PART I

MAGNETIC AND OPTICAL SYSTEMS

EQUATIONS OF MOTION FOR VORTICES IN 2-D EASY-PLANE MAGNETS

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The dynamics of individual and pairs of vortices in a classical easy-plane Heisenberg spin model is studied. There are two types of vortices possible: in-plane, with small out-of-plane spin components present only at nonzero velocity, and out-of-plane, with large out-of-plane spin components even when at rest. As a result, the two types are governed by different equations of motion when in the presence of neighboring vortices. We review the static spin configurations and the changes due to nonzero velocity. An equation of motion introduced by Thiele and used by Huber will be re-examined. However, that equation may be inadequate to describe vortices in the XY model, due to their zero gyrovector. An alternative dynamical equation is developed, and effective mass and dissipation tensors are defined. These are relevant for models with spatially anisotropic coupling in combination with easy-plane spin exchange.

INTRODUCTION

A model for the dynamic correlations of vortices in easy-plane two-dimensional magnets has been presented, that uses the idea of an ideal gas of weakly interacting vortices.[1] Assuming a Boltmann velocity distribution, and if the velocity-dependent spin field of the vortices is known, then the dynamic structure function $S^{\alpha\alpha}(\vec{q},\omega)$ can be determined. At the microscopic level we would like to investigate the time-dependent motion of a single vortex, to understand how the neighboring vortices cause forces and accelerations, and to have a clear picture of how equilibrium is achieved.

Huber[2,3] has done such an analysis for diffusive motion of so-called "out-of-plane" vortices, ones that possess large out-of-easy-plane spin components. However, it is now realized that there are two type of vortices possible,[4,5] depending on the strength of the easy-plane anisotropy.[6,7] The stable vortices of the XY model, for example, are so-called "planar" vortices that only have small out-of-plane spin components. In that case the equation of motion that was used[3,8] is found to be inapplicable because these planar vortices have a zero gyrovector, to be discussed below. Here we propose an alternative dynamic equation of motion that applies to both types of vortices.

We begin by summarizing the properties of the two types of vortices allowed in the easy-plane anisotropic ferromagnetic Heisenberg model. The derivation of the equation of motion introduced by Thiele,[8] in terms of conserved force densities, will be sketched out, and the breakdown for planar vortices will be discussed. An alternative formalism using a canonical momentum for the vortex is developed. The new equation of motion includes the effects of vortex shape changes that are the result of acceleration. This leads to definition of an effective mass tensor, and, the gyrovector also re-appears. The new equation allows for a consistent description of both types of vortices.

Anisotropic Heisenberg Ferromagnet

The model system is the nearest neighbor 2D Heisenberg ferromagnet with easy-plane anisotropic exchange, characterized by a parameter $0 \leq \lambda < 1$; the Hamiltonian is

$$\mathbf{H} = -J\sum_{ij}\left(S_i^x S_j^x + S_i^y S_j^y + \lambda S_i^z S_j^z\right) . \tag{1}$$

J is an energy scale and the $\vec{S}_i$ are classical spins with fixed length. The XY model is given when $\lambda=0$, the Heisenberg model when $\lambda=1$. The spin dynamics is described by the Landau-Lifshitz equation,[9,10]

$$\dot{\vec{S}}_i = \{\vec{S}_i, H\} - \alpha\, \vec{S}_i \times \dot{\vec{S}}_i = \vec{S}_i \times \left(\vec{H}_i - \alpha\dot{\vec{S}}_i\right) \tag{2a}$$

$$\vec{H}_i = J\sum_{(ij)}\left(S_j^x\hat{x} + S_j^y\hat{y} + \lambda S_j^z\hat{z}\right) \tag{2b}$$

The sum is only over the neighbors of site i. A Landau-Gilbert damping term of strength α has been included. At any given time each spin is instantaneously precessing about the effective field $(\vec{H}_i - \alpha\dot{\vec{S}}_i)$. Initially the vortices will be described in the absence of damping, $\alpha=0$, which can be later re-introduced at the phenomenological level.

Static Vortices

The spins are parametrized in terms of an in-plane angle $\phi(\vec{x},t)$ and an out-of-plane angle $\theta(\vec{x},t)$ (or we use $S^z = S\sin\theta$),

$$\vec{S}(\vec{x},t) = S(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta) . \tag{3}$$

Then in a continuum limit including terms up to 2nd order in gradients the equations of motion are[4,5,7]

$$\dot{\theta} = JS\,(\cos\theta\, \nabla^2\phi - 2\sin\theta\, \nabla\theta\cdot\nabla\phi) \tag{4a}$$

$$\dot{\phi}\cos\theta = -\frac{1}{2} JS\,\{[\delta(|\vec{\nabla}\theta|^2 - 4) + |\vec{\nabla}\phi|^2]\sin 2\theta + 2(1-\delta\cos^2\theta)\nabla^2\theta\} \tag{4b}$$

where $\delta \equiv 1-\lambda$. Using polar coordinates (r,φ), and assuming a spatially isotropic solution, $\phi=\phi(\varphi)$, while $\theta=\theta(r)$, static vortices always have an in-plane angle satisfying Laplace's equation,

$$\phi(\vec{x}) = q\varphi + \phi_0 = q \tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) + \phi_0 . \tag{5}$$

The charge q is an integer, and ϕ_0 is a constant of integration.

The two types of vortices are distinguished by the out-of-plane angle θ. The static planar vortices have an out-of-plane angle that is zero, $\theta=0$. This solution exists independent of the anisotropy λ. However, when placed on a lattice,[6,7] it is found to be unstable for $\lambda>\lambda_c$, where λ_c is a critical anisotropy depending on the lattice ($\lambda_c=0.72$ for a square lattice). Planar vortices are stable only for $\lambda<\lambda_c$.

Out-of-plane vortices have a nonzero out-of-plane angle with asymptotic behavior[5,7]

$$\sin\theta \sim \begin{cases} p(1-ar^2) & \text{as } r\to 0 \\ \sqrt{\frac{r_v}{r}}\, e^{-r/r_v} & \text{as } r\to\infty \end{cases} \tag{6a}$$

$$r_v = \frac{1}{2}\sqrt{\frac{\lambda}{1-\lambda}}, \tag{6b}$$

where r_v is a characteristic vortex radius and a is a constant. The charge p is ±1 which determines whether the spin at the vortex center points along $+\hat{z}$ or $-\hat{z}$. When this solution is placed on a lattice,[6,7] it is found to be unstable for $\lambda<\lambda_c$ and stable only for $\lambda>\lambda_c$. Thus we have a situation where either planar ($\lambda<\lambda_c$) or out-of-plane ($\lambda>\lambda_c$) vortices are stable, and we expect that the dynamics may also reflect this as a crossover point in other quantities.

Dynamic Vortices

The equilibrium correlations between vortices can be found in an ideal gas phenomenology using the known spin profiles given above.[1] However, for correlations of the z spin components, for $\lambda<\lambda_c$, static planar vortices can contribute nothing to $S^{zz}(\vec{q},\omega)$. Then the lowest order vortex contribution must come from moving planar vortices, which do have nonzero S^z components. One can determine the perturbation due to a constant velocity $\vec{v}$ by assuming a solution $\vec{S}(\vec{x}-\vec{v}t)$. For planar vortices, with $\lambda<<1$, to first order in $\vec{v}$ we have no change in ϕ. The change in θ is given by[6,7]

$$\sin\theta = \frac{-\vec{v}\cdot\vec{\nabla}\phi}{JS(4\delta-|\nabla\phi|^2)} = \frac{q}{4JSr^2}(\vec{v}\times\vec{r})_z \qquad (7)$$

in the asymptotic $r\to\infty$ regime, and $\vec{r}$ is measured from the instantaneous center position of the vortex. A similar change in $\sin\theta$ occurs for moving out-of-plane vortices, but it is small compared to the large out-of-plane profile already present in the static out-of-plane vortex.

Thiele's Equation of Motion

We review Thiele's vortex equation of motion[8] and the definition of the gyrovector, which vanishes for planar vortices. The equation is based on an interesting force-density interpretation of the Landau-Lifshitz equation, first rewritten in equivalent form,

$$\vec{S}\times\vec{H}_{net} = 0, \qquad (8a)$$

$$\vec{H}_{net} = \vec{S}\times\dot{\vec{S}} + \vec{H} - \alpha\,\dot{\vec{S}}\,. \qquad (8b)$$

$\vec{H}$ is analogous to that in Eq.(2b), representing the effective local field from neighboring spins. The other terms in $\vec{H}_{net}$ are dynamic and damping terms, respectively. In this notation the dynamics is "simple," in that each spin remains parallel to its instantaneous local field $\vec{H}_{net}$. Thus we could write $\vec{S}=\beta\vec{H}_{net}$ where $\beta(\vec{x},t) = S^2/(\vec{S}\cdot\vec{H})$. Combinations of $\vec{H}_{net}$ with gradients of $\vec{S}$ have dimensions of force per unit volume. Applying the operator $\cdot\, \partial_j\vec{S}\ \hat{e}_j \equiv \cdot\vec{\nabla}\vec{S}$, (sum over repeated indices j=1,2) and realizing $\vec{S}\cdot\vec{\nabla}\vec{S} = 0$, there results the statement of conserved force density,

$$\vec{H}_{net}\cdot\vec{\nabla}\vec{S} = \left(\vec{H} + \vec{S}\times\dot{\vec{S}} - \alpha\dot{\vec{S}}\right)\cdot\vec{\nabla}\vec{S} = 0 \tag{9}$$

where the contraction is over spin components.

To apply this to a vortex we assume a travelling wave $\vec{S}(\vec{x} - \vec{v}t)$, and rewrite time derivatives using $\dot{\vec{S}} = -v_k\partial_k\vec{S}$.

There results

$$\vec{H}\cdot\vec{\nabla}\vec{S} + \vec{S}\cdot\left(\partial_j\vec{S}\times\partial_k\vec{S}\right)\hat{e}_j v_k + \alpha\left(\partial_j\vec{S}\right)\cdot\left(\partial_k\vec{S}\right)\hat{e}_j v_k = 0\,. \tag{10}$$

This then motivates the definition of the gyrovector $\vec{G}$,

$$G_i = -\frac{1}{2}\,\epsilon_{ijk}G_{jk}, \tag{11a}$$

$$G_{jk} \equiv -\int d^2x\ \vec{S}\cdot\left(\partial_j\vec{S}\times\partial_k\vec{S}\right) \tag{11b}$$

and the symmetric dissipation tensor $\tilde{D}$

$$D_{jk} = -\int d^2x\ \alpha\ \left(\partial_j\cdot\vec{S}\right)\cdot\left(\partial_k\vec{S}\right)\ . \tag{12}$$

The gyrovector is derived from an antisymmetric tensor G_{jk}. An equivalent expression for $\vec{G}$ is

$$\vec{G} = S^2\int d^2x\ \vec{\nabla}\phi\times\vec{\nabla}S^z. \tag{13}$$

The remaining term concerns reversible effects. It is taken to give the effective force acting on the vortex,

$$\vec{F} = -\int d^2x\ \vec{H}\cdot\vec{\nabla}\vec{S}. \tag{14}$$

Then the Thiele equation of motion is

$$\vec{F} + \vec{G} \times \vec{v} + \tilde{D} \cdot \vec{v} = 0 . \tag{15}$$

This equation can be used to describe the motion of out-of-plane vortices, for example, interacting in pairs, with a force $F=2\pi JS^2 q_1 q_2/r_{12}$. The gyrovector is found to be $\hat{G}=2\pi pq\hat{z}$. In the absence of damping, the pair will move in a circle if the gyrovector are parallel ($p_1q_1=p_2q_2$) or they will have a parallel translational motion if the gyrovectors are antiparallel ($p_1q_1=-p_2q_2$).

A problem arises for planar vortices. The gyrovector for static and moving planar vortices is zero.[11] This is due to the asymmetry of the S^z component about the direction of motion. The Thiele equation of motion becomes singular because the dynamic term $\vec{G}\times\vec{v}$ vanishes. This is most obvious with no damping, then the equation reads $\vec{F}=0$, which is not necessarily true for a vortex in the field of its neighbors. This leads to a conceptual difficulty in making a vortex ideal gas description for $\lambda<\lambda_c$.

Vortex Momentum

The problem seems to be that the analysis above does not allow for shape changes of a vortex in response to external fields as will occur for an accelerated vortex. That is, S^z for a planar vortex depends approximately linearly on its velocity.[6,7] If it accelerates it changes shape by developing more spin tilting out of the easy plane. On the other hand, the out-of-plane vortices have a large S^z component even at zero velocity so velocity-dependent changes in S^z may have a lesser effect.

An alternative viewpoint is to define a canonical momentum[12] $\vec{P}$ for the vortex, conjugate to its position $\vec{r}$, and then use the equation of motion $\dot{\vec{P}} = -\partial H/\partial \vec{r} = \vec{F}$. A Lagrangian that gives the correct spin-dynamics equations of motion is

$$\mathcal{L} = \int d^2x\, S^z\dot{\phi} - H \tag{16}$$

because S^z is the field momentum conjugate to ϕ. This then suggests that we take the definition of the vortex momentum to be

$$\vec{P} = -\int d^2x\, S^z\, \vec{\nabla}\phi \tag{17}$$

and then $\mathcal{L}=\vec{v}\cdot\vec{P}-H$ for a vortex of velocity $\vec{v}$. This definition is analogous to the canonical momentum developed for describing solitons in 1-D magnets[12,13,14] (generator of translations). For a planar vortex we get

$$\vec{P} = \frac{\pi}{4JS}\,\ln(R/a_o)\,\vec{v} \tag{18}$$

where R is a large distance cutoff (system size), and a_o is a short distance cutoff (≈ lattice constant). The effective mass seen here is identical to that found from the kinetic energy of the planar vortex.[7]

An equation of motion results by conserving momentum,[15] $\dot{\vec{P}} = \vec{F}$, and using

$$\dot{\vec{P}} = -\int d^2x\left(\dot{S}^z\,\vec{\nabla}\phi + S^z\,\vec{\nabla}\dot{\phi}\right). \tag{19}$$

The vortex shape is determined by its velocity $\vec{v} = \dot{\vec{r}}(t)$ as well as its position $\vec{r}(t)$, so we assume $\vec{S} = \vec{S}(\vec{x} - \vec{r}(t), \vec{v}(t))$. Therefore the time derivative is replaced by space $(\partial_j \equiv \partial/\partial x_j)$ and velocity $(\tilde{\partial}_j \equiv \partial/\partial v_j)$ gradients,

$$\frac{d}{dt} = -v_j\partial_j + a_j\tilde{\partial}_j\,. \tag{20}$$

In the absence of damping the total rate of change of momentum is

$$\dot{\vec{P}} = -\hat{e}_j K_{jk} v_k + \hat{e}_j M_{jk} a_k = -\tilde{K}\cdot\vec{v} + \tilde{M}\cdot\vec{a} \tag{21}$$

where $\tilde{K}$ and $\tilde{M}$ are an effective gyrovector and mass tensor,

$$K_{jk} = -\int d^2x\, \partial_k (S^z\, \partial_j \Phi) \tag{22a}$$

$$M_{jk} = -\int d^2x\, \tilde{\partial}_k (S^z\, \partial_j \Phi)\ . \tag{22b}$$

The generalized gyrovector $\tilde{K}$ is related to Thiele's gyrotensor $\tilde{G}$. We separate $\tilde{K}$ into a symmetric and an antisymmetric part,

$$K_{jk} = g_{jk} + L_{jk}, \tag{23a}$$

$$g_{jk} = \frac{1}{2}\,(K_{jk} - K_{kj}), \tag{23b}$$

$$L_{jk} = \frac{1}{2}\,(K_{jk} + K_{kj})\ . \tag{23c}$$

Then the equation of motion now becomes

$$\vec{F} = -\vec{g}\times\vec{v} - \tilde{L}\cdot\vec{v} + \tilde{M}\cdot\vec{a} \tag{24a}$$

where

$$\vec{g} = -\frac{1}{2}\epsilon_{ijk}\hat{e}_i g_{jk} = \frac{1}{2}\vec{G} \tag{24b}$$

$\vec{g}$, is the gyrovector, smaller by a factor of 1/2 from equation 11. Note that the origin must be excluded from the integrals giving $\tilde{K}$ and $\tilde{M}$. The symmetric tensor $\tilde{L}$ has no effect for vortices, it is zero for moving planar and out-of-plane vortices. The mass tensor $\tilde{M}$, determined by mixed space and velocity derivatives, exhibits the dynamic effect that changes in velocity cause changes in the internal structure of the vortex spin field. The force need not be parallel to $\vec{g}\times\vec{v}$ or $\vec{a}$.

For <u>planar</u> vortices, $\vec{g}=0$, $\tilde{L}=0$, $M_{ij}=(\pi/4JS)\ell n(R/a_o)\ \delta_{ij}$. Then the equation of motion is Newtonian,

$$\vec{F} = M\vec{a} \tag{25}$$

stating that a pair of interacting vortices move straight away or straight toward each other in the absence of damping. This behavior is seen in simulations. This is not qualitately different from Thiele's equation <u>with</u> damping, but this new approach remains applicable even when the damping is removed. In this way we now have a microscopic dynamics for the vortices in the XY model, or, whenever $\lambda<\lambda_c$.

For <u>out-of-plane</u> vortices, $\vec{g}=\pi pq\hat{z}$, $\tilde{L}=0$, and the mass is equivalent to the planar vortex mass. Then the equation of motion is

$$\vec{F} + \vec{g} \times \vec{v} = M\vec{a} \ . \tag{26}$$

We can consider the interaction of a pair of out-of-plane vortices, with force $F=A/r_{12}$, $A\equiv 2\pi JS^2 q_1 q_2$. If the gyrovectors are antiparallel ($p_1q_1=-p_2q_2$), then the motion is unaccelerated parallel translation as mentioned earlier. However, if the gyrovectors are parallel ($p_1q_1=p_2q_2$), the motion can be circular, but the angular frequency depends on whether the pair's interaction is repulsive ($A>0$) or attractive ($A<0$). For small mass ($AM/G^2 r^2_{12} \ll 1$) the frequency is found to be

$$\omega = \frac{A}{g r_{12}^2}\left(1 + \frac{AM}{g^2 r_{12}^2}\right) \ . \tag{27}$$

The repulsive interaction gives the larger frequency, and has the angular velocity $\vec{\omega}$ parallel to $\vec{g}$. The attractive interaction, with smaller frequency, has $\vec{\omega}$ antiparallel to $\vec{g}$. This new effect is related to a competition between an intrinsic vortex momentum (similar to $\vec{g}$) and the orbital angular momentum of the pair. For a given size of force transverse to the direction of motion, a larger acceleration occurs when the force is parallel to $\vec{g} \times \vec{v}$. The path of the vortex is more easily bent in the $\vec{g} \times \vec{v}$ direction than in the $-\vec{g} \times \vec{v}$ direction (i.e., left turns are easier than right turns for an "up" vortex, $\vec{g}=g\hat{z}$).

Conclusion

An equation of motion developed by Thiele[8] and Huber[2,3] is limited to cases where the vortex shape is fixed as a function of velocity, excluding application to the important XY model. An alternative equation of motion was developed here, based on finding the time rate of change of vortex momentum. The new equation alleviates the difficulty encountered when $\vec{G}=0$ (for planar vortices) and indicates new behavior for interacting out-of-plane vortices. The differences with this new equation of motion are summarized as follows: 1) an equation of motion exists for planar and out-of-plane vortices even for zero damping; 2) the new mass term accounts for velocity-dependent shape changes that result from acceleration; 3) a pair of out-of-plane vortices with parallel gyrovectors and $q_1=q_2$ (vortex-vortex) orbits faster than a pair with $q_1=-q_2$ (vortex-antivortex), all other things being equal. It should have important applications for microscopic vortex dynamics, especially for models with spatial anisotropy, whose vortices will possess an anisotropic mass tensor.

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