INTERACTIONS OF SOLITONS in (2+1) DIMENSIONS

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Abstract: We consider instanton solutions of the CP^N models in two Euclidean dimensions as solitons of the same models in (2+1) dimensions. We find that, in general, the solitons tend to shrink so to stabilise them we add special potential and skyrme-like terms. We show that in head-on collisions the solitons scatter at 90° to the direction of their original motion and that they also undergo a shift along their trajectories.

1. INTRODUCTION

Consider the scattering of two solitons in any relativistic (1+1) dimensional model, such as the well known Sine-Gordon model. Imagine that one soliton moves with velocity v_1, the other with velocity v_2 and that, at some time $t = t_0$, they are well separated and their separation is δ. If the velocities of the solitons are such that they approach each other then after a certain length of time they will interact (and at that time the description of the system in terms of two isolated solitons ceases to be applicable) but then, some time later, they will emerge from the scattering region and later on will become well separated again. Looking at the positions of the solitons at this time, we find that each is at a place different from where it would have reached, had there been no interaction between them. We see that one effect of their interaction is to shift the solitons along their trajectories, the direction and the magnitude of this shift being determined by the strength of the interaction. All this is well known and has been observed in many models.

Let us now increase the spatial dimension by one and look at the scattering of solitons in (2+1) dimensions. What would be the corresponding properties? To answer this question we observe, that on purely kinematical grounds, there are more possibilities. As solitons correspond to extended structures, we see, that as they approach each other they can either experience:

1) a head on,

2) a small impact parameter or

3) a large impact parameter collision.

In the latter case, if the impact parameter is larger than the size of each soliton (in particular if there are no net forces acting on the solitons) we would expect the solitons to pass each other experiencing only small perturbations due to their interactions.

The most interesting are clearly head-on collisions and the scattering at very small impact parameters. So what happens with solitons in such scatterings? To try to answer this question we have to decide what we mean by a soliton in (2+1) dimensions. Clearly, we would expect it to represent at each value of time a localised but spatially extended structure of finite energy. So what model should we use and will the results depend on its choice?

Most of the properties of solitons in (1+1) dimensions are associated with the integrability of the underlying theory. Unfortunately, although several integrable models in (2+1) dimensions are known, none of them is relativistically covariant. But why should we consider a relativistically covariant model? There are many reasons; let us mention only that static extended structures arise often in many such theories and, in particular, many properties of the physical proton follow quite naturally from its description in terms of such an extended structure in a phenomenologically successful "Skyrme model of the proton" (of course, in a relativistic model in (3+1) dimensions). So if we want to consider solitons in a relativistic model we cannot rely on the integrability of the model for the properties of their scattering.

The simplest relativistic model in which we can study various properties of solitons in (2+1) dimensions is the S^2 σ model, also called the CP^1 model, which involves one real vector field of 3 components, $\vec{\phi} \equiv (\phi^1, \phi^2, \phi^3)$. In (2+1) dimensions $\vec{\phi}$ is a function of the space-time coordinates (t, x, y) which we will also write as (x^0, x^1, x^2). The model is defined by the Lagrangian density

$$\mathcal{L} = \tfrac{1}{4}(\partial^\mu \vec{\phi}) \cdot (\partial_\mu \vec{\phi}) \,, \tag{1}$$

together with the constraint $\vec{\phi} \cdot \vec{\phi} = 1$, *i.e.* $\vec{\phi}$ lies on a unit sphere S^2_{ϕ}. In (1) the Greek indices take values $0, 1, 2$ and label space-time coordinates, and ∂_μ denotes partial differentiation with respect to x^μ. Note that we have set the velocity of light, c, equal to unity, so that in all our calculations we can use dimensionless quantities. The Euler-Lagrange equations derived from (1) are

$$\partial^\mu \partial_\mu \vec{\phi} + (\partial^\mu \vec{\phi} \cdot \partial_\mu \vec{\phi}) \vec{\phi} = \vec{0} \,. \tag{2}$$

For boundary conditions we take

$$\vec{\phi}(r, \theta, t) \to \vec{\phi}_0(t) \quad \text{as} \qquad r \to \infty \,, \tag{3}$$

where (r, θ) are polar coordinates and where $\vec{\phi}_0$ is independent of the polar angle θ. In two Euclidean dimensions (*i.e.* taking $\vec{\phi}$ to be independent of time) this condition ensures finiteness of the action and in (2+1) dimensions it leads to a finite potential energy.

It is convenient to express the $\vec{\phi}$ fields in terms of their stereographic projection onto the complex plane W

$$\phi^1 = \frac{W + W^\star}{1 + |W|^2}, \quad \phi^2 = i\frac{W - W^\star}{1 + |W|^2}, \quad \phi^3 = \frac{1 - |W|^2}{1 + |W|^2}. \tag{4}$$

The W formulation is very useful, because it is in this formulation that the static solutions take their simplest form; namely, as originally shown by Belavin and Polyakov[1] and Woo,[2] they are given by W being any rational function of either $x + iy$ or of $x - iy$. In this formulation the Lagrangian density is given by

$$\mathcal{L} = \frac{\partial_\mu W \partial^\mu W^\star}{(1 + |W|^2)^2}, \tag{5}$$

where * denotes complex conjugation and the equations of motion are given by

$$\partial_t^2 W = \frac{2W^\star((\partial_t W)^2 - (\partial_x W)^2 - (\partial_y W)^2)}{1 + |W|^2} + \partial_x^2 W + \partial_y^2 W. \tag{6}$$

The simplest nontrivial static solution of (6) is given by

$$W = \lambda \frac{z - a}{z - b}, \tag{7}$$

where $z = x + iy$ and a, b and λ are arbitrary complex numbers. It is easy to calculate the energy density, E, corresponding to the static solution (7). We find

$$E = \frac{8|\lambda|^2\,|a - b|^2}{(|z - b|^2 + |\lambda|^2\,|z - a|^2)^2} \tag{8}$$

and so we see that the extended structure of this solution has a bell-like shape, with its position and size determined by

$$\frac{a|\lambda|^2 + b}{|\lambda|^2 + 1} \quad \text{and} \quad \frac{|\lambda|^2\,|a - b|^2}{(|\lambda|^2 + 1)^2}$$

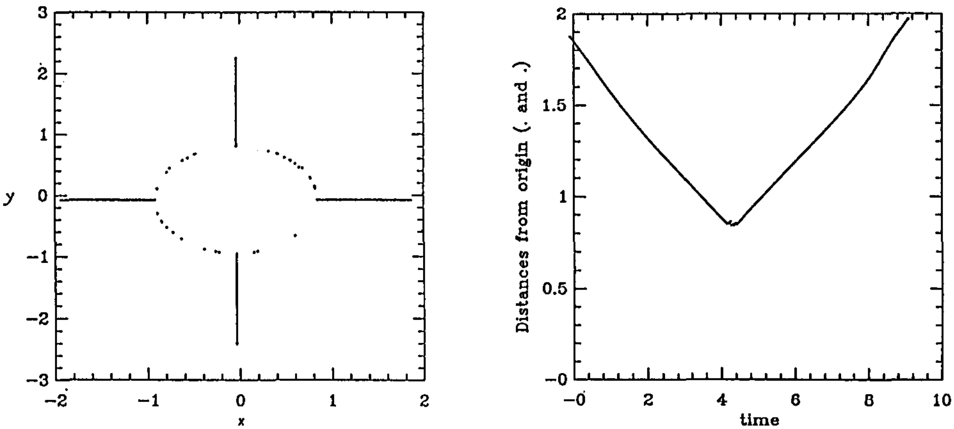
respectively. Taking the limit $\lambda \to \infty$, $b \to \infty$ while keeping their ratio fixed (and still called λ) allows us to consider $W = \lambda(x + iy - a)$ as our candidate for a soliton of "size" λ, which is positioned at a. In the same sense $W = \lambda\frac{(x+iy-a)(x+iy+a)}{2a}$, provides us with a field configuration describing two such solitons (positioned at $\pm a$).

These solutions, strictly speaking, can be considered as soliton configurations for the (2+1) dimensional model only if they are stable and do not desintegrate when we consider their time evolution. So what is their evolution? We performed several numerical studies using a 4th order Runge-Kutta method of simulating time evolution. The calculations were performed on fixed lattices which varied from 201×201 to 512×512, with lattice spacing $\delta x = \delta y = 0.02$. The time step was 0.01. We used fixed boundary conditions for most of our simulations involving the $\vec{\phi}$ fields (with some absorption on the boundaries) and the extrapolated boundary conditions for the simulations involving the W_i fields. We have tested our results by changing the lattice size and varying the boundary conditions and we are reasonably confident as to their validity.

2. CP^1 RESULTS

The obtained results reveal that when the solitons are sent initially at zero impact parameter they scatter at 90°. At the same time, however, they are unstable in the sense that they tend to change their size. We have analysed this problem in some detail[3] [4] . The observed instability is due to the fact that the pure S^2 model (has no intrinsic scale and so admits the existence of solitons of arbitrary size. Hence under small perturbations the solitons can either expand indefinitely or shrink to become infinitely tall spikes of zero width. Our simulations have shown that this is exactly what happens in this model. In fact, as soon as the solitons are purturbed, *e.g.* start moving, they start shrinking. This is true not only in the full simulation of the model but also[6] in the approximation to the full simulation provided by the so-called "collective coordinate" approach in which the evolution is approximated by the geodesic motion on the manifold of static solutions.

However, even though the extended structures shrink, we can still look at their scattering properties as they become very spiky only some time after emerging from the scattering region. We can lengthen this time by letting them expand as they move towards each other (in this case they shrink less as they come out). In all cases the scattering proceeds through the same intermediate stages; first the extended structures come towards each other, then when they are close together they form a ring and finally they emerge out of the ring at 90° to the original direction of motion. We can follow their trajectory in the x, y plane; an example of a typical simulation is shown in fig 1a; in fig 1b we show a plot of the time dependence of the relative distance between the extended structures. Looking at fig 1a we see that when the solitons are close together it is really difficult to talk about their positions (they are really in the shape of a ring); when we think of their speeds and the positions they started at and they find themselves at when they have emerged from the interaction region we see that, in analogy to what happens in (1+1) dimensions, their positions are translated forward along their trajectories. How much are they shifted along their trajectories? This question is difficult to answer as, due to the ring structure of their intermediate state, we do not understand their trajectories. What happens when the two solitons are on top of each other? Such a configuration would correspond to $W = \mu(x + iy - a)^2$. As its easy to see the energy density of such a configuration is in the shape of a ring centered at $x + iy = a$. Thus it would seem natural to assume that the two solitons come on top of each other before they scatter at 90°.



To study the importance of the ring formation and/or the shrinking of solitons we decided to go beyond the simplest S^2 model. To overcome the shrinking one has to break the conformal invariance of the model. This can be done by introducing a "Skyrme-like" and a potential term to the Lagrangian. Such a model was introduced in ref. [5] and used for many simulations. The "Skyrme-like" term of that model was later shown to be unique[7] so that the only arbitrariness resides in the form of the potential term. Of course, the form of the potential term does effect the details of the scattering, but all our simulations have shown[8] that the gross features are always the same. The introduction of the new terms generates some forces between the solitons; in particular, if we stick to the model of ref [5] these forces are repulsive and the only static solution corresponds to one soliton of a fixed "size" (whose value is determined by the parameters of the additional terms). The existence of the repulsive forces introduces a "critical" velocity into the model; in head-on collisions below a certain value of the velocity the solitons scatter back to back, above it they scatter at 90°. If one assumes that the scattering at 90° proceeds through the intermediate stage of the solitons "coming on top of each other" one can estimate the value of this critical velocity by performing an energy balance; such a calculation can be found in ref [9] where it was shown that this estimate agrees well with the results obtained in numerical simulations.

Similar results have also been found in the models based on other potentials[8] . In all cases, at sufficiently high velocities (to overcome all the repulsive forces), the elementary head-on collisions between two solitons exibit the 90° scattering. At lower velocities, and in interactions involving more solitons the scattering properties are more complicated. For lack of space we will present the results of our investigations of these cases elsewhere.

3. CP^2 MODEL

To go beyond the formation of a ring, when the solitons are on top of each other, we have to consider a model with a larger target manifold space; the simplest such model is the CP^2 model. This model involves two complex W fields (like the W field of the CP^1 model) and the Lagrangian density is given by

$$\mathcal{L} = \frac{\partial_\mu W_1 \partial^\mu W_1^\star + \partial_\mu W_2 \partial^\mu W_2^\star + (W_1 \partial_\mu W_2 - W_2 \partial_\mu W_1)(W_1 \partial^\mu W_2 - W_2 \partial^\mu W_1)^\star}{(1+|W_1|^2+|W_2|^2)^2}, \tag{9}$$

and the equations of motion are given by

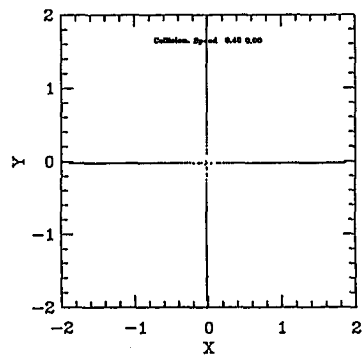
$$\begin{aligned} \partial_t^2 W_1 &= \frac{2W_1^\star\Big((\partial_t W_1)^2 - (\partial_x W_1)^2 - (\partial_y W_1)^2\Big)}{1+|W_1|^2+|W_2|^2} \\ &+2W_2^\star\Big(\frac{(\partial_t W_1)(\partial_t W_2) - (\partial_x W_1)(\partial_x W_2) - (\partial_y W_1)(\partial_y W_2)}{1+|W_1|^2+|W_2|^2}\Big) + \partial_x^2 W_1 + \partial_y^2 W_1 , \end{aligned} \tag{10}$$

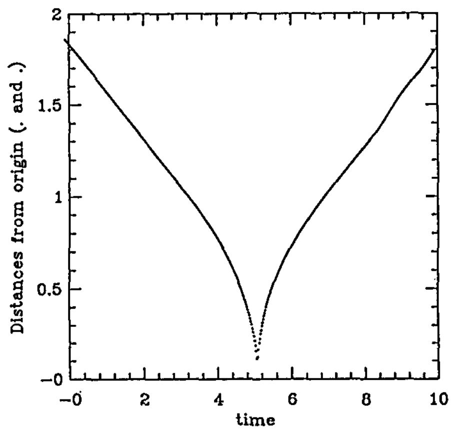
and a similar equation for W_2, obtained from (10) by the interchange $(1 \leftrightarrow 2)$.

It is easy to see that $W_1 = \lambda z^2$, $W_2 = \mu z$ is a static solution of the equations of motion and describes two solitons on top of each other (and located at $z = 0$). For a general choice of the parameters λ and μ the energy density of the configuration has a ring-like structure (like in the CP^1 case); however, when the parameters μ and λ satisfy $\mu^2 = \sqrt{2}\lambda$ the energy density takes the shape of a single peak (*i.e.* the ring becomes a peak). We can displace the solitons initially by choosing $W_1 = \lambda(z^2 - a)$ for some reasonable value of a, and then taking W_2 as above with $\mu^2 =$

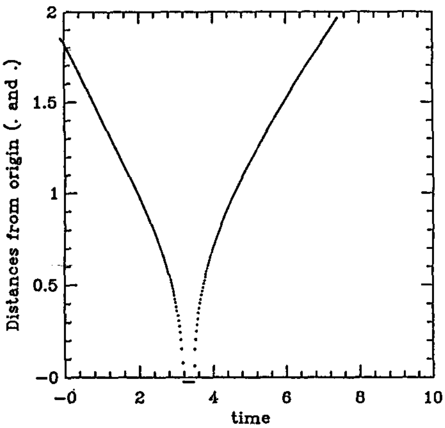
$\sqrt{2}\lambda$, set the two solitons moving towards each other by starting the simulation off with $\frac{dW_1}{dt} = aV$, $\frac{dW_2}{dt} = 0$. With such an initial value problem the solitons are set to expand as well so that when they emerge out of the interaction region they do not shrink too fast.

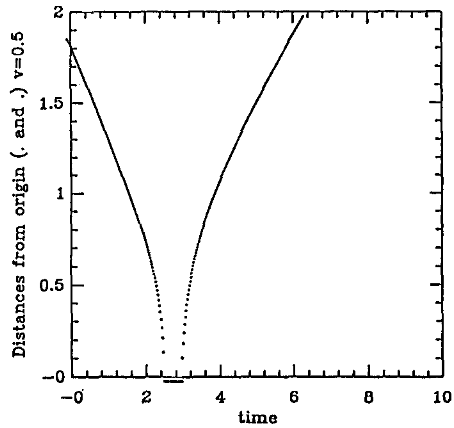
We have performed many simulations corresponding to different values of the the initial velocity V. All our simulations showed a 90° scattering. Moreover, they also showed a shift along the trajectory as seen initially in the CP^1 case. In fig2 we display typical trajectories of our solitons and in fig3 a,b and c we show the time dependence of the distance between the solitons for simulations started with three values of V. We clearly see a shift along the trajectory which is similar to the one observed in the CP^1 case except that this time the interpretation is easier (our picture suggests that as the solitons are close together they speed up and then come on top of each other where they spend some time after which they separate and gradually, as they leave the interaction region, they regain their initial speed). Clearly this is only a qualitative picture of their interaction; when they are close together they loose their identity, and like in the (1+1) dimensional case, it makes little sense of talking about their trajectories. Moreover, as is easy to check, the shift along the trajectories does not depend on V (and in the case of the simulations shown in fig3. its value is $\delta = 1$, if we assume that the solitons go through the origin).





In addition we would like to add that we have also looked at some field configurations corresponding to one soliton and one antisoliton. As the forces between them are attractive, placed some distance apart, solitons and antisolitons move towards each other and then annihilate into pure radiation. The angular dependence of the outgoing radiation is not uniform; most of it is, again, sent at out at 90° to the direction of their final approach (just before the annihilation). This has also been observed in some earlier simulations[5] .





Thus we see that our solitons in (2+1) dimensions behave very much like real solitons. In the scattering processes they preserve their shape and although some radiation effects are present these effects are always very small. In conclusion, we see that the modified S^2 model, although non-integrable, is almost integrable in that it has many features in common with many integrable models. Most differences or deviations are rather small. As most physically relevant models are not integrable our results suggest that the results found in some integrable models should not be dissmissed as not relevant; it is quite likely that some of these results may also hold in models which, strictly speaking, are not integrable but whose deviations from integrability are rather small.

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