GAP SOLITONS IN 1D ASYMMETRIC PHYSICAL SYSTEMS

J.M. Bilbault, C. Tatuam Kamga and M. Remoissenet
Laboratoire O.S.C., Université de Bourgogne
6,Bd Gabriel,21100 Dijon, France

Abstract. We present a general approach for studying the nonlinear transmittance and gap solitons characteristics of asymmetric and one dimensional (1 D) systems in the low amplitude or Nonlinear Schrödinger limit. Included in this approach are some novel results on naturally asymmetric systems and systems where the symmetry is broken by an external constant force.

I. Introduction

Transmissivity near the gaps of a nonlinear system[1] of finite length exhibits bistability and can approach unity once the amplitude of the incoming sinusoidal wave is greater than a certain threshold which is frequency dependent and decreases with the length of the system. In the transmitting state, one has a nonlinear standing wave called[2,3] a "gap soliton ". Recent literature[1-3] has focused on symmetric systems, i.e. where the nonlinear potentiel (substrate or interaction potentiel) is symmetric: it only contains even powers of the characteristic field or of its gradient.

An interesting way to complete and extend our knowledge of the nonlinear response of finite systems with gaps is to analyse systems which are naturally asymmetric or systems where the symmetry is broken by the presence of an external force[4]. We present here a general approach for studying the nonlinear transmissivity and gap soliton characteristics of asymmetric 1D systems in the N.L.S. limit. We illustrate this methodology by application to the perturbed Sine-Gordon system.

II . General problem

In the one-dimensional arrangement illustrated in fig. 1, an incoming wave plane wave[1] of frequency ω, amplitude ϕ_0 and wave number $k_\ell = \omega / c_\ell$ in a linear medium propagates along the x direction and strikes at x = 0 a nonlinear medium of length L. The complex quantity R is the amplitude of the reflected wave measured with respect to ϕ_0 and similarly T is the amplitude of the transmitted wave at x = L in the linear medium (3) expressed as a fraction of that of the incident wave.

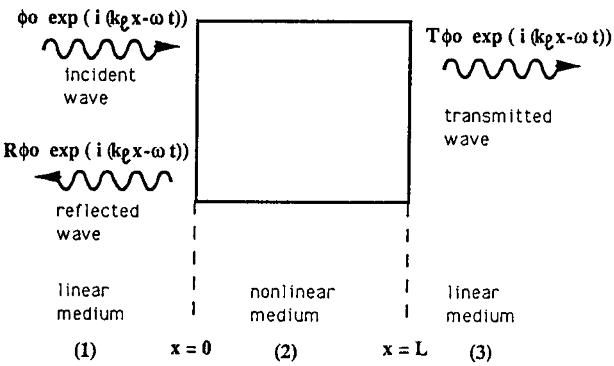


figure 1 : the incoming field $\phi(x,t) = \phi_0 \exp(i(k_\ell x - \omega t))$ strikes a nonlinear film (2) of length L. The transmitted wave emerges in the linear medium (3).

Inside the nonlinear medium we assume that the field $\Phi_n(t)$ obeys a generalized Klein-Gordon[4] (K G) lattice model equation :

$$\frac{\partial^2 \Phi n}{\partial t^2} = \frac{c_0^2}{a^2} (\Phi_{n+1} + \Phi_{n-1} - 2\Phi_n) - \omega_0^2 \frac{dV(\Phi n)}{d\Phi n} \quad . \tag{2.1}$$

Here n is the site number, $V(\Phi_n)$ is a nonlinear substrate potential ,the constants c_0 and ω_0 are the characteristic velocity and frequency of the system and a is the lattice parameter. In the low amplitude limit, we look for nonlinear collective oscillations in the bottom of the potential wells. For this purpose, assuming $\Phi_n = \varepsilon\phi_n + \Phi_0$ in eq. (2.1), where $\varepsilon << 1$ and Φ_0 is the ground state or potential minimum around which the oscillations will occur, and keeping terms to order ε^2, one gets:

$$\frac{\partial^2 \phi_n}{\partial t^2} = K (\phi_{n+1} + \phi_{n-1} - 2\phi_n) - (\omega'_0)^2 (\phi_n + \varepsilon\alpha\phi_n^2 + \varepsilon^2\beta\phi_n^3) \quad , \tag{2.2}$$

where $K = c_0{}^2 / a^2$ and the coefficients ω'_0, α and β are determined by the shape of the potential. It is interesting to note here that $\Phi_0 = 0$ and the second order term vanishes ($\alpha = 0$) when the potential wells are symmetric, as it is the case for the classical Sine-Gordon system where $dV(\Phi_n)/d\Phi_n = \sin \Phi_n$. We have $\Phi_0 \neq 0$ when the potential wells are asymmetric, which is the case in SG system perturbed[4] by an external force $\mathcal{F}$.

Let us now consider oscillating solutions of the form:

$$\phi_n(t) = F_1 e^{i\theta_n} + c.c + \varepsilon[F_0 + F_2 e^{2i\theta_n} + c.c] \tag{2.3}$$

where F_1, F_2 and F_3 are, respectively, the slowly varying amplitudes of the first harmonic, the dc and and second harmonic terms. These last two terms are introduced to take into account of the asymmetry of the potential, however we neglect higher harmonics. The phase defined as $\theta_n = kna - \omega t$ varies rapidly. Inserting (2.3) in (2.2) , equating dc, first and second harmonic terms and keeping terms to order ε^2 , we can relate F_0 and F_2 to F_1, and get :

$$\omega^2=(\omega'_0)^2 +4K \sin^2 \frac{ka}{2} + \varepsilon^2(\omega'_0)^2 \left[-4\alpha^2+ \frac{2\alpha^2}{3 + \frac{16K}{(\omega'_0)^2} \sin^4 \frac{ka}{2}} +3\beta\right] |F_1|^2 \tag{2.4}$$

Expanding now this general nonlinear dispersion relation eq. (2.4) in Taylor's serie about the carrier frequency ω_p and wave vector k_p yields[5]:

$$\omega-\omega_p = \left(\frac{\partial\omega}{\partial k}\right)(k-k_p)+\frac{1}{2}\left(\frac{\partial^2\omega}{\partial k^2}\right)(k-k_p)^2+ \left(\frac{\partial\omega}{\partial |F_1|^2}\right) |F_1|^2 \tag{2.5}$$

Setting $\Omega = \omega-\omega_p$ and $K = k-k_p$,with $\Omega << \omega_p$ and $K<< k_p$, eq. (2.5) represents the nonlinear dispersion relation $\Omega = f(K, |F_1|^2)$ of the wave envelope. In (2.5), the derivatives represent respectively the group velocity V_g, the group velocity dispersion P and the nonlinearity:

$$\left(\frac{\partial\omega}{\partial k}\right) = V_g, \qquad \left(\frac{\partial^2\omega}{\partial k^2}\right) = \left(\frac{\partial V_g}{\partial k}\right) = 2P, \qquad Q = -\left(\frac{\partial\omega}{\partial |F_1|^2}\right) \tag{2.6}$$

Substituting now the derivative operators in eq. (2.5) by the coefficients defined in eq. (2.6) yields the Nonlinear Schrödinger Equation (N.L.S.):

$$i\,[F_{1t} + V_g F_{1x}] + P F_{1xx} + Q|F_1|^2 F_1 = 0 \tag{2.7}$$

We now consider a particular case, in order to illustrate how one can determine the transmittance and the envelope behaviour of a given system.

III. An example: the perturbed Sine-Gordon system.

We consider the specific case[4] of a perturbed SG system. In this case, the Sine-Gordon potential is:

$$V(\Phi_n) = 1 - \cos\Phi_n + \frac{\mathcal{F}}{\omega_o^2}\Phi_n \tag{3.1}$$

Its minimum occurs at $\Phi_o = -\sin^{-1}(\mathcal{F}/\omega_o^2)$, while the coefficients of eq. (2.2) become[4] $(\omega'_o)^2 = \omega_o^2\cos\Phi_o$, $\alpha = -\frac{1}{2}\tan\Phi_o$ and $\beta = -\frac{1}{6}$. The force $\mathcal{F}$ lowers the linear dispersion curve with respect to the unperturbed case ($\mathcal{F} = 0$), as shown on fig.2. This is enforced by the nonlinearity Q in N.L.S. eq. (2.7), which is positive in this case:

$$Q = \frac{\omega_0^2\cos(\Phi_0)}{2\omega_p}\left[\tan^2(\Phi_0) - \frac{\frac{1}{2}\tan^2(\Phi_0)}{3 + \frac{16K}{\omega_0^2\cos(\Phi_0)}\sin^4\frac{k_p a}{2}} + \frac{1}{2}\right], \tag{3.2}$$

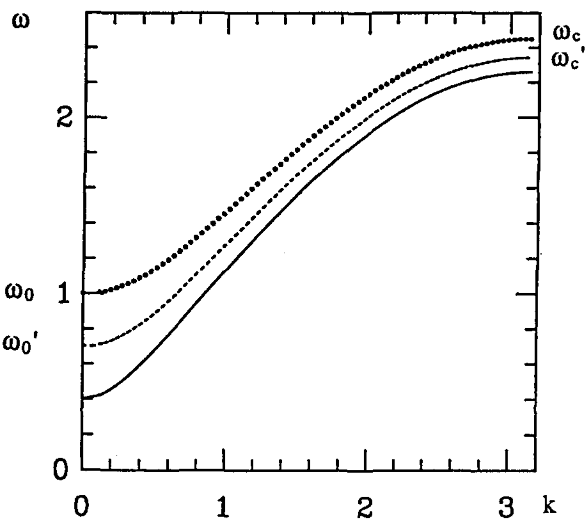


fig.2 Linear and nonlinear dispersion curves for a perturbed SG system. The external force lowers the linear curve (dashed line) with respect to the unperturbed case (dotted line), while the nonlinearity with an arbitrary amplitude enforces this lowering (solid line).

The interesting situations occur then near the gap edges of the linear dispersion curves, because the nonlinearity will change the transmission behaviour. We have:

- either $k_p = 0$, $\omega_p = \omega_0'$, $V_g = 0$ and $P = + c_0^2/2\omega_p$ near the lower gap
- or $k_p = \pi/a$, $\omega_p = \omega_c'$, $V_g = 0$ and $P = - c_0^2/2\omega_p$ near the upper gap.

In both cases,the angular frequency of the incoming wave is repered by the small detuning $\Omega = \omega - \omega_p$. We seek now envelope functions in the nonlinear medium of the form[6]:

$$F_1(x,t) = \phi_0 \sqrt{I(x)}\, \exp(i\theta(x)) \exp(-i\Omega t) \tag{3.3}$$

where the squared envelope function I(x) and the phase θ(x) are real. Putting the form (2.9) in N.L.S. eq. (2.7) gives after some calculations[6]:

$$\left(\frac{dI}{dx}\right)^2 = \mathcal{P}(I) \tag{3.4}$$

with:

$$\mathcal{P}(I) = -\left(2\frac{Q}{P}\phi_0^2 I^3 + 4\frac{\Omega}{P} I^2 - 4IB + 4A^2\right). \tag{3.5}$$

The constants of integration A and B are determined by the boundary conditions at the interfaces (see fig.1), namely the field and its first spatial derivative are continuous at x=0 and x=L. Moreover, these boundary conditions show that $I(x=L) = I_L$ is always a root of $\mathcal{P}(I)$, while limiting ourselves to low amplitude ϕ_0 and small detuning Ω, the two other roots of $\mathcal{P}(I)$ are real and positive. Finally, $\mathcal{P}(I)$ becomes:

$$\mathcal{P}(I) = -2\frac{Q}{P}\phi_0^2 (I - I_L)(I - I_+)(I - I_-) \tag{3.6}$$

where I_+ and I_- are given by:

$$I_+ = -\left(\frac{I_L}{2} + \frac{\Omega}{Q\phi_0^2}\right) + \sqrt{\left(\frac{I_L}{2} + \frac{\Omega}{Q\phi_0^2}\right)^2 + \frac{2I_L k_\ell^2 P}{Q\phi_0^2}} \tag{3.7.a}$$

$$I_- = -\left(\frac{I_L}{2} + \frac{\Omega}{Q\phi_0^2}\right) - \sqrt{\left(\frac{I_L}{2} + \frac{\Omega}{Q\phi_0^2}\right)^2 + \frac{2I_L k_\ell^2 P}{Q\phi_0^2}} \tag{3.7.b}$$

Integrating eq. (2.10) according to the ordering between I(x), I_L, I_+ and I_- and the sign of P, Q and Ω, leads to a Jacobi Elliptic[7] function expression with the parameter I_L. Using once more[6] the boundary conditions gives the following condition:

$$\left[\left(\frac{dI}{dx}\right)_{x=0}\right]^2 + 4k_\ell^2 (I_L + I(x=0))^2 - 16\, k_\ell^2\, I(x=0) = 0\ , \tag{3.8}$$

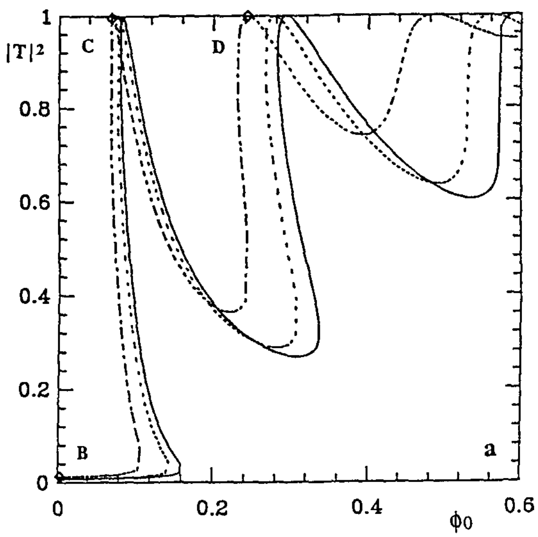
which permits to keep by numerical calculations the suitable values of I_L. Once I_L is determined, one can get I(x). Then, from eqs. (2.3), (3.3) and continuity at x = L, we get the transmissivity coefficient $|T|^2 = I_L$.

IV. Results for the perturbed Sine-Gordon system.

Using the approach presented in the previous sections, we calculate numerically the transmissivity and the square of the envelope function. We consider successively the lower gap and the upper gap of the dispersion curve (see fig.2), where the results are quite different.

A. Lower gap: we consider a system where L = 60 a, the velocities (defined in section II) are $c_0 = c_\ell = 4.5$ and: $\Omega = \omega - \omega_0' = -0.01$.

Our results, represented on fig.3.a , agree with the previous works[1,6] obtained for SG systems, i.e. the system presents bistabilities and hysteresis cycles. The external force lowers the threshold values, as seen on fig.3.a. This can be easily understood because Q, given by eq. (3.2) is a growing function of Φ_0 and $\mathcal{F}$.



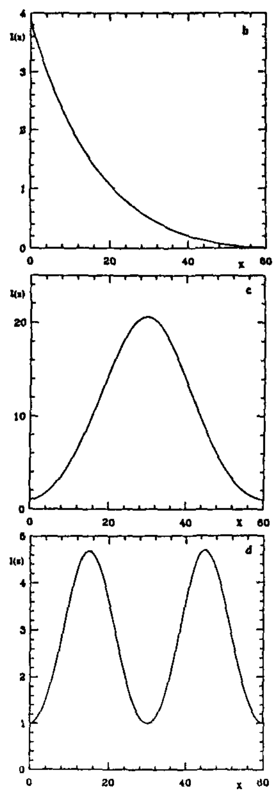


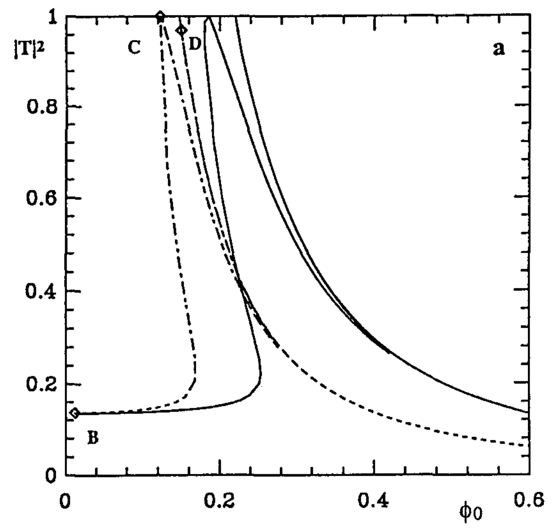
Fig.3.a Transmissivity $|T|^2$ versus the amplitude ϕ_0 of the incoming wave, when the frequency ω lies just below the lower gap edge (Ω=–0.01) for the unperturbed S-G system (solid line) and for the perturbed S-G system with a force $\mathcal{F}$ =0.39 (dashed line) or $\mathcal{F}$=0.72 (dotted line).

Fig.3.b,c and d: For $\mathcal{F}$= 0.72, the shape of I(x) at points B, C and D of fig.3.a is represented versus the coordinate x.

When ϕ_0 is weak, i.e. in the linear limit, the wave envelope is evanescent (fig.3.b). When ϕ_0 increases, the system reaches a certain threshold, which depends on the value of $\mathcal{F}$.

Then the system switches to a transmitting state and the envelope is that of a nonlinear standing wave (called a gap soliton). In this case, one has P.Q > 0. Then if one further increases ϕ_0, the transmissivity reaches successively several maxima, which correspond to different resonant modes (standing waves of fig.3.c and d) described by Jacobi Elliptic functions. For one resonant mode (fig.3.c), the maximum is at L/2 = 30.

B. Upper gap: the nonlinear system length is still L = 60 a , but now $c_0 = 2.$, $k\ell = 0.015$ and $\Omega = \omega - \omega_c' = -0.003$. Our results, represented on fig.4.a, still show that the threshold decreases with the external force. This can be explained by the fact that Q increases with $\mathcal{F}$. The squared envelope function, represented on fig.4.b, is characteristic of a standing wave behaviour: it now corresponds to P.Q < 0.By contrast to the previous case, for one resonant mode (standing wave on fig.4.c) at x = L/2 = 30 one has now a minimum. When ϕ_0 is further increased, the envelope finally becomes evanescent (fig.4.d); this can be explained by considering the dispersion curve (see fig. 2) and remarking that the nonlinearity tends to lower the curves. Then, ω lies inside the gap.



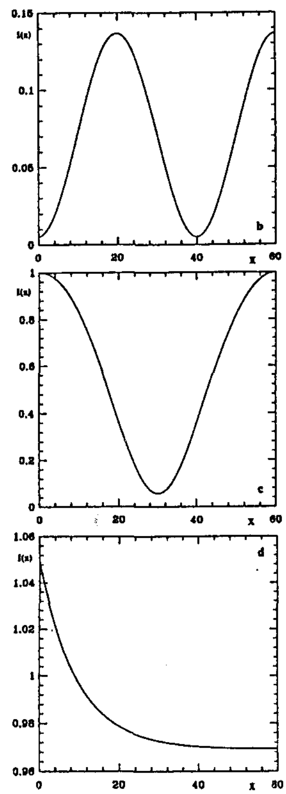


Fig.4.a Transmissivity $|T|^2$ versus the amplitude ϕ_0 of the incoming wave when the frequency ω lies just below the upper gap edge ($\Omega = -0.003$) for the unperturbed S-G system (solid line) and for the perturbed S-G system with a force $\mathcal{F}$ =0.72 (dotted line).
Fig.4.b,c and d: For $\mathcal{F}$= 0.72, the shape of I(x) at points B,C and D of fig.4.a is represented versus x.

The method presented here allows to investigate asymmetric systems. If the asymmetry results from a symmetry breaking, as for the perturbed S-G system, the external force allows to control the bistability or nonlinear switching. Note that our approach can also be used for a natural asymmetric system[4] like "ϕ^4 ".

References

[1] D.Barday and M.Remoissenet,Phys.Rev.B **41**, 10387 (1990).

[2] D.L.Mills and S.E.Trullinger,Phys.Rev.B **36**,947 (1987).

[3] C.Martijn de Stercke and J.E.Sipe,Phys.Rev.A **38**, 5149 (1988)

[4] M.Remoissenet,Phys.Rev.B **33**, 2386 (1986).

[5] Whitham , Linear and nonlinear waves (Academic Press, New York, 1972)

[6] W.Chen and D.L.Mills,Phys.Rev.B **35**, 524 (1987).

[7] P.F.Byrd and M.D.Friedman,Handbook of Elliptic integrals(Springer, Berlin, 1954)