SPIRAL WAVES IN EXCITABLE MEDIA

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INTRODUCTION

The problem of the formation of spiral waves in two-dimensional excitable media is one of the classical problems of nonequilibrium pattern-forming system which remains unsolved. It is clear now , from numerical simulations, that reaction diffusion sytems are a good physical basis for the explanation of the formation and of the characteristics of the spiral waves. However, the mathematical understanding of these solutions is far from being complete. A first problem is to determine the characteristics of the steadily rotating spiral wave , rotational frequency and radius of the unexcited circle lying at the center of the spiral.
A second problem is the stability of the spiral wave. In a range of control parameters, an instability mode called meandering is found. Meandering of spiral waves is a significant deviation of the spiral tip from circular trajectories and is now well confimed by experiments [1],[2]. Meandering is also observed in numerical simulations of standard reaction-diffusion models showing that the explanation of this fascinating behaviour must be found in the frame of these reduced systems. Contrary to experiments transition to meandering is well defined. It is a supercritical Hopf bifurcation [3].

In order to understand these phenomena, the simplest model from which one can start consists of a set of two coupled reaction diffusion equations with two very different time scales, one for the trigger variable c_1 which varies on the fast scale, the other for the recovery variable c_2 , on the slow scale. Numerical simulations of this model exhibit steadily rotating fields , with a well determined frequency ω, the region where the gradient of c_1 is sharp being located on a spiral shape. In order to understand these solutions and determine the frequency of rotation of the spiral, one simplifies the model and assimilates the regions where c_1 varies rapidly to a closed moving contour which delimits an excited region. Outside this contour , in the refractory region , the slow recovery variable decays until the occurence of another excitation. Moreover, aroud the center of rotation of the spiral lies an unexcited region with a well determined radius r_0 [4],[5],[6].

In a first part we describe this simplified model and discuss how it can be determined from the set of two coupled reaction-diffusion equations. In a second part, we study the solutions of the model corresponding to steadily rotating spirals. We determine uniquely the angular velocity ω and the hole of radius r_0 as a function of the control parameters of the model, i.e. ε the ratio between fast reaction time and refractory time and δ the excitability [6]. In a third part we discuss the stability of the spiral waves in a limiting case where a first analysis can be performed [7].

THE MODEL

The basis of W.F.M. is the classical piece-wise linear model for the following reaction-diffusion system:

$$\varepsilon \frac{\partial c_1}{\partial t} = \varepsilon^2 \Delta c_1 + f(c_1, c_2) \qquad (1)$$

$$\frac{\partial c_2}{\partial t} = c_1 - \delta \qquad (2)$$

where $f(c_1, c_2)$ is the piece-wise linear function drawn on Fig.1:

$$f(c_1, c_2) = \begin{cases} -c_1 - c_2 + 1 , & 0 < c_1 \\ -c_1 - c_2 - 1 , & c_1 < 0 \end{cases} \qquad (3)$$

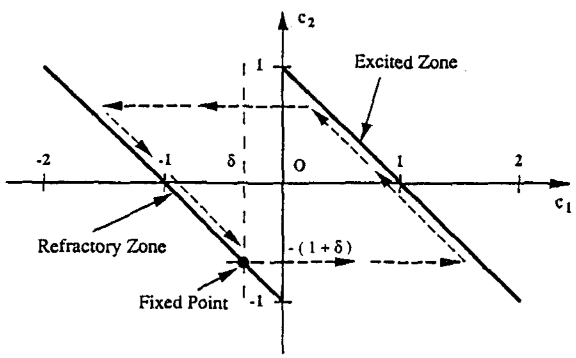

Fig.1 Piecewise linear model.

and δ a real and negative number larger than -1 which characterizes the excitability of the system. Here, $\varepsilon = \tau / T$ is the ratio between fast reaction time τ and the refractory time T. Time is scaled with T and lengths by the diffusive length $(D\tau)^{1/2} / \varepsilon$, where D is the diffusive coefficient of the trigger variable c_1. It is assumed here that only the trigger variable can diffuse in the system.

It is well known that when ε is small , the reaction-diffusion system admits one-dimensional solutions corresponding to the propagation of a sequence of stable pulses. Each pulse of the sequence is characterized by two waves (the front and the back) propagating with the same velocity separated by an excited region. In each wave , the concentration of the recovery variable c_2 can be assumed constant . The propagation velocity $c(c_2)$ of the wave can be found as a function of c_2 after integration of eqn.1 in a frame moving with constant velocity $c(c_2)$, in the new space variable $\xi = (x - c(c_2)t) / \varepsilon$. In the case of the piece-wise linear model (3), this velocity is found as

$$c (c_2) = - \frac{2c_2}{\sqrt{1 - c_2^2}} + O (\varepsilon) \qquad (4)$$

As introduced in Zykov [8] book p.5, an important quantity caracterizing the front is the maximum relative rate of increase of c_1 , E_{max} / A = | (∂c_1 / ∂t) $|_{max}$/ A ,where A is the amplitude of the wave. In the case of the simple piece-wise linear model defined above, this quantity is found as | c_2 | / ε. Its inverse corresponds to the transit time of the wave. In the case of the single pulse c_2 = - (1 + δ), sothat E_{max} / A = | 1 + δ | / ε .When the pulse is isolated, the duration of the pulse (transit time of the excited region) is found as D = - Log | δ |. Experimentalists are more familiar with the quantities E_{max} / A and D . The model is defined with the two independant parameters ε and δ. In the following we will use alternatively these two kinds of parameters.

When the pulse propagates in an inhomogeneous field c_2, the equiconcentration waves of the trigger variable c_1 become distorted and the normal velocity (4) of the waves are modified by transverse concentration fluxes. It is well known that when the curvature radius of the wave is large compared to its thickness, the normal velocity of the wave is modified proportionally to the curvature as

$$\vec{v}.\vec{n} = c (c_2) - \varepsilon \kappa + O(\varepsilon) \qquad (5)$$

where c (c_2) is determined by relation (4). We will now explain the model studied in the paper. We call this model " Wave Front Interaction Model " (W.F.M.) since it describes the motion of two fronts , the front and the back , which move with a normal velocity given by relation (5).These two waves interact with a relaxational field c_2 which satisfies the equation

$$\frac{\partial c_{2\pm}}{\partial t} = - c_{2\pm} \pm 1 - \delta \qquad (6)$$

where the subscript + (resp. -) means that the relaxational field c_2 is calculated in the excited region (resp. refractory region). This last equation is simply deduced from relation (2) with the additional relation c_1 = ± 1 - c_2 which holds respectively in excited and refractory regions. The common end point of the two curves must have a zero normal velocity. This model appears as a free-boundary problem in the spirit of the one proposed by Fife [3] and Tyson and Keener [4] , but simpler and thus more tractable since the diffusion coefficient of the recovery variable is assumed here to be zero. It contains two control parameters , the ratio between fast reaction time and refractory time ε , a real positive number and the excitability δ, a real negative number larger than -1. It is clear from the beginning that a first validity condition of the model is that the two waves will be well separated, which imposes that $0 < \varepsilon << 1$. Secondly, that the curvature radius of the wave is much larger than the front thickness, which implies $c << 1$. From relation (4), this condition can be satisfied if δ goes to -1.

STEADILY ROTATING SPIRALS

We look for solutions of W.F.M. corresponding to clockwise spirals rotating at constant angular velocity ω , around a hole of radius r_0. Consider a polar coordinate system (r, θ) rotating with constant angular velocity ω. Then , the shapes of the front and the back , respectively θ_F (r) and θ_B (r) satisfy eqn.5 , i.e.

$$\frac{\omega r}{\sqrt{1+\Psi_{F,B}^2}} = \pm c\,(c_{2F,B}) - \varepsilon\left(\frac{\frac{d\Psi_{F,B}}{dr}}{\sqrt{1+\Psi_{F,B}^2}^{\,3}} + \frac{\Psi_{F,B}}{r\sqrt{1+\Psi_{F,B}^2}}\right) \qquad (7)$$

Here , $\Psi_{F,B} = r\, d\,\theta_{F,B}(r)/dr$, the sign + (resp. -) corresponds to the subscript F (resp. B). and c (c_2) is determined by relation (4). The two curves meet tangentially to the circle of radius r_0 sothat $\Psi_F(r_0) = -\Psi_B(r_0) = -\infty$, or from eqn. (7) their curvature reaches their critical value $\kappa_{cr} = c\,(c_{2F,B})/\varepsilon$. At large distance from the tip, front and back behave as phase-shifted Archimedean spirals, i.e. $\theta_B(r) = kr = \theta_F(r) + \text{constant}$, where k is a positive number . In each part of the domain delimited by the curves, the concentration of the recovery variable c_2 satisfies

$$\omega \frac{dc_{2\pm}}{d\theta} = -c_{2\pm} \pm 1 - \delta \qquad (8)$$

where the subscripts + and - are respectively associated to the excited and refractory regions. On the front , $c_{2+}(\theta_F(r)) = c_{2-}(\theta_F(r)) = c_{2F}$ and on the back, $c_{2+}(\theta_B(r)) = c_{2-}(\theta_B(r)) = c_{2B}$.

From Eqn.(8), one can deduce c_{2F} and c_{2B} as :

$$c_{2F} = 1 - \delta + \frac{2\left(\exp\left(\frac{\theta_B - \theta_F - 2\pi}{\omega}\right) - 1\right)}{\left(1 - \exp - \frac{2\pi}{\omega}\right)} \qquad (9)$$

$$c_{2B} = 1 - \delta + \frac{2\left(\exp - \frac{2\pi}{\omega} - \exp\left(\frac{\theta_F - \theta_B}{\omega}\right)\right)}{\left(1 - \exp - \frac{2\pi}{\omega}\right)} \qquad (10)$$

Numerical Results

For the details of the resolution of the system composed by eqns. (7), (9),(10) see the full paper by Pelcé and Sun [6]. For convenience we introduce the radial coordinate scaled with the tip radius $R = r\,\kappa_{tip} = r\,c/\varepsilon$.

Angular velocity ω and hole radius r_0

Hole radius r_0

The smaller is the ratio between fast reaction time and refractory time, the smaller is the radius of the circle around which the spiral tip rotates at constant angular velocity Fig.2. This curve diverges at the limiting value $\varepsilon_{max} = 6.8$. This means that solutions for steadily rotating spirals exist only for values of ε less than 6.8. For this particular value of δ, large hole radii are found for values of ε which are not small so that results obtained from W.F.M. may differ from the one obtained from the complete reaction-diffusion system. On the other hand solutions corresponding to small radii which are obtained for small values of ε may be in good agreement with the one obtained from complete simulations of the reaction-diffusion system.

Angular velocity ω

It is a decreasing function of ε Fig.3.

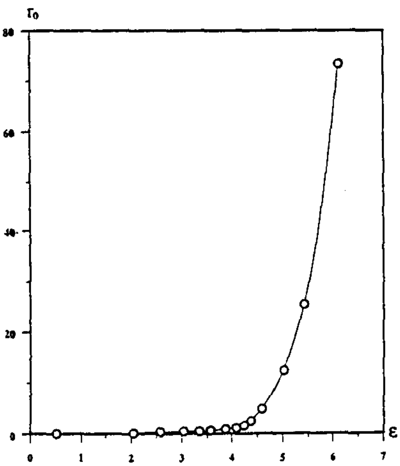

Fig.2 Hole radius r_0 as a function of ε (δ = - 0.1).

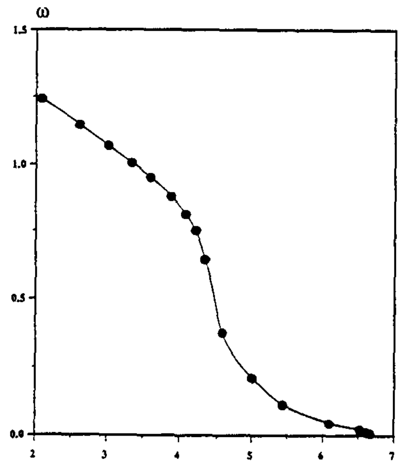

Fig.3 Angular velocity ω as a function of ε (δ = - 0.1).

The spiral shape

Spiral shapes are drawn on Fig.4 for R_0 = 1. and R_0 = 10. for the same value of the excitability δ = -.1.

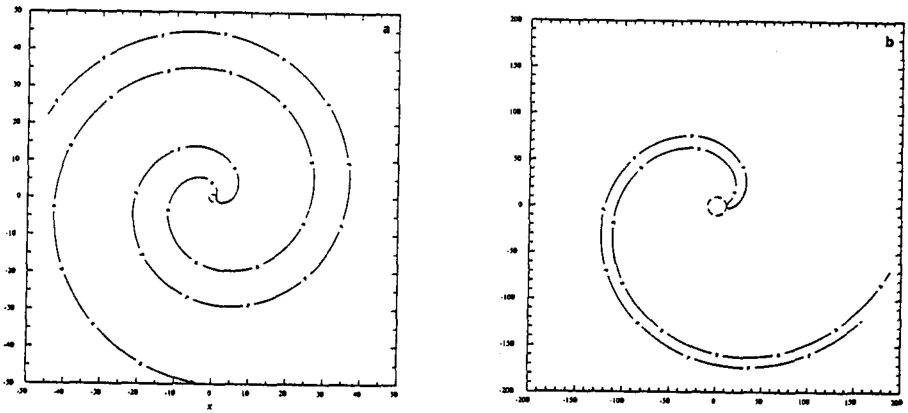

Fig.4 Shapes of spirals (δ = -.1): left R_0 = 1. ; right R_0 = 10

Steadily rotating spirals in the diagram (E_{max} /A , D)

It is more convenient to draw the steady state curves for rotating spirals in the diagram (maximum relative rate of increase of the trigger variable c_1, pulse duration D) (Fig.7). This kind of diagram is well discussed in the Zykov book [8] and very useful as far as tip meandering problem is posed in the cardiologist context. For this, we take different values of ε and δ , determine the corresponding values of E_{max} /A and D and compute the hole radius R_0 for these values in a similar way as what was done in section II.a) . Then, curves of equiradius are drawn in the diagram (E_{max} /A, D) . As was found for the approximate solutions of Zykov [8], for a fixed hole radius, the maximum relative rate of increase of the trigger variable c_1 decreases when the pulse duration D increases. On this diagram, we draw the "validity line" of W.F.M. which limits the region where 0 < ε < .5.

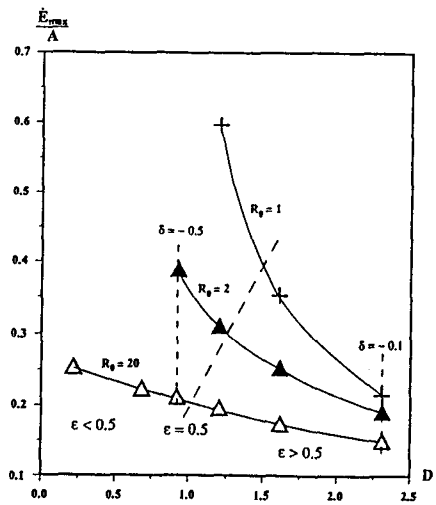

Fig.7 Steady state rotating spirals in the (E_{max} / A, D) diagram. The dashed lines limits the region where 0 < ε < .5.

FIRST APPROACH FOR A LINEAR STABILITY ANALYSIS.

Before to perform the whole linear stability analysis involving perturbations of the two waves, it is more convenient to analyse a simpler situation [7]. It was observed from the analysis of the steady states that for an excitability δ slightly larger than a minimum value δ_{min} (ε) the radius of the central unexcited region (hole radius) is very large [8],[9]. Below this value spiral wave retracts and no steady rotation is possible. When the spiral tip rotates on a trajectory of large curvature radius, the time between two consecutive excitations is very large and the refractory fiels has time to come back to its steady value. Thus the front can be considered as moving in an unexcited medium at equilibrium. It is shown [8] that in this case, the structure of the wave is that of a pulse moving with a well determined normal velocity c terminating on a free end with zero normal velocity and constant tangential velocity c. It is known from experiments and numerical simulations that in this case, the uniform spiral rotation is stable. Even in this simple situation it is not evident to explain the reason for this stability. If at large distance from the tip it appears clear that perturbations of the spiral shape must be smoothed out because of the stabilizing effect of the curvature on the normal velocity, it is not so clear why , if the tip penetrates in the unexcited region (Fig.8) , it is repelled towards the steady tip trajectory.

Thus, we perform a linear stability analysis of the uniform steady rotation of an opened curve moving with a normal velocity linear function of the curvature and with a free end moving with zero normal velocity and constant tangential velocity c. As expected, we show that this uniform rotation is stable , i.e., all the eigenvalues of the stability spectrum are negative (Fig.9). Furthermore, as it is often the case for linear stability problems in semi-infinite space, this spectrum is found to be discrete.

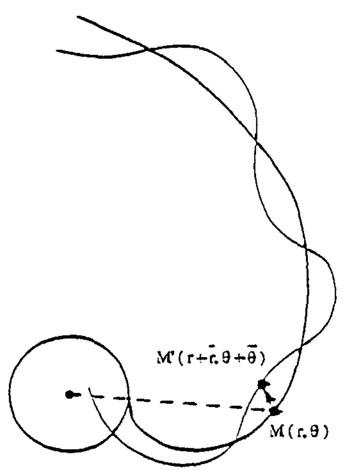

Fig.8: Sketch of a perturbation of the steadily rotating spiral.

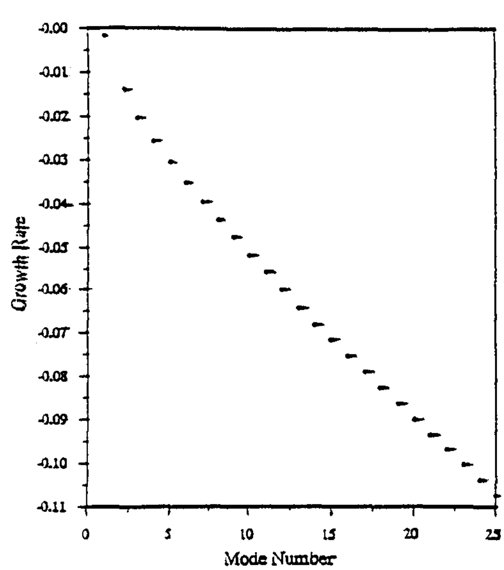

Fig.9: Spectum of growth rates

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