PART IV

TWO-DIMENSIONAL STRUCTURES

SELF-ORGANIZATION AND NONLINEAR DYNAMICS WITH SPATIALLY COHERENT STRUCTURES

K.H. Spatschek, P. Heiermann, E.W. Laedke, V. Naulin, and H. Pietsch
Institut für Theoretische Physik I, Heinrich-Heine-Universität Düsseldorf
D-4000 Düsseldorf, F.R. Germany

Abstract: In near-integrable soliton-bearing systems spatially coherent states can play an important role. In this contribution we briefly review some of the main phenomena for physically relevant situations. We start with the well-known soliton formation in integrable systems which can be interpreted as the first appearence of self-organization in physics. It is shown here that also in non-integrable Hamiltonian systems solitary waves can self-organize. For dissipative systems, the self organization hypothesis is presented and tested for 2d drift-waves. A socalled self-organization instability is found which shows the growth of a spatially coherent (solitary) structure even in the presence of turbulence. The other finding in this respect, the absence of (Anderson) localization in nonlinear disordered systems, is also briefly mentioned. The soliton, as a collective excitation, can overcome individual chaotic motion. A recent result for the proton motion in two Morse-potentials under the influence of oscillations of the heavy ions, is discussed showing the importance of solitons to create ordered structures and collective transport. Nevertheless, solitary waves can also be the constituents of deterministic (temporal) chaos as shown in the final part of this contribution.

1. Self-Organization of spatially coherent structures

The constructive proof [1] of integrability of the 1d KdV-equation can be considered as a milestone in the development of nonlinear physics. As a by-product of the proof, self-organization in the form of stable solitons appears. This is very fascinating and can be considered as an important contribution to the new discipline "synergetics". From the physical point of view the question arises whether this self-organization phenomenon is an artefact of the integrable systems. Integrability can be broken by several means, e.g. higher space dimensions, dissipation, driving, etc. In the following we shall present four examples for self-organization of solitary waves in non-integrable systems. The results follow from numerical simulations, but can be understood by analytical theory.

Let us start with self-organization in KdV-systems. We take as an example the non-integrable 2d KdV equation

$$\partial_t u + u\partial_z u + \partial_z \nabla^2 u = 0 \tag{1}$$

in the Zakharov-Kuznetsov form. Here, $\nabla^2 = \partial_z^2 + \partial_x^2$. It reduces for only one relevant space coordinate z to the celebrated KdV-equation [1]. As has been shown [2], 1d soliton solutions $u_s = 12\eta^2 \operatorname{sech}^2[\eta(z - z_0 - 4\eta^2 t)]$ of (1) are transversely unstable. The growth rate γ_k can be calculated by variational principles to yield

$$\gamma_k^2 = \sup_{\varphi} \frac{\langle\varphi|\partial_z H_k \partial_z|\varphi\rangle}{\langle\varphi|H_k^{-1}|\varphi\rangle} = \inf_{\varphi} \frac{\langle\varphi|\partial_z H_k \partial_z H_k \partial_z H_k \partial_z|\varphi\rangle}{\langle\varphi|\partial_z H_k \partial_z|\varphi\rangle} \,. \tag{2}$$

Here $H_k = -\partial_z^2 + 4\eta^2 - u_s + k^2$. The growth rate depends on the transverse wavenumber k; a cut-off appears at $k = k_c \equiv \sqrt{5}\eta$, and the growth rate has its maximum for $k^2 \approx 1.7\eta^2$. In Fig. 1a this dependence is shown by constructing numerically upper and lower bounds from (1). The exact growth rate curve lies within the shaded area. Also the small-k and small-(k_c-k) expansions are shown, respectively. In a 2d numerical simulation [3] we identified this instability and followed its time evolution. A typical result is shown in Fig. 1b. We can interpret this finding in the following

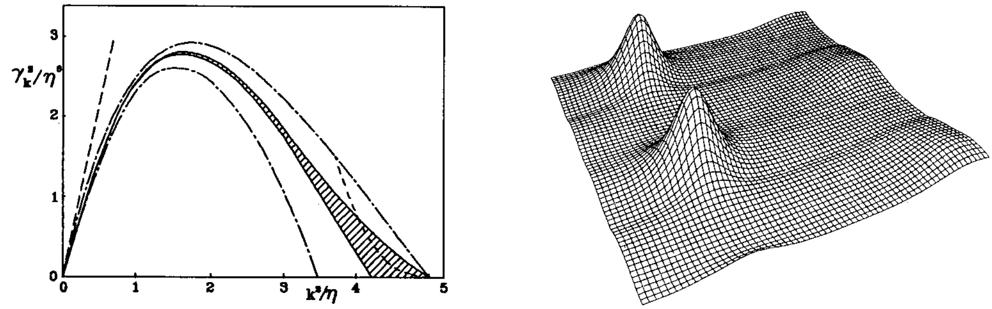

Fig. 1 (a) Transverse instability growth rate γ_k vs. wavenumber k for a 1d KdV soliton.
(b) Appearence of stable 2d KdV-solitons in a numerical simulation of (1). We started with a one-dimensional soliton in the z-direction.

way. In a narrow channel of width d, small k-values cannot occur and the (along the channel) 1d KdV soliton is (transversely) stable. At $d = 2\pi/k_c$ a bifurcation occurs and the 2d KdV solitary wave is the new self-organized state. It is shown in Fig. 1b as one of the humps.

For the 2d soliton solution of (1) a Liapunov functional can be presented [4] in the form $L = L_p\{u\} - L_p\{\bar{u}_M\}$, where

$$L_p\{u\} := \int d^2r[(\nabla u)^2 - \frac{1}{3}u^3 + 4\eta^2 u^2] \,, \tag{3}$$

and $\bar{u}_M \epsilon S$ belongs to the invariant set S defined with respect to space translations $\vec{\xi}, \bar{u}_M = u_s(\vec{r} - \vec{\xi})$. The functional (3) proves the stability of the 2d stationary localized solitary wave solution of (1). The procedure is standard: for the first variation $\delta L = 0$ and for the second variation $\delta^2 L > 0$ can be shown. [When instead of two space dimensions the three-dimensional case is considered, the stability of a 3d localized solitary wave solution can be proven in a similar manner!]

However, one should be cautious in generalizing these results. If, for example, the case of the cubic nonlinear Schrödinger (NLS) equation

$$i\partial_t q + 2|q|^2 q + \nabla^2 q = 0 \tag{4}$$

is investigated, again the one-dimensional case shows self-organization into 1d solitons $q_s = \eta \,\mathrm{sech}(\eta x) \exp(i\eta^2 t)$. This fact follows from the inverse scattering solutions by Zakharov and Shabat [5]. The soliton solutions are two-dimensionally unstable, with a transverse instability growth rate [6]

$$\gamma_k^2 = \sup_\varphi \frac{-\langle\varphi|H_-|\varphi\rangle}{\langle\varphi|H_+^{-1}|\varphi\rangle} = \inf_\varphi \frac{-\langle\varphi|H_- H_+ H_-|\varphi\rangle}{\langle\varphi|H_-|\varphi\rangle} \,. \tag{5}$$

In the second expression, the variation of φ is restricted to the subspace $\langle\varphi|H_-|\varphi\rangle < 0$. Here, the operators H_+ and H_- are defined as $H_+ = -\partial_x^2 - k^2 - 2|q_s|^2 + \eta^2$ and $H_- = H_+ - 4|q_s|^2$, respectively. The cut-off wavenumber is $k_c = \sqrt{3}\eta$. When again considering $d = 2\pi/k$ as the bifurcation parameter,

at $d = 2\pi/k_c$ a bifurcation occurs but instead of a stationary (unstable) 2d Schrödinger soliton a new time-depending (collapsing) solution appears. When the initial state is close to a (unstable) stationary 2d Schrödinger solitary wave, the following theorem can be proven [7]:
Let us assume in 2d that $H\{q\} \leq 0$ holds, where H is the energy functional $H = \int d^2r(|\nabla q|^2 - \frac{1}{2}|q|^4)$. Then, up to translation in space and phase shifts, we can find for every ϵ a δ_ϵ such that

$$\| q(x, t=0) - G(x) \|_{W^{1,2}} \leq \delta_\epsilon \;\Rightarrow\; \| q(x,t) - \mu(t)G[\mu(t)x] \|_2 \leq \epsilon \text{ holds,}$$

$$\text{with } \mu(t) \equiv \frac{\|\nabla q\|_2}{\|\nabla G\|_2},\ \mu(t=0) = 1 \text{ and } 0 \leq t < t_c.$$

Here G is the 2d solitary wave solution. It is very interesting to see that, although the stationary solitary wave solution is not stable, the new bifurcating state is connected to the 2d solitary wave solution: it is a solitary wave solution with time-varying parameters, i.e. the width is decreasing with time, leading to a singularity within a finite time. Because of space limitations , we cannot present more details and numerical results here. They will be published elsewhere [7]. We should note that the area of collapsing solutions is a very active one and for arbitrary initial conditions the question of the collapse as an effective dissipation mechanism in plasmas is still open.

Next, we turn to an essentially non-integrable problem (with dissipative and driving terms) to discuss self-organization in nonlinear drift-waves. When dissipative and driving terms are ignored, the basic equation is the Hasegawa-Mima-equation [8] for the normalized electrostatic potential ϕ:

$$\partial_t(1 - \nabla^2)\phi - \kappa_n\partial_y\phi = \hat{z} \times \nabla\phi \cdot \nabla\nabla^2\phi\,. \tag{6}$$

Here κ_n is a normalized density-gradient-coefficient. Equation (5) is non-integrable, but has 2d dipolar vortex solutions. The latter (in general) do not interact elastically, but show a surprising stability against small perturbations. We now generalize (6) by including self-consistently driving and damping terms due to collisions in the same way as done by Kono and Miyashita [9]. In plasma physics the corresponding linear instability is known as the collisional drift instability. Instead of (6) then

$$\partial_t(1 - \nabla^2 - \frac{\kappa_n}{Dk_\|^2}\partial_y)\phi + [-\kappa_n\partial_y + \frac{\kappa_n^2}{Dk_\|^2}\partial_y^2 + \mu\nabla^4]\phi = \hat{z} \times \nabla\phi \cdot \nabla\nabla^2\phi \tag{7}$$

appears. In (7), $D = \Omega_e/\nu_e$ characterizes the collisional contributions and $k_\|$ is an effective parallel wavelength. A numerical simulation [9,10] of (7) shows the self-organization of an arbitrary initial state into a dipolar vortex. The maximum vortex with respect to the (numerically prescribed) box size appears. This end-result is shown in Fig. 2a. For this simulation we started at time $t = 0$ with random noise of low level. The unstable (linear) modes grow, transfer energy via mode-coupling to other modes, and a parametric instability amplifies small-k contributions. This numerical behavior can be understood analytically. A key-role in the interpretation of the final result plays the socalled self-organization hypothesis [11]. It is formulated for nonlinear partial differential equations with dissipation which contain two (or more than two) quadratic (or higher order) conserved quantities in the absence of dissipation. In the case of (6) we shall apply the self-organization hypothesis for the conserved quantities energy $E[\phi] = \int d^2r[(1-\nabla^2)\phi]^2$ and enstrophy $K[\phi] = \int d^2r[(1-\nabla^2)\nabla^2\phi]^2$. The hypothesis is formulated under the following two conditions: (i) There exists a selective dissipation process among the conserved quantities E and K when the dissipation is introduced. That is, one conserved quantity K decays faster than the other E. (ii) The nature of the mode-coupling through the nonlinear terms in the equation is such that the modal cascade in the quantity E is towards small wavenumbers. Then it is assumed (and justified by numerics as, e.g., shown in Fig. 2a) that the following hypothesis holds: The randomly excited field ϕ is expected to reach a quasi-stationary state in which ϕ is described by a deterministic field equation. The latter is obtained by minimizing K within the constraint that E is kept constant: $\delta K - \lambda \delta E = 0$. It is straightforward to show that this variational principle leads to the (quasi-stationary) equation for dipole solutions of (6) [in the absence of dissipation].

We have developed a method and derived concrete equations to explain the self-organization

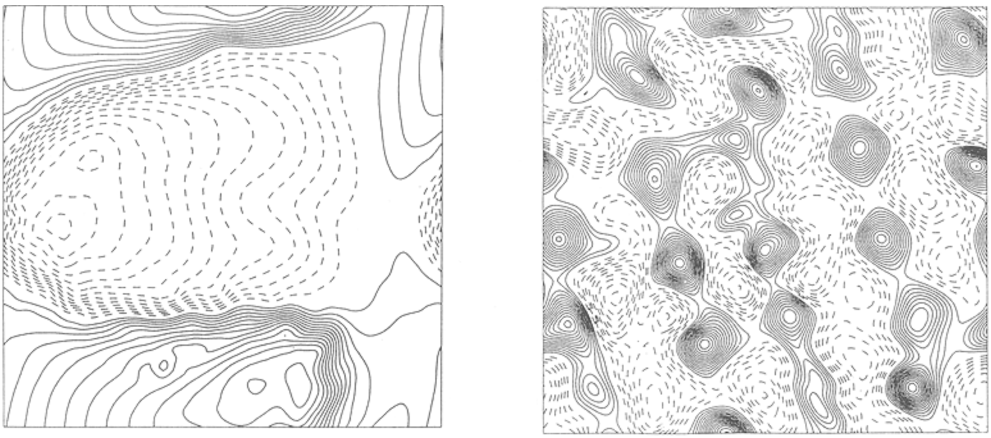

Fig. 2 (a) Self-organization of a big dipolar vortex as a result of numerical simulation of (7). (b) A chain of 2d KdV-type solitary waves appear when (12) is solved.

hypothesis from first principles [12]. Abbreviating the linear operator appearing on the left-hand-side of (7) by $\hat{L}$ we rewrite (7) in the form

$$\hat{L}\phi + \{\phi, \nabla^2\phi\} = 0 \, . \tag{8}$$

Here, $\{...,...\}$ denotes the Poisson-bracket. Next we separate the normalized potential ϕ into a regular (ϕ^R) and a turbulent (ϕ^T) component in the usual way by making use of a turbulent ensemble and the corresponding averaging denoted by $\langle ... \rangle$. Thus when introducing $\phi = \phi^R + \phi^T$ we assume $\langle \phi^T \rangle = 0$. Within this concept we obtain from (8) the coupled equations

$$\hat{L}\phi^R + \{\phi^R, \nabla^2\phi^R\} = -\langle\{\phi^T, \nabla^2\phi^T\}\rangle \, , \tag{9}$$

$$\hat{L}\phi^T + \{\phi^T, \nabla^2\phi^T\} - \langle\{\phi^T, \nabla^2\phi^T\}\rangle = -\{\phi^R, \nabla^2\phi^T\} - \{\phi^T, \nabla^2\phi^R\} \, . \tag{10}$$

In the absence of turbulence ($\phi^T \equiv 0$), the left-hand-side of (9) determines in the usual way the regular structures. On the other hand, in the absence of regular structures, i.e. when the right-hand-side of (10) is zero, the last equation will be similar to that known from (weak) turbulence theory [13]. Linearizing (9) and (10) leads after some tedious algebra [12] to the growth rate

$$\gamma_k = \pi \frac{k k_y^2}{1+k^2} \int_0^\infty k_1^3 W_{k_1} dk_1 > 0 \, , \tag{11}$$

where $W_k = \frac{1}{2}(1+k^2)\langle|\phi^T|^2\rangle_k^0$ is the zeroth order turbulent spectral energy density, which has been assumed, in lowest order and for demostration, to be isotropic.

Besides this analytical attempt to justify the self-organization hypothesis we performed a numerical simulation for a slightly different model equation compared to (6). In the presence of a temperature gradient also a KdV-type nonlinearity appears in the basic equation [10]

$$\partial_t(1-\frac{\kappa_n}{Dk_{\|}^2}\partial_y)\phi+[-(u+\kappa_n)\partial_y+\frac{\kappa_n^2}{Dk_{\|}^2}\partial_y^2+\mu\nabla^4]\phi+u\partial_y\nabla^2\phi+\kappa_T\phi\partial_y\phi=\hat{z}\times\nabla\phi\cdot\nabla\nabla^2\phi\,. \quad (12)$$

Here κ_T is the temperature-gradient-coefficient, and we have transformed (with velocity u) into a co-moving frame. The interesting difference with respect to (6) is that (in the dissipationfree case) the relevant conserved quantities change from $E[\phi]$ and $K[\phi]$ to $\tilde{E}[\phi]=\int d^2r[(\nabla\phi)^2-\frac{\kappa_T}{3u}\phi^3]$ and $\tilde{K}[\phi]=\int d^2r\phi^2$, respectively. Now, the self-organization hypothesis yields the variational principle $\delta\tilde{E}-\lambda\delta\tilde{K}=0$ whose solutions are 2d monopole structures of the KdV-type [see Fig. 1b]. And indeed a numerical simulation of (12) confirms this conjecture. In contrast to the single big dipolar vortex for (7) a chain of 2d solitary waves (zonal flow) appears for the model (12) as shown in Fig. 2b. The stability of the 2d solitary waves, even in the presence of the twisting nonlinearity, can be proven, whereas the dipolar vortex is structurally unstable with respect to perturbations in form of a scalar nonlinearity.

2. Self-organized solitary waves as constituents of nonlinear dynamics

We now turn to the question whether solitary waves are "robust". There are three aspects connected with the definition of robustness. The first one is related to linear and nonlinear stability of the exact solutions within the corresponding models. Also the elasticity or inelasticity of collisions falls into this first category. The second one consists of the question whether solitary collective excitations can overcome individual chaotic motion, disturbances due to external fluctuations, etc. The third one is mainly considered here and goes one step further. Can solitary waves (as a whole) behave chaotically in time so that we can consider them as constituents of deterministic chaos?

Let us start with a few remarks with respect to the second aspect. (The first one was already touched in the previous section 1.) A simple example might be helpful. In hydrogen-bonded chains solitary waves are found as the solutions of, e.g., the two-component model [14]

$$\frac{d^2u_n}{dt^2}=u_{n+1}-2u_n+u_{n-1}-\omega_o^2\frac{\partial U(u_n;\rho_n)}{\partial u_n}\,, \quad (13)$$

$$\frac{d^2\rho_n}{dt^2}=\rho_{n+1}-2\rho_n+\rho_{n-1}-\Omega_n^2(\rho_n-\Delta_n)\,. \quad (14)$$

Here, U is in general a double-well potential for the hydrogen-bonded proton; it is created by heavy ions. The potential U is assumed to be a function of two variables: the displacement u_n of the n-th proton from the middle of the hydrogen bridge and the relative displacement ρ_n of the neighbouring heavy ions creating this potential. Solitary wave solutions consist of kinks (or anti-kinks) for the proton displacement and are accompanied by disturbances in the heavy ion sublattice. From the *individual* (proton) point of view, the motion of the particle in an unharmonic potential is driven by the external motion of the heavy ions. Thus, when *collective* solitary wave excitations are not present, the position of the proton can be random. We [15] have verified this statement by a model calculation for the motion of a proton in the superposition of two Morse-potentials created by the two neighbouring heavy ions (see Fig. 3). The equation

$$\frac{d^2u}{dt^2}+\frac{\partial}{\partial u}\tilde{U}(u,\rho)=\gamma\frac{du}{dt} \quad (15)$$

was solved, where γ is a damping decrement and $\tilde{U}$ has the form shown in Fig. 3. The superposition of two Morse-potentials depends on the (normalized) coordinate ρ. For the latter we have assumed an harmonic time-dependence $\rho = \rho_o \sin(\Omega t)$ to simulate the dynamical behavior of the heavy sublattice in the absence of collective solitary wave excitations. As as result, similar to the finding for the Duffing oscillator, chaos can appear. This shows that solitary wave solutions are extremely important (e.g. for transport) since they can override the otherwise chaotic behavior. In this sense they can be called robust.

Another important feature in this respect is the fact that solitary waves can even render an effective transport through random media. As is well-known [16], in linear systems disorder causes Anderson localization, i.e. an exponential decay of the transmission coefficient with the system length. But it has been shown that nonlinearity can lead via soliton formation to an effective transmission mechanism. We have investigated this phenomenon for a similar model as that one originally treated by Caputo et al. [17]. Especially in biological systems, where the environment always causes irregularities in the chain, the formation of solitary waves, their propagation characteristics, and stability are now under active investigation [18].

Here we would like to discuss in more detail the other type of robustness which qualifies solitary

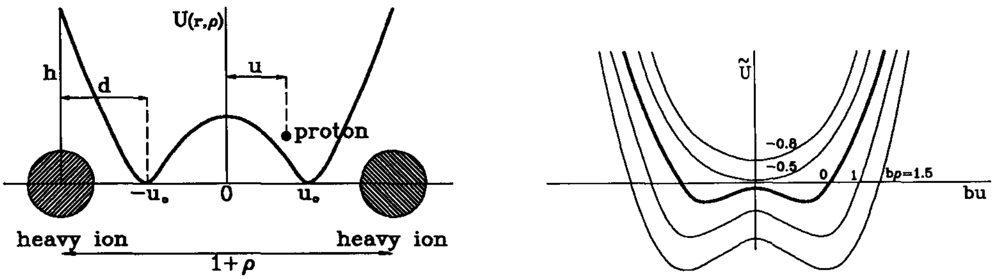

Fig. 3 (a) Motion of a proton in the superposition of two Morse-potentials created by two heavy ions. (b) Changes of the potential with ρ as a parameter.

waves as constituents for nonlinear dynamics, with possible temporal chaos. Let us demonstrate this on the paradigm of a perturbed NLS equation. As has been first demostrated by Nozaki and Bekki [19], for a model of damped nonlinear Langmuir waves driven in a rf capacitor field,

$$i\partial_t q + \partial_x^2 q + 2|q|^2 q = -i\gamma q - iae^{i\omega t} , \tag{16}$$

the period-doubling route to temporal chaos occurs for phase-locked solitary waves. Analyzing (16), we can derive the existence condition for a phase-locked solitary wave as $2\gamma\omega^{1/2}/\pi a \ll 1$. The stability of this phase-locked solitary wave was investigated analytically [20]; at finite driving amplitudes (and for fixed damping rate γ and prescribed frequency ω) an instability in form of a Hopf bifurcation takes place and a regulary pulsating solitary wave appears. In a reduced phase-space, the phase-locked solitary wave corresponds to a limit-cycle. With increasing values of driving amplitudes, the system undergoes a series of torus-doubling bifurcations for which the universal Feigenbaum constants $\delta_\infty = 4.6692...$ and $\alpha_\infty = 2.50291...$ could be recovered quite accurately. The situation changes when two space dimensions are taken into account. Then the collapsing solutions can be new attractors as has been discussed in Sec. 1. On the other hand, the whole scenario depends on the form of the "perturbations". If, e.g., we change from (16) to

$$i\partial_t q + \partial_x^2 q + |q|^2 q = -i\alpha q - \beta q - \gamma q^\star \tag{17}$$

or

$$i\partial_t q + \partial_x^2 q + p|q|^2 q = 1 - xq \tag{18}$$

for nonlinear modulated cross-waves in Faraday resonance [21] or radiation in laser irradiated inhomogeneous plasmas [22], respectively, we can find different nonlinear dynamical behaviors with spatial coherence. The first one shows bifurcations into cnoidal-wave-like functions whereas for the second one the quasi-periodic route to temporal chaos occurs. Both models have, in certain parameter regimes, stable solutions [21-23] with spatial coherence; see. Fig. 4. Here, we would like to

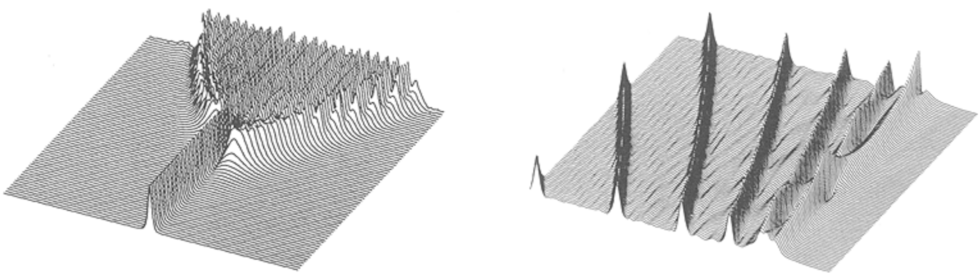

Fig. 4 (a) Appearence of a cnoidal-wave-type stable attractor for $\beta = -1$, $\alpha = 1$, *and* $\gamma = 1.6$ *in (17). (b) Space-time-plot of a solution of (18) for p = 1. A similar regular emission and (accelerated) propagation occurs for* $0.7 \leq p \leq 1.2$.

emphasize a new point in the region of a stable solitary solution to (17). When the driving amplitude is time-modulated [24], i.e. $\gamma = \gamma_o \cos \Omega t$, similar to (16) a phase-locked solitary wave appears which can take part *in toto* in the nonlinear dynamics as a spatially coherent structure. At the first glance, this looks similar to the phenomena detected in (16) and indeed all the tools used there can also be applied here. However, because of the possible bifurcation in space, an interesting interplay between nonlinear dynamics with spatial coherence and simultaneous bifurcation in space can take place. Details will be published elsewhere [25].

3. Summary and conclusions

In this contribution we have given an overview over the possibilities of self-organization and subsequent nonlinear dynamics with spatially coherent structures. The presentation is based on several new and original results which will be published in more details in subsequent publications. The main conclusions are the following: (i) In non-integrable systems stable solitary wave structures are formed by self-organization. (ii) The solitary (and spatially coherent) structures are robust in the sense that they can override individual chaotic behavior and contribute to transport even in disordered systems. (iii) Interesting and generic nonlinear dynamics takes place, with the spatially coherent structures as constituents.

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