KADOMTSEV-PETVIASHVILI AND (2+1)-DIMENSIONAL BURGERS EQUATIONS IN THE BÉNARD PROBLEM

R.A. KRAENKEL, S.M. KURCBART, J.G. PEREIRA
Instituto de Física Teórica
Universidade Estadual Paulista
0145-SAO PAULO, SP, BRESIL

and

M.A. MANNA
Laboratoire de Physique Mathématique
Université MONTPELLIER II, Sciences & Techniques du Languedoc
34095-MONTPELLIER CEDEX, FRANCE

ABSTRACT. We study the surface perturbation of a viscous fluid adequately heated from below. We shown that under appropriate perturbations to the static solution the system exhibit oscillatory instabilities governed by the Kadomtsev--Petviashvili equation or by the (2+1) dimensional Burgers equation.

I. INTRODUCTION

The system formed by a fluid heated from below, the so called Bénard problem, has been along the years a standard model for many studies in fluid dynamics [1]. In most cases, however, the main interest has been concentrated in convection phenomena. While considering the same system, our concern here will be quite different since we shall be interested in the study of surface waves and in situations for which the Rayleigh number R is well below that determined by the onset of convection. Furthermore, we shall only consider systems for which the upper boundary is a two-dimensional surface. Yet, we shall restrict ourselves to the study of long surface waves, on whose description the slow variables play a very important role. By using appropriated slow space and time variables we shown that nearly one-dimensional undamped waves, described by the Kadomtsev-Petviashvili[2] equation, may propagate in a shallow viscous fluid, provided the Rayleigh number of the system satisfy the condition $R = 30$. This extend the result obtained by Alfaro and Depassier[3], in (1+1) dimensions. Furthermore, it will be shown that changing appropriately the perturbation scaling and the slow variables, the evolution equation governing the surface displacement is the (2+1)-dimensional Burgers[4] equations provided the Rayleigh number satisfy the condition $R \neq 30$.

II. THE SYSTEM

Let us consider a Bénard system consisting of a fluid bounded below by a plane stress-free perfect thermally conducting medium at z=0 and temperature $T = T_b$ and above by a free surface which, at rest, lies at z=d. The depth d is such that the buoyancy effect is predominant when compared to the influence of the surface tension. This is the reason we assume a vanishing surface tension. The equations governing the hydrodynamical flow of a viscous fluid can be simplified considerably by using the Boussinesq approximation. The origin of this simplification is the smallness of the coefficient of thermal expansion γ. As for mostly situations of practical occurence γ is indeed small, ranging usually from 10^{-3} to 10^{-4}, this approximation does not impose severe restrictions from the physical point of view. In this approximation, the equations describing the motion of a fluid are given by

$$\vec{\nabla}.\vec{V} = 0 \tag{1}$$

$$\rho_0 \frac{d\vec{V}}{dt} = - \vec{\nabla}p + \mu \nabla^2 \vec{V} + \vec{g}\rho \tag{2}$$

$$\frac{dT}{dt} = \kappa \nabla^2 T \tag{3}$$

$$\rho = \rho_0[1 - \gamma(T-T_0)] \tag{4}$$

where $d/dt = \partial/\partial t + \vec{V}.\vec{\nabla}$ is the convective derivative, $\vec{V} = (u,v,w)$ is the fluid velocity, and p is the pressure. The viscosity μ, thermal diffusivity κ, and coefficient of thermal expansion γ, are constant. T_0 and ρ_0 are a reference temperature and density, respectively.

On the upper free surface $z= d+\eta(x,y,t)$ the boundary conditions are[5]

$$\eta_t + u\eta_x + v\eta_y = w \tag{5}$$

$$(p-p_a)\eta_x-\mu \{2 u_x\eta_x - (u_z+w_x) + (u_y+v_x) \eta_y\} = 0 \tag{6}$$

$$p-p_a + \mu \{(w_x + u_z)\eta_x - 2w_z + (w_y+v_z)\eta_y\} = 0 \tag{7}$$

$$(p-p_a)\eta_y - \mu \{(v_x+u_y)\eta_x - (v_z+w_y) + 2v_y\eta_y\} = 0 \tag{8}$$

and

$$\hat{n} \cdot \vec{\nabla}T = -\frac{F}{k} \tag{9}$$

where $\hat{n}$ is the unit vector normal to the free surface, given by

$$\hat{n} = (-\eta_x, -\eta_y, 1)/N \quad ; \quad N = \left(1 + \eta_x^2 + \eta_y^2\right)^{1/2},$$

F is the normal heat flux, k is the thermal conductivity, and p_a is a constant pressure exerted on the upper free surface.

An important point is the dynamical boundary condition to be satisfied at the lower plane. We suppose that the sliding resistance between two portions of the fluid is much greater than between the fluid and the plane[6]. Under this condition, it is reasonable to assume a stress-free lower surface, which implies[1]

$$w = u_z = v_z = 0 \tag{10}$$

for $z = 0$.

The static solution to these equations depends only on the coordinate z and is given by

$$T_s = T_0 - \frac{F}{k}(z-d) \quad , \quad \rho_s = \rho_0\left[1 + \frac{\gamma F}{k}(z-d)\right]$$

$$p_s = p_a - g\,\rho_0\left[(z-d) + \frac{\gamma F}{2k}(z-d)^2\right] \quad .$$

In order to get the dimensionless form of the equations, boundary conditions and static solutions, we adopt d as unit of length, d^2/κ as unit of time, $\rho_0 d^3$ as unit of mass, ant Fd/k as unit of temperature. Furthermore, we introduce three dimensionless parameters : the Prandtl number $\sigma = \mu/\rho_0\kappa$, the Rayleigh number $R = \rho_0 g\gamma F d^4/k\kappa\mu$ and the Galileo number $G = gd^3\,\rho_0^2/\mu^2$.

III. THE KADOMTSEV-PETVIASHVILI EQUATION

In order to obtain the Kadomtsev-Petviashvili (KP) equation we need introduce the following slow variables

$$\xi = \epsilon\,(x - ct) \quad ,$$
$$\zeta = \epsilon^2 y \quad ,$$
$$\tau = \epsilon^3 t \quad ,$$

with ϵ a small parameter, chosen so that the amplitude η of the surface perturbations is $O(\epsilon^2)$:

$$\eta = \epsilon^2\left(\eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \ldots\right) \quad .$$

Introducing now the expansions

$$u = \epsilon^2 \left(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots\right) \quad ,$$

$$v = \epsilon^3 \left(v_0 + \epsilon v_1 + \epsilon^2 v_2 + \ldots\right) \quad ,$$

$$w = \epsilon^3 \left(w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots\right) \quad ,$$

$$P - P_s = P_0 + \epsilon\, P_1 + \epsilon^2\, P_2 + \ldots \quad ,$$

$$T - T_s = \theta_0 + \epsilon\, \theta_1 + \epsilon^2 \theta_2 + \ldots \quad ,$$

where all quantities are dimensionless, we can obtain an order by order solution to the equations (1)-(10). In order ϵ^0, the solution is given by

$$\theta_0 = P_0 = 0 \quad .$$

In order ϵ^1 it is

$$\theta_1 = P_1 = 0 \quad , \quad \eta_0 = f(\xi,\zeta,\tau)/c \quad ,$$

$$u_0 = f(\xi,\zeta,\tau), \; w_0 = -z\, f(\xi,\zeta,\tau) \quad ,$$

with $f(\xi,\zeta,\tau)$ an arbitrary function. In order ϵ^2 we find

$$\theta_2 = 0 \quad , \quad P_2 = \sigma^2 G\, \eta_0 \quad , \quad \eta_1 = g(\xi,\zeta,\tau)/c \quad ,$$

$$u_1 = g(\xi,\zeta,\tau) \quad , \quad v_0 = r(\xi,\zeta,\tau) \quad ,$$

$$w_1 = -\, z\, g_\xi(\xi,\zeta,\tau) \quad ,$$

with $g(\xi,\zeta,\tau)$ and $r(\xi,\zeta,\tau)$ arbitrary functions. At this order, the solubility condition implies

$$c^2 = \sigma^2\, G \quad .$$

In the next order the solution is given by

$$\theta_3 = \frac{1}{6}\, f_\xi\, (z^3 - 3z) \quad ,$$

$$P_3 = \frac{1}{24}\, \sigma\, R\, f_\xi (z^4 - 6z^2 + 5) - 2\sigma\, f_\xi + \sigma^2\, G\, \eta_1 \quad ,$$

$$u_2 = \frac{1}{24} f_{\xi\xi}(z^6 - 15\, z^4 + 39z^2) + h(\xi,\zeta,\tau) \quad ,$$

$$v_1 = s(\xi,\zeta,\tau) \quad ,$$

$$w_2 = - \frac{1}{168} f_{\xi\xi\xi}(z^7 - 21\, z^5 + 91\, z^3) - z(h_\xi + r_\zeta) \quad ,$$

with $h(\xi,\zeta,\tau)$ and $s(\xi,\zeta,\tau)$ arbitrary functions. The solubility condition in this order determines the critical Rayleigh number $R_c = 30$. On the other hand, the boundary conditions yield now two relations among the arbitrary functions $f(\xi,\zeta,\tau)$, $h(\xi,\zeta,\tau)$ and $r(\xi,\zeta,\tau)$:

$$ch_\xi - c^2(\eta_2)_\xi = - f_\tau - 2\, ff_\xi - \frac{71}{168} cf_{\xi\xi\xi} - cr_\zeta \quad , \tag{11}$$

$$r_\xi = f_\zeta \quad . \tag{12}$$

Finally, in order ϵ^4 the expression for θ_4, P_4 and the boundary conditions, yields the relation

$$ch_\xi - c^2(\eta_2)_\xi = f_\tau + \frac{30 + \sigma G}{\sigma G} ff_\xi + \frac{272\sigma - 15}{168} cf_{\xi\xi\xi} \quad . \tag{13}$$

The requirement of compatibility of eqs. (11)-(13) provides an evolution equations for the function $f(\xi,\zeta,\tau)$:

$$\left(f_\tau + \frac{3(10 + \sigma G)}{2\sigma G} ff_\xi + c\left(\frac{17}{21}\sigma + \frac{1}{6}\right) f_{\xi\xi\xi}\right)_\xi = - \frac{1}{2} cf_{\zeta\zeta} \quad .$$

By transforming the variables as

$$\xi \longrightarrow \Lambda^{-1}\, \xi \ , \ \zeta \longrightarrow \left(\frac{c}{6\Lambda}\right)^{1/2} \zeta \quad , \quad f \longrightarrow - \frac{4G\sigma}{(10+G\sigma)\Lambda} f \quad ,$$

with

$$\Lambda = \left[c\ \left(\frac{17}{21}\sigma + \frac{1}{6}\right)\right]^{-1/3}$$

we get

$$(f_\tau - 6\, ff_\xi + f_{\xi\xi\xi})_\xi + 3\, f_{\zeta\zeta} = 0 \quad . \tag{14}$$

which is the KP equation[7].

The solitary-wave solution of (14) is given by[8]

$$f(\xi,\zeta,\tau) = A\ \mathrm{sech}^2\left\{\left[\frac{1}{2}(m-\ell)\ \xi - \frac{1}{2}(m^2-\ell^2)(6/\Lambda c)^{1/2}\zeta - 2(m^3-\ell^3)\ \tau/\Lambda\right]\Lambda-\beta\right\} \quad (15)$$

with m and ℓ real numbers, $\beta = \frac{1}{2}\ \ell n\ \left(-\frac{m}{\ell}\right)$, and

$$A = \frac{2G\sigma}{\Lambda(10 + G\sigma)}\ (m-\ell)^2 \quad .$$

If $\ell \neq -m$, eq. (15) represents an oblique solitary wave which moves at a certain angle to the ξ-axis and does not decrease along the direction defined by the equation

$$\xi = (\ell+m)(6/\Lambda c)^{1/2}\zeta \quad ,$$

as ξ, $\zeta \longrightarrow \infty$. If, however, $\ell = -m$, the solution (15) is converted into the Korteweg-de Vries soliton. In this way we see that, the solitary-wave solution of the KP equation describes a wave whose pattern consists of a horizontal streamline. These solitary waves, associated with oscillatory instabilities, are sustained by the adverse temperature gradient applied and just come into play because the amount of energy released by buoyancy exactly compensates the amount dissipated by viscosity.

IV. THE (2+1)-DIMENSIONAL BURGERS EQUATION

Here, we shall obtain a non-linear evolution of the surface displacement governed by the (2+1)-dimensional Burgers equation. To this end let us consider surface perturbation and slow variables given by

$$\eta = \epsilon\ \left(\eta_0 + \epsilon\ \eta_1 + \epsilon^2\ \eta_2\right) \quad ,$$

$$\xi = \epsilon(x - ct) \quad ,$$

$$\zeta = \epsilon^{3/2}y \quad ,$$

$$\tau = \epsilon^2\ t \quad .$$

Furthermore, we introduce the expansions

$$u = \epsilon\ \left(u_0 + \epsilon\ u_1 + \epsilon^2\ u_2 + \ldots\right) \quad ,$$

$$v = \epsilon^{3/2}\left(v_0 + \epsilon\ v_1 + \epsilon^2\ v_2 + \ldots\right) \quad ,$$

$$w = \epsilon^2\left(w_0 + \epsilon\ w_1 + \epsilon^2 w_2 + \ldots\right) \quad ,$$

$$P-P_s = P_0 + \epsilon\ P_1 + \epsilon^2\ P_2 + \ldots \quad ,$$

$$T-T_s = \theta_0 + \epsilon\, \theta_1 + \epsilon^2\, \theta_2 + \ldots \quad .$$

In the lowest order, the solution of equation (1)-(10) is given by

$$\theta_0 = 0 \quad , \quad P_0 = 0 \quad , \quad u_0 = f(\xi,\zeta,\tau) \quad ,$$

$$w_0 = -z\; f_\xi(\xi,\zeta,\tau) \quad ,$$

with $f(\xi,\zeta,\tau)$ an arbitrary function. In the next order it is

$$\eta_0 = \frac{1}{c}\, f \quad , \quad v_0 = h(\xi,\zeta,\tau) \quad ,$$

$$\theta_1 = 0 \quad , \quad P_1 = G\sigma^{\,2}\eta_0 \quad ,$$

$$u_1 = g(\xi,\zeta,\tau) \quad , \quad w_1 = -z(g_\xi + h_\zeta) \quad .$$

with $g(\xi,\zeta,\tau)$ and $h(\xi,\zeta,\tau)$ arbitrary functions.
In the order ϵ^2 we get

$$\theta_2 = \frac{1}{6}\, f_\xi(z^3 - 3z) \;\; , \;\; P_2 = \left[\frac{R\sigma}{24}\;\; (z^4 - 6z^2+5) \; - \; 2\sigma\right] f_\xi \; + \; \frac{R\sigma}{2}\, \eta_0^2 \; + \; G\sigma^2\; \eta_1 .$$

The boundary condition on the upper free surface yield the equation

$$c^2\eta_{1\xi} \; - \; cg_\xi = f_\tau + 2\; ff_\xi + c\; h_\zeta \quad . \tag{16}$$

At this order, there appear a solubility condition giving

$$c^2 = G\sigma^2 \quad .$$

In the next order the expressions for θ_3, P_3 and the remaining boundary condition yield the equation

$$-\, c^2\; \eta_{1\xi} + cg_\xi = f_\tau + \left[1 + \frac{R}{G\sigma}\right] ff_\xi \; - \; \sigma \left[4 \; - \; \frac{2R}{15}\right] f_{\xi\xi} , \tag{17}$$

as well as a relation between the arbitrary functions $f(\xi,\zeta,\tau)$ and $h(\xi,\zeta,\tau)$

$$f_\zeta = h_\xi \quad . \tag{18}$$

The requirement of compatibility of Eqs (16) (17) and (18) provides an evolution equation for f :

$$\left[f_\tau + \left(\frac{3G\sigma+R}{2G\sigma}\right) ff_\xi + \nu\; f_{\xi\xi}\right] = -\, \frac{c}{2}\, f_{\zeta\zeta} \quad ,$$

where

$$\nu = 2\sigma - \frac{R\sigma}{15} \quad .$$

By transforming f according to

$$f \longrightarrow \frac{2G\sigma}{3G\sigma+R} f \quad ,$$

we obtain the equation

$$(f_\tau + ff_\xi - \nu f_{\xi\xi})_\xi = - \frac{c}{2} f_{\zeta\zeta} \quad . \tag{19}$$

This is the (2+1)-dimensional Burgers equation. The complete integrability of (19) is currently under investigation[9]. Eq (19) has a progressive wave solution of the form

$$f(\Lambda) = f(\mathbb{A}\xi + \mathbb{B}\zeta - \mathbb{C}\tau) \quad ,$$

whose explicit form is

$$f(\Lambda) = \frac{(3G\sigma+R)(2\mathbb{C}-c\mathbb{B}^2)}{4G\sigma} \left\{1 - \mathrm{tgh}\left[\frac{(2\mathbb{C}-c\mathbb{B}^2)}{4\nu}(\Lambda-\Lambda_0)\right]\right\} \quad .$$

where $\mathbb{A},\mathbb{B},\mathbb{C}$ are constants depending on the initial condition. It represents a nearly one-dimensional Burgers shock wave. This shock wave is propagated without any overtaking and it is substained by the adverse temperature gradient provided the Rayleigh number of the system satisfy the condition $R \neq 30$. As the Rayleigh number approaches the point $R = 30$, the coefficient ν approaches to zero, nonlinearity is not compensated by dissipation and the wave begins to break.

It should be noticed, however, that this solution is not valid for an arbitrary large R, since in this region other phenomena which are not considered by our approach, may take place.

V. CONCLUSIONS

We have extended a (2+1) dimensions the result that a solitary wave may propagate in a viscous fluid subject to an adverse temperature gradient. We found that the surface displacement obey the Kadomtsev-Petviashvili equation.

On the other hand we shown that a much larger surface perturbation will be governed by the (2+1) dimensional Burgers equation. We predict the existence of a nearly one dimensional kink, provides the Rayleigh number satisfy the condition $R \neq 30$.

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