algebras. Let V be a complex Hilbert space of finite dimension. A linear map $D: A \to L(V)$, (the algebra of linear transformations on V), is called a representation if

$$D([x,y]) = D(x)D(y) - D(y)D(x), \quad (x,y) \in A \ .$$

D is called unitary if, in addition, $D(x) = -D(x)^*$. D is irreducible if it contains no invariant subspace. Two unitary representations D and D' are equivalent if there exists an isometry $U: V \to V'$ such that $UD(x) = D'(x)U$ for all $x \in A$.

Theorem II: Let D be unitary. Then V splits into orthogonal subspaces V_i, $V = \oplus V_i$ such that D restricted to V_i is irreducible.

This shows that D is equivalent to a direct sum of irreducible representations.

Theorem III (classification of unitary representations): Let D be a unitary irreducible representation in V. There is a nonzero maximal vector $x \in V$, which is uniquely determined up to scalar multiplication,

together with a unique linear function $\Lambda \in H^*$, called the maximal weight of D, such that the equations

$$(D(A_\omega) - iD(B_\omega))x = 0 \tag{6}$$

$$D(h)x = i\Lambda(h)x \tag{7}$$

hold for all $\omega \in \phi$, $h \in H$. Two irreducible representations are equivalent if and only if their maximal weights coincide.
All unitary representations are, therefore, uniquely classified by maximal weights $\Lambda \in H^*$. The following theorem states how the maximal weights look like:
Theorem IV (maximal weights): Set $r = \dim H$. Define the so-called fundamental weights f_i, $i = 1,\ldots,r$ by the formula

$$2\langle f_i, \omega_j \rangle_{H^*} / \langle \omega_j, \omega_j \rangle_{H^*} = \delta_{ij} \ . \tag{8}$$

Then each maximal weight Λ can be written in the form

$$\Lambda = \sum_{i=1}^{r} m_i f_i \tag{9}$$

with integers $m_i \geq 0$. Moreover, to each r-tuple of non-negative integers m_i belongs a unitary irreducible representation of A with a maximal weight given by the last formula.

The last two theorems classify the unitary, irreducible representations D of A completely up to equivalence in terms of their maximal weights. Hence it is to be expected that any property of D which is invariant under equivalence transformation can be expressed in terms of the maximal weight. One such quantity is certainly the dimension of V. Indeed the following equation holds:

$$\dim V = \prod_{\omega \in \phi} \langle \Lambda + \delta, \omega \rangle_{H^*} / \langle \delta, \omega \rangle_{H^*} \tag{10}$$

with $\delta = \frac{1}{2} \sum_{\omega \in \phi} \omega$.

Another quantity of this type is the eigenvalue $\lambda(D)$ of the Casimir-operator C(D) of D. This operator can be defined for any unitary representation D by the formula:

$$C(D) = \sum_{i,j=1} \gamma_{ij} D(C_i) D(C_j) + \sum_{\omega \in \phi} (D(A_\omega)^2 + D(B_\omega)^2) \qquad (11)$$

It commutes with all element $D(x)$, $x \in A$; hence, by Schur's lemma, it must be proportional to the identity if D is irreducible, i.e.

$$C(D) = \lambda(D) 1 \ .$$

In terms of Λ we have the formula

$$\lambda(D) = -\langle \Lambda + 2\delta, \Lambda \rangle_{H^*} \ . \qquad (12)$$

3. The Lie algebra of SU(n)

In this section we specify A to be the Lie algebra of SU(n), i.e. the Lie algebra of antihermitean, trace-free matrices d. In this case the Killing form $\langle , \rangle$ can be expressed in terms of an ordinary matrix trace; more precisely if d,d' are antihermitean matrices, then

$$\langle d, d' \rangle_A = -2n \ \mathrm{tr} \ dd' \ . \qquad (13)$$

This formula shows that $\langle , \rangle_A$ is indeed positive definite, whence A is compact. As a basis for the Cartan algebra H we can choose the diagonal matrices C_k, $(k = 1, \ldots, n-1)$, whose matrix elements $C_{k;\alpha\beta}$ read as follows:

$$C_{k;\alpha\beta} = i(\delta_{k\alpha}\delta_{k\beta} - \frac{1}{n} \delta_{\alpha\beta}) \ . \qquad (14)$$

The matrix γ_{ij} in equ. (3) can then be computed:

$$\gamma_{ij} = \frac{1}{2n} (\delta_{ij} + 1) \ , \qquad (15)$$

whence, according to equ. (3)

$$\langle f, g \rangle_{H^*} = \frac{1}{2n} \sum_{i=1}^{n-1} f(C_i) g(C_i) + \frac{1}{2n} \left(\sum_{i=1}^{n-1} f(C_i) \right) \left(\sum_{j=1}^{n-1} g(C_j) \right) \qquad (16)$$

for all $f, g \in H^*$.

Next we consider the antihermitean, trace-free matrices A_{kl}, B_{kl}; $(1\leq k<l\leq n)$, whose matrix elements $A_{kl;\alpha\beta}$, $B_{kl;\alpha\beta}$ are defined as follows:

$$A_{kl;\alpha\beta} = \frac{1}{2\sqrt{n}} (\delta_{k\alpha}\delta_{l\beta} - \delta_{l\alpha}\delta_{k\beta}) , \tag{17}$$

$$B_{kl;\alpha\beta} = \frac{i}{2\sqrt{n}} (\delta_{k\alpha}\delta_{l\beta} + \delta_{l\alpha}\delta_{k\beta}) . \tag{18}$$

The matrices A_{kl} and B_{kl} form indeed an orthonormal basis of H. Moreover, if we define the linear functions $\omega_{kl} \in H^*$ by the equation

$$\omega_{kl}(C_m) = \delta_{km} - \delta_{lm} , \quad (1\leq k<l\leq n,\ 1\leq m<n) \tag{19}$$

then it is easily checked that the quantities A_{kl}, B_{kl}, ω_{kl} satisfy equ. (4). It follows that the set ω_{kl} is the space of positive roots for the Lie algebra of SU(n). We now set

$$\omega_i = \omega_{i,i+1} \tag{20}$$

Evidently, for $1\leq k<l\leq n$,

$$\omega_{kl} = \sum_{i=k}^{l-1} \omega_{i,i+1} = \sum_{i=k}^{l-1} \omega_i ,$$

which proves that ω_i is in fact a root basis. We can now insert these ω_i and the expression for $\langle , \rangle_{H^*}$ given by equ. (16) into the equation (8) in order to determine the fundamental weights f_k. The result is

$$f_i(C_k) = \begin{cases} -i/n & i<k , \\ (n-i)/n & i\geq k . \end{cases} \tag{21}$$

Next let D be a unitary, irreducible representation of the Lie algebra of SU(n). Let the maximal weight be

$$\Lambda = \sum_{i=1}^{n-1} k_i f_i .$$

It follows from equ. (21) that

$$\Lambda(C_k) = -\frac{1}{n} \sum_{i=1}^{n-1} i k_i + \sum_{i=k}^{n-1} k_i . \tag{22}$$

The equations (19) and (22) together with the expression (16) for $\langle , \rangle_{H^*}$ can now be inserted into the general formulas (10) and (11) for

dim V and the eigenvalue $\lambda(D)$ of the Casimir operator. We find the results

$$\dim V = \prod_{1\leq i<j\leq n} (j-i + \sum_{m=i}^{j-1} k_m)/(j-i) \ . \tag{23}$$

$$\lambda(D) = -\frac{1}{2n}\left[\sum_{i=1}^{n-1} i(n-i)k_i(1+k_i/n) + \frac{2}{n}\sum_{1\leq i<j\leq n-1} k_i k_j i(n-j)\right] \tag{24}$$

These formulas can now be applied to the representation introduced in section 3 of part I. Recall that this representation was defined by assigning to each trace-free, antihermitean matrix d with matrix elements $d_{\alpha\beta}$ the operator $\sum_{\alpha,\beta=1}^{n}\sum_{r=1}^{3} d_{\alpha\beta} a^+_{r\alpha} a_{r\beta}$. Hence we have to set $D(d) = \hat{d}$. Using the formulas (15), (17) and (18) we find the identities:

$$D(C_\mu) = i\Big(\sum_{r=1}^{3} a^+_{r\mu} a_{r\mu} - \frac{1}{n} N_{op}\Big), \quad (\mu = 1,\ldots,n-1) \tag{25}$$

$$D(A_{\mu\nu}) = \sum_{r=1}^{3} (a^+_{r\mu} a_{r\nu} - a^+_{r\nu} a_{r\mu})/2\sqrt{n} \ , \quad (1\leq\mu<\nu\leq n) \tag{26}$$

$$D(B_{\mu\nu}) = i\sum_{r=1}^{3} (a^+_{r\mu} a_{r\nu} + a^+_{r\nu} a_{r\mu})/2\sqrt{n}, \quad (1\leq\mu<\nu\leq n) \tag{27}$$

A vector x is maximal, if according to equation (6),

$$(D(A_{\mu\nu}) - iD(B_{\mu\nu}))x = 0 \ .$$

From equations (26) and (2) we find that

$$D(A_{\mu\nu}) - iD(B_{\mu\nu}) = \frac{1}{\sqrt{n}} \sum_{r=1}^{3} a^+_{r\mu} a_{r\nu} \ . \tag{28}$$

Consider now the vector (compare section 3 of part I for the notation),

$$x = \prod_{\alpha=1}^{N} B^+(\alpha,\alpha,\alpha)|0\rangle = \prod_{\alpha=1}^{N} (\varepsilon_{rst} a^+_{r\alpha} a^+_{s\alpha} a^+_{t\alpha})|0\rangle \ ,$$

which is certainly in the colourfree 3N-quark space V_N considered in section 3 of part I. With the help of (28) it is easy to check that x is indeed maximal. Since we know that the representation D acts irreducibly on the colour-free states, theorem III tells us that it must be the only maximal vector (up to a scalar factor) in that space. With the help of (25) we next compute that

$$D(C_\mu)x = i\lambda_\mu x$$

with

$$\lambda_\mu = \begin{cases} 3(1-N/n), & \mu \leq N \ , \\ -3N/n \quad , & \mu > N \ : \end{cases}$$

Comparing this with equation (22) we find that

$$\lambda_\mu = 3f_N(C_\mu) \ ;$$

hence $3f_N$ is the maximal weight of the SU(n)-representation in the space V_N. In order to compute the dimension of V_N, we can now use formula (23) with $k_N = 3$ and $k_m = 0$ otherwise. The result turns out to be

$$\dim V = 2\binom{n}{N}\binom{n+1}{N}\binom{n+2}{N}/(N+1)^2(N+2) \ . \tag{29}$$

We consider now the Casimir operator of D. After inserting the formulas (15), (25), (26) and (27) into equation (11) we find the simple formula

$$C(D) = -\frac{1}{2n}\Big(\sum_{\alpha,\beta=1}^{n}\ \sum_{r,s=1}^{3} a^+_{r\alpha}a_{r\beta}a^+_{s\beta}a_{s\alpha} - \frac{1}{n}N^2_{op}\Big) \tag{30}$$

On the other hand, the eigenvalue $\lambda(D)$ in the space V_N of colourfree 3N-quark states can now be computed by using formula (24) with $k_N = 3$ and $k_m = 0$ otherwise. It follows that

$$\lambda(D) = -3(1+3/n)N(n-N)/2n \tag{31}$$

4. The Lie algebra of SO(n)

Throughout this section we consider only the case that n is even i.e. $n = 2m$. The Lie algebra of SO(n) can be defined as the set of trace-free antihermitean matrices d whose matrix elements $d_{\alpha\beta}$ satisfy the additional relation

$$d_{\alpha\gamma}g_{\gamma\beta} + d_{\beta\gamma}g_{\alpha\gamma} = 0 \ , \tag{32}$$

where $g_{\alpha\beta}$ is a symmetric matrix with

$$g_{\alpha\beta} = \pm\delta_{n+1-\alpha,\beta} \ . \tag{33}$$

The Killing form can be written as follows:

$$<d,d'>_A = -(n-2)\,\mathrm{tr}dd' \ .$$

A basis $\tilde{C}_i$ for the Cartan algebra H is given by (see equ. (14)),

$$\tilde{C}_i = C_i - C_{n+1-i}, \quad (i = 1,\ldots,m) \ . \tag{34}$$

It follows that

$$\gamma_{ij} = \delta_{ij}/2(n-2) \ . \tag{35}$$

Hence the formula

$$<f,g>_{H^*} = \frac{1}{2(n-2)} \sum_{i=1}^{m} f(\tilde{C}_i)\,g(\tilde{C}_i) \tag{36}$$

yields the scalar product for arbitrary linear functions $f,g\in H^*$. Next we define an orthonormal basis of H by the elements, $(1\leq k<m,\ k<l<n+1-k)$,

$$\tilde{A}_{kl} = (A_{kl} - \sum_{p,q} g_{lp}g_{kq}A_{pq})\cdot(\frac{n}{n-2})^{1/2}$$

$$\tilde{B}_{kl} = (B_{kl} - \sum_{p,q} g_{lp}g_{kq}B_{pq})\cdot(\frac{n}{n-2})^{1/2} \ , \tag{37}$$

and check that these quantities, together with the linear functions $\tilde{\omega}_{kl}$, $(1\leq k<m,\ k<l<n+1-k)$, given by

$$\tilde{\omega}_{kl}(\tilde{C}_i) = \delta_{ki} - \delta_{li} + \delta_{l,n+1-i} \ , \tag{38}$$

satisfy equ. (4). Therefore, the set $\{\tilde{\omega}_{kl}\}$ defines the positive roots of the Lie algebra of SO(n). A root basis $\tilde{\omega}_i$ is given by

$$\tilde{\omega}_i = \tilde{\omega}_{i,i+1} \ , \qquad i = 1,\ldots,m-1$$

$$\tilde{\omega}_m = \tilde{\omega}_{m-1,m+1} \ , \tag{39}$$

and the fundamental weights $\tilde{f}_i$ can now be computed by inserting the expressions (39), (38) and (36) into formula (8). We obtain in this

way the coefficients

$$\tilde{f}_i(\tilde{C}_k) = \theta_{ik} - \frac{1}{2}(\delta_{i,m} + \delta_{i,m-1}) , \tag{40}$$

with

$$\theta_{ik} = \begin{matrix} 0 & i<k \\ 1 & i\geq k \end{matrix} . \tag{41}$$

Next let D be an irreducible representation of SO(n) with maximal weight

$$\Lambda = \sum_{i=1}^{m} k_i \tilde{f}_i \tag{42}$$

From the last equations we conclude that

$$\Lambda(\tilde{C}_k) = \sum_{i\geq k}^{m} k_i - (k_m+k_{m-1})/2 \tag{43}$$

The eigenvalue $\lambda(D)$ of the Casimiroperator and the dimension of the representation D can now be computed by inserting (43), (38) and (36) into the general formulas (10) and (12). We obtain in this way the formulas:

$$\dim V = \prod_{1\leq i<j\leq m} (j-i + \sum_{r=i}^{j-1} k_r)/(j-i) \tag{44}$$

$$\cdot \prod_{\substack{1\leq i<m \\ m<j<n+1-i}} (j-i-1 + \sum_{r=i}^{m} k_r + \sum_{r=n+1-j}^{m} k_r - k_m - k_{m-1})/(j-i-1),$$

$$\lambda(D) = -\frac{1}{2(n-2)} \Big[\sum_{i=1}^{m} ik_i(k_i-k_m-k_{m-1}+n-i-1) + 2 \sum_{1\leq i<j\leq m} ik_ik_j \tag{45}$$

$$+ m(k_m+k_{m-1})(k_m+k_{m-1}-2m+2)/4 \Big] .$$

We now apply these general formulas to the particular representation D of the Lie algebra of SU(n) in the space V_N of colourfree N quark--states. This representation restricts to a representation of SO(n), which is reducible but, according to theorem II splits into irreducible components. Our aim is to determine these components together with their maximal weights. In order to make contact with the physical application, we note that the metric $g_{\alpha\beta}$ in formula (32) has to be identified with the expression given in section 4 of part I. We begin

by writing down the formulas for $D(\tilde{C}_\alpha)$, $D(\tilde{A}_{\alpha\beta})$, $D(\tilde{B}_{\alpha\beta})$:

$$D(\tilde{C}_\alpha) = \sum_{r=1}^{3} a^+_{r\alpha}a_{r\alpha} - \sum_{r=1}^{3}\sum_{\mu,\nu} g_{\alpha\mu}g_{\alpha\nu}a^+_{r\mu}a_{r\nu}\ , \quad (\alpha = 1,\ldots,m) \tag{46}$$

$$D(\tilde{A}_{\alpha\beta}) = [\sum_{r=1}^{3} (a^+_{r\alpha}a_{r\beta} - a^+_{r\beta}a_{r\alpha} - \sum_{r=1}^{3}\sum_{\mu,\nu} g_{\beta\mu}g_{\alpha\nu}(a^+_{r\mu}a_{r\nu} - a^+_{r\nu}a_{r\mu})]/2\sqrt{n-2}$$

$$D(\tilde{B}_{\alpha\beta}) = i[\sum_{r=1}^{3} a^+_{r\alpha}a_{r\beta} + a^+_{r\beta}a_{r\alpha} - \sum_{r=1}^{3}\sum_{\mu,\nu} g_{\beta\mu}g_{\alpha\nu}(a^+_{r\mu}a_{r\nu} + a^+_{r\nu}a_{r\mu})]/2\sqrt{n-2}$$

$(1\leq\alpha<m;\ \alpha<\beta<n+1-\alpha)$.

According to equation (6) a vector x is maximal, if for all these indices α,β.

$$(D(\tilde{A}_{\alpha\beta}) - iD(\tilde{B}_{\alpha\beta}))x = 0\ .$$

From the equations (46) we conclude that in our example:

$$D(\tilde{A}_{\alpha\beta}) - iD(\tilde{B}_{\alpha\beta}) = (\sum_{r=1}^{3} a^+_{r\alpha}a_{r\beta} - \sum_{r=1}^{3} g_{\beta\mu}g_{\alpha\nu}a^+_{r\mu}a_{r\nu})/\sqrt{n-2} \tag{47}$$

Consider now the vectors $x_l \in V_N$ defined by $(l = 0,\ldots, k = \min(N,n-N))$

$$x_l = (\prod_{\alpha=N-l+1}^{N} B^+(\alpha,\alpha,\alpha) \prod_{B=1}^{N-l} C^+_\beta |0>. \tag{48}$$

(Compare sections 3 and 6 of part I for the notation; note that $x_l = 0$ for all $l>n-N$ if $N>m$.) By construction $x_l \in G_l$, the SO(n)-invariant subspace introduced in section 6 of part I. Moreover, it is easily checked with the help of equ. (47) that x_l is in fact maximal. If $N = m$ an additional set of maximal vectors $x_l \in G_l$ is found:

$$\tilde{x}_l = B^+(N+1,N+1,N+1) \prod_{\alpha=N-l+1}^{N-1} B^+(\alpha,\alpha,\alpha) \prod_{\beta=1}^{N-l} C^+_\beta |0> \tag{49}$$

for $l\geq 1$ and

$$\tilde{x}_0 = C^+_{N+1} \prod_{\beta=1}^{N-1} C^+_\beta |0> \tag{50}$$

By applying $D(\tilde{C}_\alpha)$, as given by equ. (4), to each of these vectors we find that the maximal weights Λ_l corresponding to the vectors x_l are given by the list:

$$\Lambda_l = \tilde{f}_k + 2\tilde{f}_l \quad , \quad (l \leqq k < m-1) \ ,$$

$$\Lambda_l = \tilde{f}_{m-1} + \tilde{f}_m + 2\tilde{f}_l \quad , \quad (l < k = m-1) \ ,$$

$$\Lambda_l = 3(\tilde{f}_{m-1} + \tilde{f}_m) \quad , \quad (l = k = m-1) \ ,$$

$$\Lambda_l = 2(\tilde{f}_m + \tilde{f}_l) \quad , \quad (l < m-1, \ k = m) \ ,$$

$$\Lambda_l = 4\tilde{f}_m + 2\tilde{f}_{m-1} \quad , \quad (l = m-1, \ k = m) \ ,$$

$$\Lambda_l = 6\tilde{f}_m \quad , \quad (l = k = m) \ , \tag{51}$$

whereas the maximal vectors $\tilde{x}_l$ given by (49) and (50) correspond to

$$\tilde{\Lambda}_l = 2\tilde{f}_{m-1} + 2\tilde{f}_l \quad , \quad (l < m-1) \ ,$$

$$\tilde{\Lambda}_l = 4\tilde{f}_{m-1} + 2\tilde{f}_m \quad , \quad (l = m-1) \ ,$$

$$\tilde{\Lambda}_l = 6\tilde{f}_{m-1} \quad , \quad (l = m) \ . \tag{52}$$

From this list we can immediately read off the coefficients k_i entering into the general formulas (44) and (45) for the eigenvalues λ_l of the Casimiroperator and the dimension d_l of the irreducible subspace, to which x_l belongs. The actual computation of these numbers gives the following result:

$$\lambda_l = -\,[l(3+n-l) + N(n-N)/2]/(n-2) \tag{53}$$

$$d_l = \binom{n+2}{N}\binom{n+4}{l}\binom{n+4}{l+1}\,\frac{(N+1-l)(N+2-l)(n+1-N-l)(n+2-N-l)(n+3-2l)}{(N+1)(N+2)(n+1)(n+2)(n+3)(n+4)} \tag{54}$$

These formulas are also valid for the weights $\tilde{\Lambda}_l$ in the exceptional case N = m. The Casimiroperator itself can be rewritten in the form:

$$C(D) = -\frac{1}{2(n-2)}\Big(\sum_{r,s=1}^{3}\sum_{\alpha,\beta=1}^{n} a^{+}_{r\alpha}a_{r\beta}a^{+}_{s\beta}a_{s\alpha} - \sum_{r,s=1}^{3}\sum_{\alpha,\beta,\mu,\nu=1}^{n} a^{+}_{r\alpha}a_{r\beta}a^{+}_{s\mu}a_{s\nu}\, g_{\mu\alpha}\cdot g_{\nu\beta}\Big) \tag{55}$$

which is obtained after inserting the expressions (46) and (35) into formula (11).

Up to now we are not sure that all maximal weights and vectors are indeed found. We have by section 6 of part I an orthogonal, SO(n)--invariant decomposition $V_N = \bigoplus_{l=0}^{N} G_l$. In each subspace G_l we have found one maximal vector x_l when N<m, two maximal vectors x_l and $\tilde{x}_l$ when N = m. When N>m, we have found one maximal vector in G_l, if $l \leq N-m$ and no maximal vector when l>N-m. Let us now assume that, in the last case, G_l is in fact zero for l>N-m, that each G_l is in fact irreducible for N ≠ m and splits into two irreducible components for N = m. This conjecture is equivalent to the statement that the formulas (48), (49) and (40) give all the maximal vectors in V_N, because theorem III forbids any more candidates. On the other hand the conjecture is true if and only if the sum of the dimensions d_l associated to the maximal weights λ_l equals the dimension of V_N, i.e. we must check that

$$\dim V_N = \sum_{l=0}^{k} d_l$$

for N ≠ m, and

$$\dim V_N = 2 \sum_{l=0}^{k} d_l$$

for N = m. By using the formulas (54) and (29) for d_l and dim V_N, these identities can indeed be verified.

We conclude this section by rewriting the pairing force P of part I as a sum of Casimir operators for SU(n) and SO(n). From section 4 of part I we have the expression

$$P = -\frac{g}{4}\Big(\sum_{i,k,l,k',l'=1}^{3} \sum_{\alpha,\beta,\mu,\nu=1}^{n} \varepsilon_{ikl}\varepsilon_{ik'l'} g_{\alpha\beta} g_{\mu\nu} a^{+}_{l'\nu} a^{+}_{k'\mu} a_{k\alpha} a_{l\beta}\Big)$$

By using the identity $\sum_i \varepsilon_{ikl}\varepsilon_{ik'l'} = \delta_{kk'}\delta_{ll'} - \delta_{kl'}\delta_{lk'}$, we rewrite P in the form

$$P = -\frac{g}{2}\Big(\sum_{k,l=1}^{3} \sum_{\alpha,\beta,\mu,\nu=1}^{n} g_{\alpha\beta} g_{\mu\nu} a^{+}_{l\nu} a_{l\beta} a^{+}_{k\mu} a_{k\alpha} - N_{op}\Big)$$

By comparing this with the expressions (55) and (30) for the Casimir-operators C_1 and C_2 of SU(n) and SO(n), respectively, we see that in fact

$$P = -g[(n-2)C_2 - nC_1 + \frac{1}{2n} N_{op}^2 - \frac{1}{2} N_{op}] \qquad (56)$$

It follows that all vectors in $G_1 \subset V_N$ are eigenvectors of P; the eigenvalue $\lambda(P)$ is obtained by replacing in (56) N_{op} by N and C_1 and C_2 by their eigenvalues (31) and (53), respectively. It follows that

$$\lambda(P) = -g[N(n+3-N) - l(l+3-l)] \; . \qquad (57)$$

This is the result which we have anticipated in part I, when diagonalizing our quark Hamiltonian. The mathematical aspects of our model are now completely described.

References

1) K. Bleuler, H. Hofestädt, S. Merk and H.R. Petry, Z. Naturforsch. 38a, (1983) 705
2) A. Chodos, R.L. Jaffee, K. Johnson, C.B. Thorn and V.F. Weisskopf, Phys. Rev. D9, (1974), 3471
3) H. Weyl, "The theory of groups and quantum mechanics" Dover Publications (1931)
4) D.E. Littlewood, "The theory of group characters", Oxford University Press (1970)
5) N. Jacobson, "Lie algebras", J. Wiley and Sons (1965)
6) H. Weyl, "The classical groups", Princeton University Press (1945)