Chaotic motion of solitons in the PDE model of long Josephson junctions

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The Josephson junction [1] is an extremely interesting solid-state device both for applied physics and non-linear physics. It consists simply of a sandwich of two superconductive films separated by a thin layer, generally of few Å, of an insulator. Depending on the thickness of the insulator the 'macroscopic' wave functions of the two superconductors overlap in the insulator region leading to a coupling that constitutes the essence of Josephson effect [2]. The phase difference $\phi(x,t)$ between the two wave functions can show, temporally and/or spatially, a highly correlated behavior. In particular it can be shown, by means of a simple electrical model [3], that the phase satisfies a partial differential equation (PDE) known as the Perturbed Sine-Gordon equation. This equation, if only one spatial dimension is longer than the so-called Josephson penetration depth λ_J and in the so-called 'inline' configuration, is:

$$\phi_{tt} - \phi_{xx} + \sin\phi = \alpha\phi_t - \beta\phi_{xxt}. \tag{1}$$

We have in addition two time-dependent boundary conditions for a current biased junction irradiated by a microwave field [4,5]:

$$\phi_x(0,t) + \beta\phi_{xt}(0,t) = -\chi + \eta(t), \tag{2a}$$

$$\phi_x(l,t) + \beta\phi_{xt}(l,t) = \chi + \eta(t). \tag{2b}$$

In these formulas all the distances are normalized to λ_J, and times to $\omega_J = \bar{c}/\lambda_J$, the plasma frequency, where $\bar{c}$ is the speed of light in the junction. $\eta(t)$ is the normalized external magnetic field at the edges of the junction, α and β are loss parameters, and χ is the normalized bias-current supplied to the junction.

It is well known that Eq.(1) admits solitonic solutions [1,4]; in the context of Josephson junctions such solitons are named 'fluxons' because they carry a flux quantum $h/2e$. In the absence of any time dependent signal, $\eta(t) = 0$, the fluxons propagate back and forth in the junction, the energy dissipated in the propagation being replenished by the bias current at the boundary χ. Thus the frequency of the fluxon ω_{free} is dependent on the bias current χ via a non-linear relation. From the second Josephson equation it is possible to link the frequency of the fluxon to the d.c. voltage at the end of the junction; the branches obtained on the current-voltage (I-V) d.c. characteristics are called Zero Field Steps (to recall the absence of any external field, also static), and correspond to the propagation of one or more fluxons; we often refer to them simply as the unperturbed I-V curves. If we write for the external signal the form $\eta(t) = \eta_0 \sin(\omega_s t)$ [4,5] a single fluxon can be forced, for some values of the bias current, to oscillate in phase with this signal at the same frequency ω_s, i.e. the soliton is phase-locked to the external signal (the values of the bias current for which we have this effect are called phase-locking range). More generally a phase-locking state is obtained if after m periods of fluxon and n periods of the external signal the same phase-relation is reached, i.e. the frequencies of the signal and of the fluxon are in a fractional ratio [6]. Thus we have the following link between the soliton frequency and d.c. voltage at the end of the junction (in normalized units) [6]:

$$V_{n,m} = \frac{2n}{m}\omega_s. \tag{3}$$

In the phase-locking range of bias-current this phenomenon gives rise to constant-voltage steps on the I-V characteristics of the junction. Steps on I-V characteristic were observed in several experiments [7]; in many cases this observation is referred to the $m = 1, n = 1$ case (fundamental frequency of the junction), that has, as can be demonstrated (see Ref. [6]) the biggest phase-locking range. Experimentally subharmonic steps, $m > 1$, appear to be [8] much smaller, and, in general, are difficult to observe, though the theory predicts phase-locking ranges smaller than the case of the fundamental frequency, but surely in principle not unobservable. As we will see this experimental difficulty can be ascribed to chaotic behavior in long junctions pumped with a subharmonic frequency.

Kautz [9] first observed chaotic behavior in small (with respect to λ_J) Josephson junctions both numerically and experimentally. For such a junction equation (1) reduces to the equation of a forced pendulum; the short junction can be locked to the external signal (again producing constant-voltage steps on the I-V curves), but for suitable values of the amplitude and frequency of the external signal the phase-lock becomes unstable and a period two solution appears. Since now a frequency of the system is not well defined Eq. (3) can be substituted by

$$V = \frac{4\pi}{\langle T_p \rangle_{ave}}, \tag{4}$$

where T_p is the period of pendulum oscillations. Continuing to increase the amplitude of the signal the system undergoes a standard bifurcation cascade until a chaotic behavior is reached. It is interesting to note that constant-voltage steps on the I-V curves again exist also if the pendulum motion is chaotic for a short range of values of amplitude of the external signal after the beginning of the chaotic region in the parameter space [9]; sometimes this phenomenon is referred to as 'phase-locking chaos' or 'frequency locking' [5,10], but we prefer the more correct name of 'voltage locking'.

In the long junction case chaotic behavior has been observed numerically in many cases [11,12,13]. In all these cases the chaos is both spatial and temporal (turbulence), i.e. it appears essentially as a higher-dimensional phenomenon clearly related to the infinite degrees of freedom of the full equation (1).

On the other hand we can proceed in another direction: applying the McLaughlin-Scott [14] perturbative approach we can treat the motion of a single fluxon in the junction as the motion of a relativistic particle in the interior of the junction. Giving to this particle-fluxon a sharp energy supply at the boundary [15] we can, if the frequency of energy supply is in a fractional relation to the time of flight of fluxon T_k in the junction, phase-lock the soliton to the external field. This method is called 'map approach' after [6] because we can write, solving the equation of motion of the particle-fluxon in the junction, an (analytic) bidimensional map directly in terms of time of flight T_k and energy y_k (here the label indicates the k-reflection of the fluxon at one edge of the junction). We do not write explicity the map here; the interested reader can find it in ref. [16]. Obviously if we know the times of flights of the fluxon, we can obtain the d.c. voltage by means of the formula (4) merely identifying T_p with T_k cfr. [6]. If the external signal makes n periods while the fluxon makes m oscillations in the junction, the Eq. (4) reduces to Eq. (3). Since the map is analytic an analysis of the stability of phase-locked solutions can be easily conducted: if the amplitude of the external signal is sufficiently high again phase-locked solutions are unstable and 2-times of flight solution sets in. Then continuing to increase the amplitude, via a bifurcation cascade, the system becomes chaotic, i.e. fluxons travel in the junction with a chaotic distribution of times of flight.

The features of this chaotic motion are studied in detail in the ref. [10,17], here we limit ourselves to report the main aspects only:

i) the amplitude of the external signal necessary to arrive at the chaos decreases with m, i.e. chaotic states appear mainly on subharmonic steps $m \geq 3$;
ii) in terms of the bias current χ the chaos begins to develop on the unperturbed I-V characteristic at the center of the phase-locking step;
iii) increasing the losses, α and β, the stability of the system is increased;
iv) the system appears to be 'voltage locked', i.e. though the motion of fluxons is chaotic, the average of time of flight in Eq.(4) is such that V remains unchanged at a phase-locked value, so on the I-V characteristic the step remains vertical;
v) continuing to increase the amplitude of the external signal the system produces very long TOFs, that consequently annihilate the fluxon after a certain value $\eta_{0,a}$.

It is natural that the low-dimensionality of this chaos is an intrinsic feature of the perturbative approach used to reduce a PDE to an ODE equation. In fact the hypothesis of a single particle-fluxon propagating in the junction is essential in the perturbative approach to obtain all the above mentioned results. Nothing can be asserted about the full equation (1), i.e. we cannot say that (1) shows this type of low-dimensional chaos except by directly attacking it.

Our procedure was first to choose a region of parameter space where the map shows chaotic phenomena; then we have integrated numerically Eq.(1). The method used is based on the reduction of Eq.(1) to a system of ordinary differential equations (ODE's), then integrating the ODE's using some standard method (or methods). In our case, to obtain a very careful integration, we have used two different spatial discretization formulas (3-point and 5-point formulas within the junction) and two different methods to integrate the ODE system (a simple and fast Predictor-Corrector method [18] and a Bulirsh-Stoer method [19]). The results appear to be the same, within the discretization and time-step induced errors.

Using the map prediction we search for chaotic dynamics on a junction of normalized length $l = 10$, pumped with a frequency of $\Omega = 0.4$ on the $m = 3$ subharmonic. Biasing the junction at center of phase-locking step, $\chi = 0.493$, with an external amplitude of $\eta_0 = 0.100$ the fluxon appears to be phase-locked to the external signal. In Fig.(1) we report the voltage peaks at two edges of the junction signaling the reflection of the fluxon at the edges. The time interval between two successive peaks yields the TOF of the soliton in the junctions, from Fig.(1) we see that this time is constant and equal to $T = 23.56$, which implies a phase-locking voltage of $V = 0.2666 = (2/3)\,0.4$, as follows by Eq.s (3) and (4).

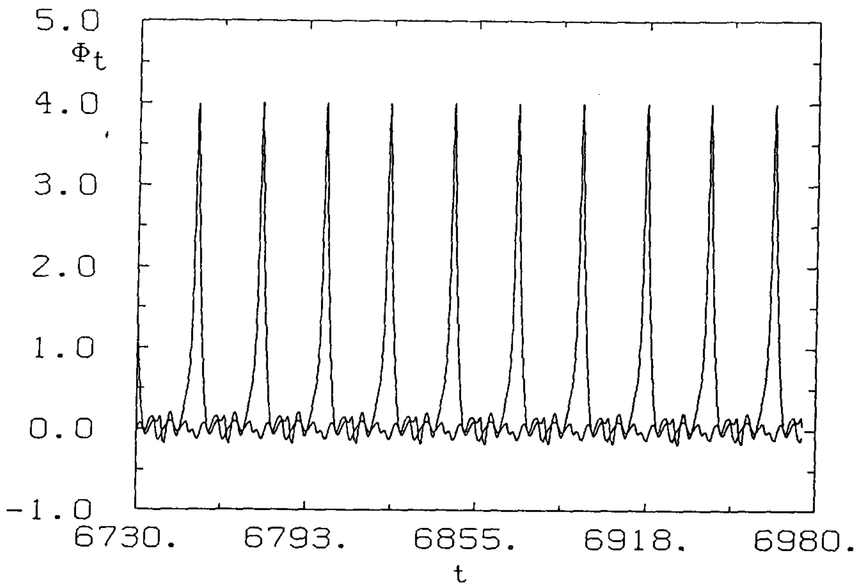

Fig.(1)

The small undulations following the reflection peaks can be ascribed to external signal effects when no soliton is present at the edges. The error, deduced from the numerical data, in the estimation of TOF is of the order of magnitude of the time-step, which is $\Delta t = 0.01$; thus we can resolve TOFs separated by more than this time-step (this fixes the above reported number of significant digits in the voltage). In Fig.(2) we have increased the value of η_0 to 0.150: as is clearly seen we have now a 2-TOFs solution in the junction. The two TOFs adjust themselves to make again $V = 0.2666$. Continuing to increase the amplitude of the external signal the 2-TOFs solution becomes unstable and the dynamics evolves via a bifurcation cascade to a chaotic state, that is shown in Fig.(3) for a field $\eta_0 = 0.190$. Though the motion is chaotic the system is again 'voltage locked' to the external signal: in fact the numerical evaluation of Eq.(4) over $n_r \sim 10^3$ reflections gives for the voltage $V = 0.2667$.

We stress that n_r is a relatively small number with respect to what might be used in a 'map approach', but in the PDE system it has to be considered a good goal, in view of the long integration time of the Eq.(1) (the number of the time steps in a typical run is $\sim 10^6$). However in the PDE system this situation is not permanent, further increase in the amplitude of the external signal leads to the loss of 'voltage locking': in fact for a field $\eta_0 = 0.260$, we obtain $V = 0.2080$, i.e. also the voltage assumes chaotic values and on the I-V characteristic the step ceases to be vertical. The bifurcation-cascade for the PDE approach is shown in Fig.(4). In Fig.(4) we have marked with an arrow the positions of η_0 relative to Fig.s (1) and (2). The dotted line is the approximate separation between the 'voltage locked' region and chaotic voltage region, occurring at an amplitude $\eta_{0,s} \sim 0.195$. After $\eta_{0,a} \sim 0.270$ the fluxon motion is unstable and the fluxon is annihilated.

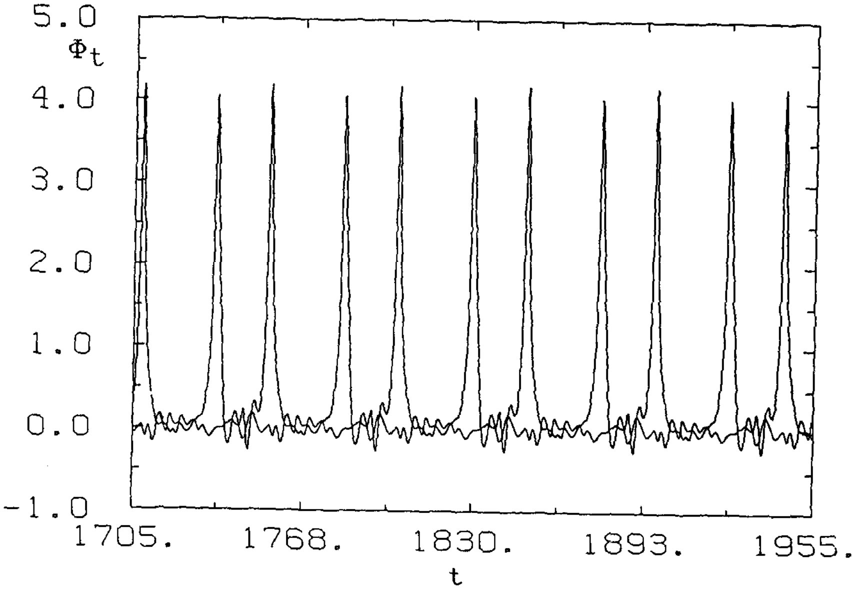

Fig.(2)

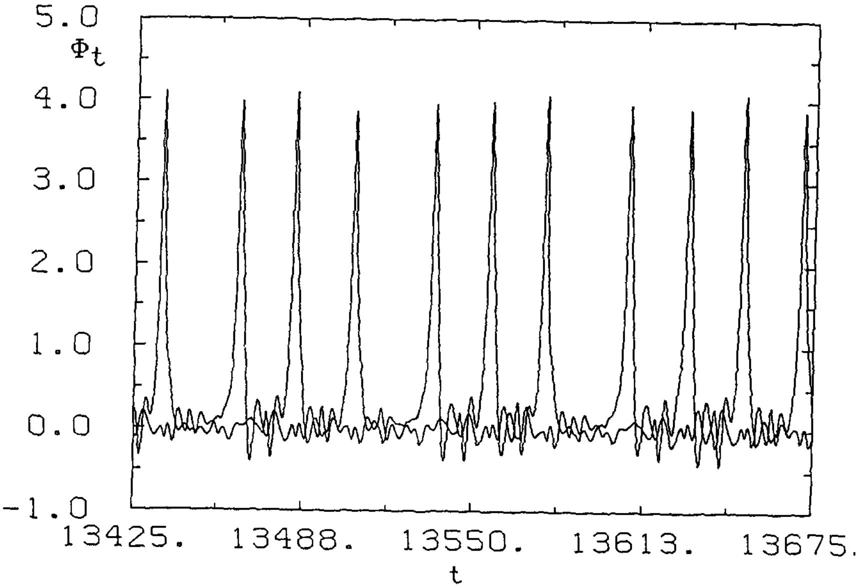

Fig.(3)

An estimate of the Feigenbaum ratio:

$$f = \lim_{n \to \infty} \frac{\eta_{0,n+1} - \eta_{0,n}}{\eta_{0,n} - \eta_{0,n-1}}$$

where $\eta_{0,n}$ is the value of n^{th}-bifurcation parameter, for the PDE data gives $f \sim 6.1$.

Comparision of Fig.(4) with the 'map approach' bifurcation [18], obtained with the same parameters, gives two quantitative differences. The first is that in the PDE model the phase-locking appears to be more stable, i.e. the value of the first bifurcation in the PDE approach is $\sim 20\%$ higher than in the map case; this can be partially explained with the differences in boundary conditions in the two methods: in the map context the external signal is an 'abrupt' delta function whereas in the PDE it is a 'smeared' sinusoidal signal, which implies that a strict comparison between the two amplitudes is not possible.

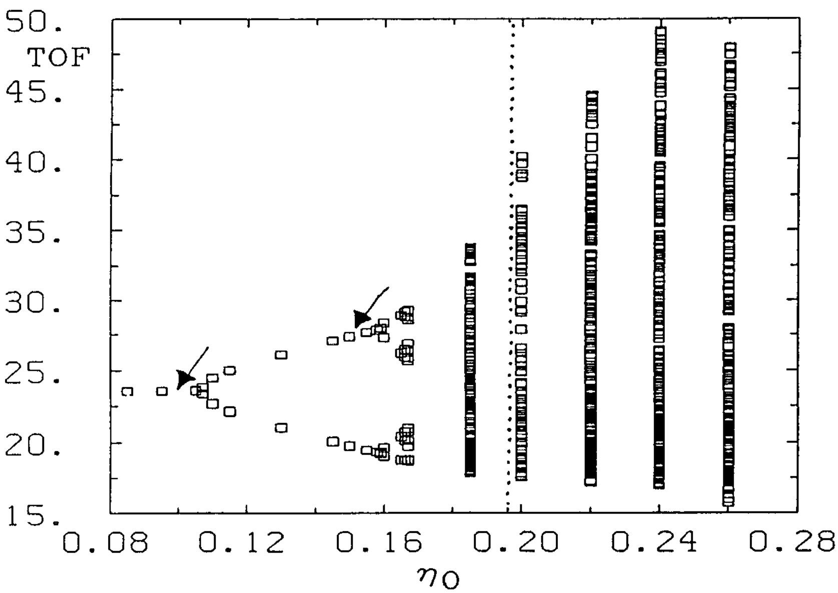

Fig.(4)

Moreover the bifurcation tree in the PDE is 'longer', i.e. the interval of η_0 in which the fluxon is stable (including the chaotic regime) is about 2 times the same interval with the map. On the other hand a comparison between map and PDE strange attractors shows that both objects are structurally very similar [20] confirming that the map approach grasps the substantial dynamics of Eq.(1). The main features of this chaotic motion of fluxon (PDE low-dimensional chaos) can be summarized as follows:

i) also here the amplitude of the external signal necessary to arrive at chaos decreases with m, i.e. in all runs chaotic states appear only on subharmonic steps $m \geq 3$; on the steps at the fundamental frequency an amplitude of $\eta_0 \sim 0.5$ in general destroys the 1-fluxon solution, but up to such an amplitude neither chaotic motion nor bifurcations are exibited;

ii) in terms of the bias current χ the chaos begins to appear on the unperturbed I-V characteristic at the center of the phase-locking step;

iii) increasing the losses, α and β, the stability of the system is increased;

iv) the system appears to be 'voltage locked' until an amplitude $\eta_{0,s}$, i.e. though the motion of fluxons is chaotic, the average of time of flight in Eq.(4) is such that V remains unchanged at the phase-locking value; after $\eta_{0,s}$ the 'voltage locking' state is lost;

v) a further increase of the amplitude of the external signal produces very long TOFs, that consequently annihilate the fluxon beyond a certain value $\eta_{0,a}$.

It is interesting to note that the value of $\eta_{0,s}$, depends upon the bias current χ, but the minimun is not at the center of the step, which implies that deviations from the voltage begin to develop in the low bias part of the step. A possible explanation could be the overlapping of attraction basins of more subharmonics [21].

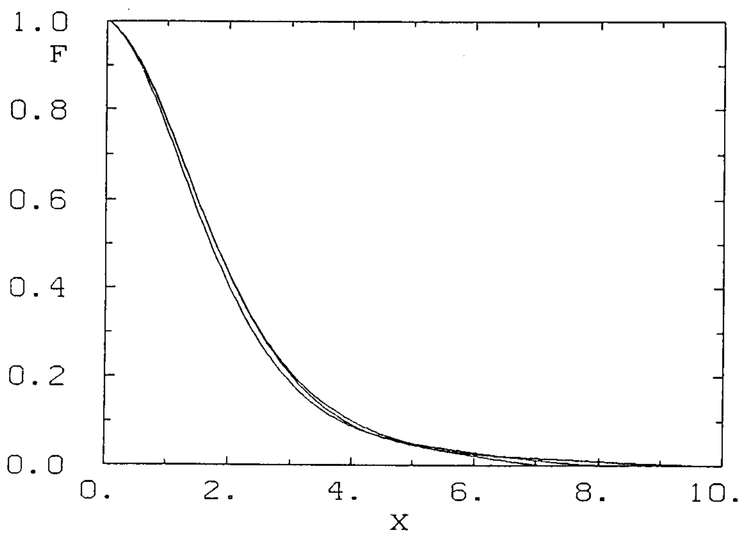

Fig.(5)

To give a measure of the permanence of a high spatial coherency also during the chaotic motion we have numerically evaluated the spatial mean autocorrelation of the soliton profile:

$$F(x) = \frac{1}{T} \int_0^T dt \, \langle \phi_t(x + x', t) \phi_t(x', t) \rangle \tag{5}$$

For a well-defined fluxon lineshape we expect correlation to extend over a distance of the order of λ_J, i.e. the typical distance of variation of the phase in the junction. A first result of this calculation is shown in Fig.(5), where we report Eq.(5) for three typical cases: phase-locking, and two chaotic solutions. As is evident from the figure correlation extends over a distance of the order of λ_J in all the cases. In Eq.(5) T was chosen of order of 10^2 periods. The small decrease in the spatial correlation of $\eta_0 = 0.190$ curves is within the accurancy of the estimate of integral (5). Thus we conclude that in the entire parameter interval the fluxon remains highly correlated though its motion is chaotic, confirming the existence of a low-dimensional chaotic attractor for the full PDE system governed by Eq.(1).

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