STOCHASTIC DYNAMICS OF SPATIAL SOLITONS ON THE PERIODIC INTERFACE OF TWO NONLINEAR MEDIA

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1.Introduction

Currently a problem of nonlinear electromagnetic wave propagation through a interface separating two dielectric media attracts much attention. It is caused by the possibilities to construct optical switchers ,scanners and other optical devices based on the applications of the beam and surface wave properties on the interface of nonlinear media.

In [1-4] a problem of interaction of plane self-focusing light channel in a self-focusing medium with cubic nonlinearity (of *spatial* soliton) with interface has been numerically and analytically considered.The main results of these papers are: there exist some regimes of complete internal reflection, trapping and transformation of a beam into nonlinear surface wave and beam passing depending upon the incident beam parameters.

The purpose of this work is to investigate nonlinear surface waves taking into account the periodic modulation of the interface.

2.Basic equations

Let us briefly consider a conclusion of the basic equations. The wave equation describing a propagation of monochromatic electromagnetic TE field in (x-z) plane takes the following form

$$\frac{\partial^2 E}{\partial z^2} + \frac{\partial^2 E}{\partial x^2} = -n^2 k^2 E \ , \qquad (1)$$

where $n^2 = n_l^2 + \alpha|E|^2 = \frac{\omega^2}{c^2k^2}[1 + 4\pi\chi^{(1)} + 4\pi\chi^{(3)}|E|^2]$

is the nonlinear refractive index of the medium, ω is the frequency of the electromagnetic field, βk is the wave number of the incident field in z, β is the waveguide mode, and $k = \omega/c$. $\chi^{(1)}$, $\chi^{(3)}$ are the linear and nonlinear responses respectively, characterizing the medium properties. We will consider a self-focusing nonlinearities when $\chi^{(3)} > 0$. The electromagnetic field is assumed to be almost monochromatic in x and z. Let

$$E = F(x,z)\, exp(i\beta kz). \qquad (2)$$

Substituting (2) into (1), making variable change $x' = kx$, $z' = kz$ and omitting a prime, one obtains

$$2i\beta \frac{\partial F}{\partial z} + \frac{\partial^2 F}{\partial x^2} - (\beta^2 - n^2)F = 0 \quad , \qquad (3)$$

where refractive index $n^2 = n_i^2 + \alpha_i|F|^2$ is discontinuous at the interface. Here the index $i = 0$ at $x < 0$ and $i = 1$ at $x > 0$. Following [1] in this work we assume that

$$\Delta = n_0^2 - n_1^2 > 0 \ , \ \alpha = \alpha_0/\alpha_1 \leqslant 1 \quad . \qquad (4)$$

Let a wave packet falls on the interface at the right side ($x > 0$). Make a variable change

$$F(x,z) = (2/\alpha_1)A(x,\tau)\, exp[i(\beta^2 - n_1^2)z/2\beta] \ , \tau = \beta k. \qquad (5)$$

Substituting (5) into (3) one obtains

$$i\frac{\partial A}{\partial \tau} + \frac{\partial^2 A}{\partial x^2} + 2|A|^2A = VA \quad , \qquad (6)$$

$$V = \begin{cases} 0 & , \quad x>0 \\ -\Delta - 2(\alpha - 1)|A|^2, & \quad x<0 \end{cases} .$$

The equation (6) is the nonlinear Schrodinger equation (NLS) with a perturbing term $V(x)A$ .At the interface absence $\alpha=1,\Delta=0$ it follows that $V = 0$. In this case the equation is exactly integrable and its general solution consists of solitons and nonsoliton component, which parameters are defined by an initial condition.

A self-focused wave channel at $x > 0$ is described by a solitonic solution which takes the following form when the perturbation is absent.

$$A(x,\tau) = 2\eta_1 \; sech[2\eta_1(x-\bar{x})] \; exp[i(\frac{vx}{2} + 2\sigma)] \quad . \tag{7}$$

It is clear, that in general case the equation (6) solution is a rather cumbersome problem but nevertheless, if the medium parameter changes are assumed to be small,i.e $\alpha^{-1} - 1 \ll 1$, $\Delta \ll 1$, the interface influence can be taken into account, considering that the soliton parameters, velocity v and amplitude η_1, change slowly with τ changing. The equation defining the soliton parameter dependence upon τ can be derived by application of the conservation laws or the perturbation theory for solitons [1-2,4]. The soliton amplitude appeared on adiabatic approximation not to depend on τ, i.e. $d\eta/d\tau = 0$, and the velocity changes are described by the following equation

$$\frac{d^2\bar{x}}{d\tau^2} = - \frac{dU}{d\bar{x}} \quad , \tag{8}$$

where

$$U = \Delta(1 - S_1^{-1}) tanh s + (\Delta/3S_1) \; tanh^3 s \; . \tag{9}$$

$$s = 2\eta_1\bar{x} \; . \; S_1 = [8(\eta_1/\Delta)^2(\alpha-1)]^{-1} \; ,$$

As seen from (8) the soliton center is moving as a particle in anharmonic potential U, and in this case the equation (8) is the Newton one for a particle, i.e we obtain a quasiparticle analog. While investigating the potential U properties one can completely describe the beam dynamics. In this case it is necessary to take into account the velocity v corresponding to the propagation angle Φ, $v = d\bar{x}/d\tau = 2\eta_1 \sin\Phi$, i.e. while considering trajectories with various initial velocities on a phase plane, indeed we investigate a problem of a beam on the at various fall angels on the interface.

3. Description of quasiparticle motion

Let us consider a potential U more detaily. First of all we start with critical point $\bar{x}_o$ where $dU/d\bar{x}=0$. With a help of calculation we find

$$2\eta_1 \bar{x}_o = \tanh^{-1}[(1-S_1)^{1/2}] ,$$

i.e. $\bar{x}_o$ exists only at $S_1 < 1$.
Let us calculate the second derivative from the potential in a point x_o,

$$\left.\frac{d^2U}{d\bar{x}^2}\right|_{\bar{x}=\bar{x}_o} = 8\Delta\eta_1^2 S_1 (1-S_1)^{1/2} > 0 ,$$

i.e. the point $\bar{x}_o$ is one of potential minimum. Hence, we have a conclusion on existence of stable stationary surface wave, which center is located in distance of $\bar{x}_o$ from the interface. At the initial velocities being not equal to 0, the soliton center will be a periodic function τ. A frequency of small oscillations w can be easily calculated by potential $U(\bar{x})$ expansion near the point x_o into degrees $(\bar{x} - \bar{x}_o)$ and preserving quadratic terms, we obtain

$$U(\bar{x}) \approx U(\bar{x}_o) + \frac{1}{2} \left.\frac{d^2U}{d\bar{x}^2}\right|_{\bar{x}=\bar{x}_o} (\bar{x}-\bar{x}_o)^2 .$$

If one denotes $\bar{x}-\bar{x}_o = y$, then the equation for particle small oscillations near a minimum of the potential well is written in the form

$$\frac{d^2y}{d\tau^2} + w^2 y = 0 \ , \qquad (10)$$

where $w^2 = 4\Delta\eta_1^2 S_1^2 (1-S_1)^{1/2}$.

If the quasiparticle initial velocity is rather different from zero the oscillations become anharmonic ones. Let us consider a complete equation. In this case, having integrated the equation (8) we obtain

$$\frac{1}{2}\left(\frac{d\bar{x}}{d\tau}\right)^2 = E - U(\bar{x}) \ . \qquad (11)$$

Let us investigate the potential $U(\bar{x})$. The quasiparticle can be trapped by a potential and make oscillating motion or reflect from it and go away at $+\infty$ depending on the initial energy. On the phase plane there also exists a separatrix trajectory which separates oscillating and reflective trajectories, and the velocity on a separatrix v_S tends to zero in infinity

$$v_s \Rightarrow 0 \ , \text{if } \bar{x} \Rightarrow \infty \ .$$

The equation (11) is integrated in quadrature

$$\tau(\bar{x}) = \int_{\bar{x}_1}^{\bar{x}} [2(E - U(\bar{x}))]^{-1/2} dx \ . \qquad (12)$$

From (12) one can obtain a period of oscillations T

$$T = \int_{\bar{x}_m}^{\bar{x}_n} [2(E - U(\bar{x}))]^{-1/2} dx \quad ,$$

where $\bar{x}_m$ and $\bar{x}_n$ are the points in reflection at the oscillating motion which depend upon the initial velocity.

Let us investigate a separatrix trajectory. We write the equation (11) in the form

$$\frac{1}{2} v^2(\tau) = E - U(\bar{x}),$$

where

$$U(\bar{x}) = a \, \tanh s + b \, \tanh^3 s \; ,$$

$$a = \Delta(1 - S_1^{-1}), \; b = \Delta/3S_1 \quad .$$

On the separatrix $v \Rightarrow 0$ at $x \Rightarrow \infty$. We define a value of the constant E from these condition

$$E = a + b \; . \tag{13}$$

Substituting (13) into (12) we obtain the equation of motion on the separatrix

$$\tau = \int_{\bar{x}_0}^{\bar{x}} [2b(1 - \tanh s)(\tanh s - m)(\tanh s - n)]^{-1/2} \, dx \; ,$$

where

$$m = -1/2 + (3/2)(1 - 4S_1/3)^{1/2} \quad ,$$

$$n = -1/2 - (3/2)(1 - 4S_1/3)^{1/2} \quad ,$$

A qualitative motion of the particle on the separatrix can be described by the following way: in a moment $\tau = -\infty$ the particle is located at the point $x = -\infty$ with the velocity $v = 0$, then it moves to the left and in a moment $\tau = 0$ it is reflected at the point $s = \tanh^{-1} m$, then it goes far away at $\bar{x} = +\infty$.The velocity on the separatrix is an odd function τ , a coordinate is an even function from τ .

4. Soliton motion along modulated interface

Let us describe soliton propagation in the case when the interface is periodically modulated. In this case the perturbation potential takes the form

$$V_1 = \theta(-x + \varepsilon\, sin\Omega\tau)[-\Delta + 2(\alpha - 1)|A|^2] \ ,$$

where ε is the amplitude of the modulation, Ω is the frequency of modulation. The equation of soliton motion in a potential takes the following form

$$\frac{d^2\bar{x}}{d\tau^2} = -(2\Delta\eta_1 sech^2 s' + 16\eta_1^3(\alpha - 1)\ sech^4 s') \quad , \qquad (14)$$

$$s' = 2\eta_1(\bar{x} - \varepsilon\ sin\Omega\tau).$$

Let us assume that $\varepsilon \ll 1$ and expand a function of the right hand of the equation (14) in the series in small parameters and preserve the first order terms . In this case we obtain the following equation for $\bar{x}$

$$\frac{d^2\bar{x}}{d\tau^2} = -\frac{dU}{d\bar{x}} + \varepsilon f(\bar{x})\ sin\ \Omega\tau \ , \qquad (15)$$

where $f(\bar{x}) = -\frac{1}{2}(d^2U/d\bar{x}^2) = 4\Delta\eta_1^2 sech^2 s\ tanh s(1 - 2S_1 sech^2 s).$

At $\varepsilon = 0$ the equation (15) is reduced to equation having been investigated in [1]. At $\varepsilon \ll 1$ the second term from the right hand (15) can be taken into account as a small periodic perturbation. As known, during the periodic perturbation influence upon the particle moving in anharmonic potential there arises a whole number of new physical phenomena, such as higher harmonic appearance, nonlinear resonances, phase oscillations, and under certain conditions, nonlinear resonance interaction and chaotic motion. An estimate of chaotic layer width near the separatrix can be found applying the

Melnikov function [5], that in our case takes the form

$$D(\tau_o) = \varepsilon \int_{-\infty}^{+\infty} \sin[\Omega(\tau - \tau_o)] f(\bar{x}) \, v(\tau) \, d\tau ,$$

where $\bar{x} = \bar{x}(\tau)$ is the separatrix trajectory. Applying $v(-\tau) = -v(\tau)$, $\bar{x}(-\tau) = \bar{x}(\tau)$, and expanding $\sin[\Omega(\tau - \tau_o)]$ we can write for $D(\tau_o)$:

$$D(\tau_o) = 2\varepsilon \cos(\Omega\tau_o) \int_{0}^{+\infty} \sin(\Omega\tau) f(\bar{x}) \, v(\tau) d\tau .$$

Let us make variable changes $v(\tau)d\tau = d\bar{x}$.

Then

$$D(\tau_o) = 2\varepsilon \cos(\Omega\tau_o) \int_{\bar{x}_m}^{+\infty} \sin[\Omega\tau(\bar{x})] f(\bar{x}) \, d\bar{x} ,$$

$$\bar{x}_m = (2\eta_1)^{-1} \tanh^{-1} m.$$

As known, the chaotic motion arises if the Melnikov function has infinite sets of zeros and this condition is realized in our case. The coefficient

$$|D| = 2\varepsilon \int_{\bar{x}_m}^{+\infty} \sin[\Omega\tau(\bar{x})] f(\bar{x}) \, d\bar{x},$$

defines a stochastic layer width. In Fig. 1 the dependence $|D|$ on Ω for specific values is presented.

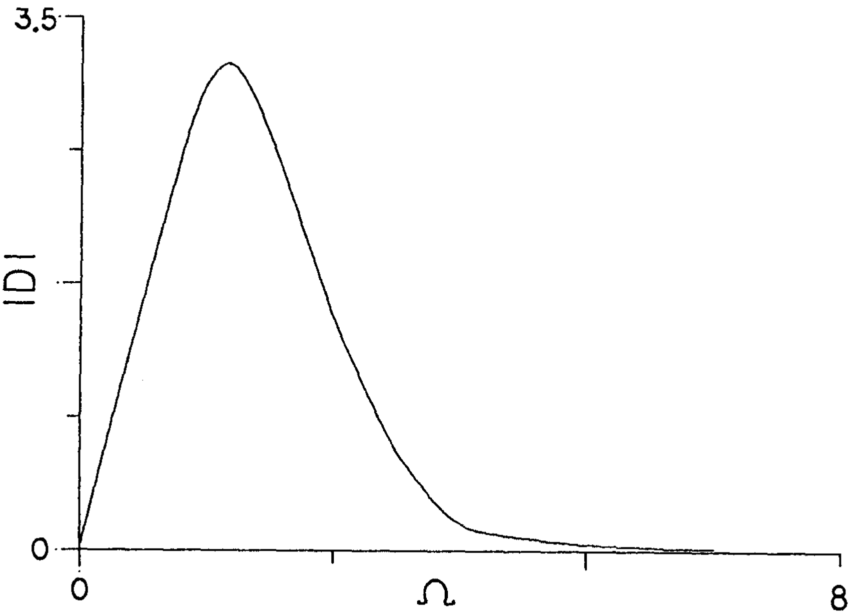

Fig.1.Dependence of the stochastic layer width versus modulation frequency at Λ = 0.1, α = 0.5, $2\eta_1$ = 1.264, ε =0.05.

As seen, |D| is rapidly decreasing with increase of Ω and has a maximum at the frequency Ω = 0.55. If the angle of the incident wave is located within the stochastic layer, then a ray is firstly trapped by the interface, slides along the interface for some distance, and the beam center is nonlinearly oscillating, then it reaches a region of chaos and reflects from the interface.

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