NOISE INDUCED BIFURCATIONS IN SIMPLE NONLINEAR MODELS

Karen Lippert, Konrad Schiele, Ulrich Behn and Adolf Kühnel
Sektion Physik der Universität Leipzig
Augustusplatz 10, O-7010 Leipzig Germany

Abstract

For two generalizations of the Stratonovich model with Gaussian white noise (GWN) and dichotomous Markovian process (DMP) the stationary probability density and moments are calculated. The bifurcation pattern of the stationary solution can be changed qualitativly by varying the deterministic or the noise parameters. We show cases where noise induced states and a subcritical bifurcation can be detected only from the knowledge of the variance.

1. Introduction

We consider systems far from equilibrium described by a simple nonlinear differential equation with few parameters and a noise term,

$$\dot{x} = f(x) + g(x)\cdot R_t \quad . \tag{1}$$

R_t denotes the noise. If the stationary solution x_s of the deterministic equation $f(x)=0$ bifurcates one observes similarities with phenomena in equilibrium phase transitions associating the position of the maxima of the stationary probability density with the order parameter. For R_t being a GWN ξ_t (with $\langle \xi_t \rangle = 0$, $\langle \xi_t \xi_{t'} \rangle = 2D\cdot\delta(t-t')$) or a DMP I_t (with $\langle I_t \rangle = 0$, $\langle I_t I_{t'} \rangle = \Delta^2 \exp\{-2\alpha|t-t'|\}$) the stationary probability distribution is calculated explicitly following standard methods [1-10]. For multiplicative coupling of the noise with the linear term the bifurcation point is shifted. For suitable chosen f(x) and g(x) the bifurcation type can be controlled by deterministic as well as by noise parameters.

For two generalizations of the Stratonovich model we show in which way the bifurcation pattern can be changed completely shifting the deterministic and the noise parameters and ask wether these changes can be detected already from the knowledge of the first and second moments. This is of interest for more complicated models wich do not allow to determine the stationary probability density.

2. Crossover from Supercritical to Subcritical Bifurcation Driven by Noise

The model is given by the equation

$$\dot{x} = ax - x^3 + (x - bx^3)\cdot R_t \quad , b > 0 \quad . \tag{2.1}$$

If the control parameter *a* changes its sign the deterministic part of (2.1) describes a supercritical (forward) bifurcation, cf. Fig.1.

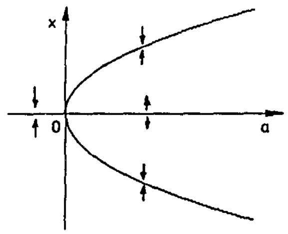

Fig.1. Bifurcation of the stationary solution of Eq.(2.1) in the deterministic case. The arrows show the direction of the flow.

2.1. The Case of GWN

The analysis of the flow yields a certain region in the (x,a)-plane (dashed in Fig. 2) which cannot be left once reached, i.e. this region is the support of the stationary probability density.

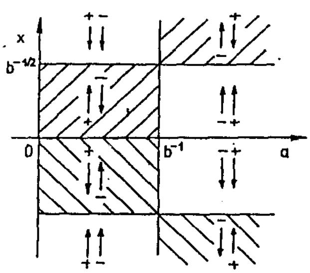

Fig.2. Support of the stationary probability density (2.2)

For a < 0 the stationary probability density P_s is degenerated to $\delta(x)$ like for the Stratonovich model. For a > 0 the stationary probability density and the moments are given by

$$P_s(x|x_o > 0) = N\cdot x^{\frac{a}{D} - 1}\cdot|1-bx^2|^{-\frac{a}{2D} - 1}\cdot\exp\left\{\frac{1}{2D}(a-b^{-1})/(1-bx^2)\right\} \quad , \tag{2.2}$$

$$
\langle x^q \rangle = \begin{cases} \dfrac{\Gamma\left(\frac{a}{2D}+\frac{q}{2}\right)}{\Gamma\left(\frac{a}{2D}\right)}\, b^{-q/2} \left(\dfrac{1-ab}{2Db}\right)^{a/2} \Psi\left(\frac{a}{2D}+\frac{q}{2}\,,\ \frac{a}{2D}+1\ ;\ \frac{1}{2D}(b^{-1}-a)\right), & a<b^{-1} \\ b^{-q/2}\, \dfrac{\Psi\left(1-\frac{q}{2}\,,\ \frac{a}{2D}+1\ ;\ \frac{1}{2D}(a-b^{-1})\right)}{\Psi\left(1,\ \frac{a}{2D}+1\ ;\ \frac{1}{2D}(a-b^{-1})\right)}\,, & a>b^{-1}. \end{cases} \tag{2.3}
$$

$\Psi(a,b;x)$ denotes the degenerated hypergeometric function [12]. Obviously in the case $a > b^{-1}$ moments of order $q \geq 2$ diverge. The extreme values of P_s obey the equation $3b^2Dx^4 - (4bD - 1)\cdot x^2 + D - a = 0$. Noise induced states are possible iff $4bD>1$. The regions of different qualitative behaviour of P_s are shown in Fig. 3.

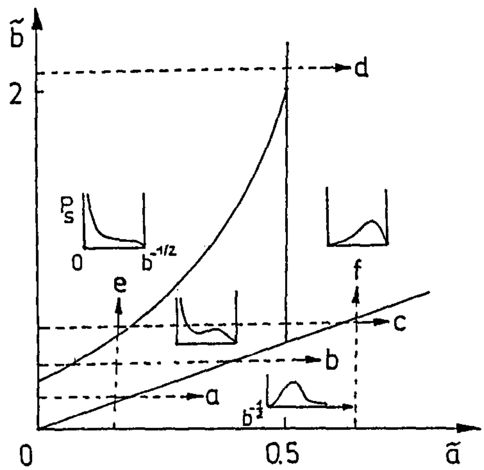

Fig.3 Phase diagram in the parameters $\tilde{a}=a/2D$, $\tilde{b}=1/2bD$.

Varying the parameters $\tilde{a}$ and $\tilde{b}$ along the arrows in Fig.3 bifurcation patterns as shown in Figs.4a-f are possible.

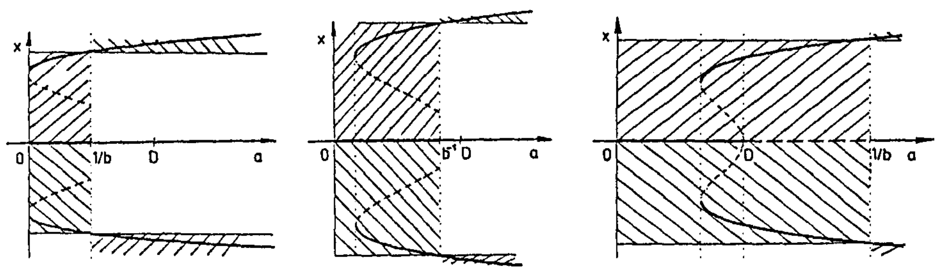

Fig.4a-c. Bifurcation patterns for varying $\tilde{a}$ along arrows a-c in Fig.3.

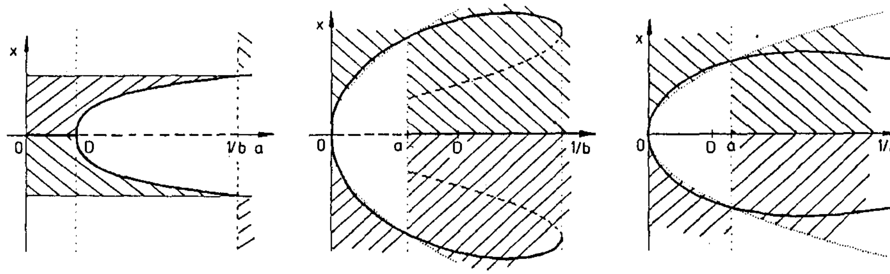

Fig.4d-f. Bifurcation patterns along arrows d-f in Fig.3.

We discuss now the shape of P_s for $a < b^{-1}$. For subcritical bifurcations P_s is bimodal in a certain parameter region. For supercritical parameters P_s is monomodal. The peak becomes sharper with increasing *a* so that the variance should decrease. It is evident that in this case the maximum of the variance must be located at values of *a* smaller than D, i.e. a subcritical maximum of the variance indicates a subcritical bifurcation. These qualitative arguments are supported by the numerical results shown in Fig.5. To distinguish between monomodal and bimodal behaviour one may also consider other integral quantities of P_s, e.g. the entropy.

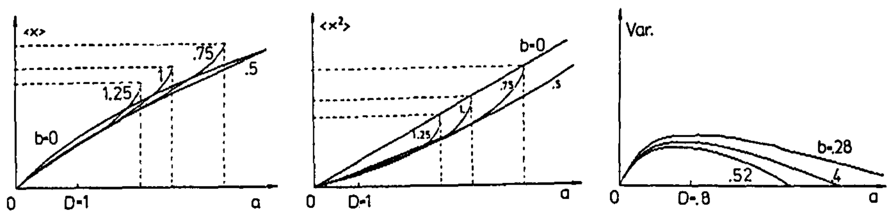

Fig.5. Moments for model (2.1).

2.2. The Case of DMP

The DMP $I_t = \Delta(-1)^{N_t}$, (N_t is a Poisson process) jumps only between two states. To determine the support of the stationary probability density one considers the flow for the realizations $+\Delta$ and $-\Delta$. The solutions of the equation $f(x) \pm \Delta g(x) = 0$ form the borders in between the system is pushed to and fro by the DMP [10]. There are two different possibilities for the shape of the support of P_s shown in Fig.6.

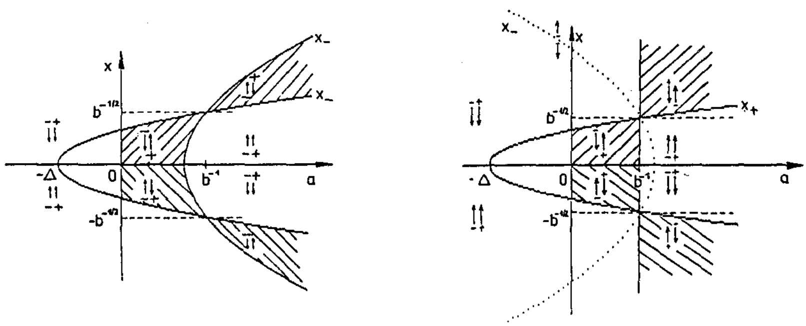

Fig.6. Support of the stationary density (2.4) for b<1/Δ (a), and b>1/Δ (b)

In the GWN limit Fig. 6b transformes into Fig. 2. P_s and the moments are calculated as

$$P_s(x|x_o > 0) = N\cdot|1-bx^2|\cdot x^{-2(\lambda+\mu)-1}\cdot|x^2-x_+^2|^{\lambda-1}\cdot|x^2-x_-^2|^{\mu-1} \quad , \tag{2.4}$$

$$\langle x^q \rangle = \begin{cases} x_+^q\,\dfrac{B(\lambda,-\lambda-\mu+q/2)}{B(\lambda,-\lambda-\mu)}\cdot\dfrac{F\left(q/2-1,\lambda,q/2-\mu;z\right) - x_+^2 b\,\frac{\lambda+\mu-1}{\mu-q/2}\cdot F\left(q/2,\lambda,q/2-\mu;z\right)}{1 + x_+^2/x_-^2\cdot\frac{\lambda}{\mu} - bx_+^2\cdot\frac{\lambda+\mu}{\mu}} \\ \text{with } z=x_+^2/x_-^2 \text{ for } a < \min\{\Delta,1/b\} \\ x_-^q\cdot\dfrac{F\left(\lambda+\mu+1-q/2,\mu,\lambda+\mu;z\right) - bx_-^2F\left(\lambda+\mu-q/2,\mu,\lambda+\mu;z\right)}{F\left(\lambda+\mu+1,\mu,\lambda+\mu;z\right) - bx_-^2F\left(\lambda+\mu,\mu,\lambda+\mu;z\right)} \\ \text{with } z=(x_-^2-x_+^2)/x_-^2 \text{ for } \Delta < a < 1/b \\ x_+^q\cdot\dfrac{bx_+^2F\left(\lambda+\mu-q/2,\lambda,\lambda+\mu;z\right) - F\left(\lambda+\mu+1-q/2,\lambda,\lambda+\mu;z\right)}{bx_+^2F\left(\lambda+\mu,\lambda,\lambda+\mu;z\right) - F\left(\lambda+\mu+1,\lambda,\lambda+\mu;z\right)} \\ \text{with } z=(x_+^2-x_-^2)/x_+^2 \text{ for } \Delta < 1/b < a \quad , \end{cases} \tag{2.5}$$

where we use the notations $2\lambda=\alpha/(a+\Delta)$, $2\mu=\alpha/(a-\Delta)$ and $x_\pm^2=(a\pm\Delta)/(1\pm b\Delta)$. $F(a,b,c;x)$ is the hypergeometric function and $B(x,y)$ the beta function. For $a < 0$ P_s in (2.4) is not normalizible. For this parameter region $P_s = \delta(x)$ as shown in [9]. The possibilities for the qualitative shape of P_s are the same as for GWN but the phase diagramm is much more complicated due to the additional parameter.The extreme values of P_s obey a cubic equation in $y=x^2$
$3b(b^2\Delta^2-1)y^3 + (2\alpha b+2ab+5-7b^2\Delta^2)y^2 + (5b\Delta^2-2\alpha b-2\alpha-6a+a^2b)y + 2\alpha a+a^2-\Delta^2 = 0.$

3. Crossover from Supercritical to Subcritical Bifurcation Driven by the Control Parameter

We consider another generalization of the Stratonovich model [10,11],

$$\dot{x} = ax + 2bx^3 - x^5 + x\cdot R_t \tag{3.1}$$

Depending on the sign of the control parameter b the stationary deterministic part of (3.1) describes a supercritical bifurcation (b < 0) or a subcritical bifurcation (b > 0), cf. Figs. 7a,b.

Fig.7. Supercritical and subcritical bifurcation of the stationary solution of Eq.(3.1) in the deterministic case.

In the following we focus our interest on the case of a subcritical bifurcation.

3.1 The Case of GWN

Depending on the initial value $x_0 > 0$ ($x_0 < 0$) the whole upper (lower) half plane is the support of P_s. For P_s and the moments we obtain

$$P_s(x|x_o > 0) = N\cdot x^{\frac{a}{D} - 1}\cdot\exp\left\{ -\frac{1}{D}\left(\frac{x^4}{4} - bx^2\right)\right\} , \tag{3.2}$$

$$\langle x^q \rangle = \frac{\Gamma\left(\frac{a}{2D} + \frac{q}{2}\right)}{\Gamma\left(\frac{a}{2D}\right)} (2D)^{-q/4} \frac{\mathcal{D}_{-a/2D-q/2}\left(-b\sqrt{D/2}\right)}{\mathcal{D}_{-a/2D}\left(-b\sqrt{D/2}\right)} , \tag{3.3}$$

where $\mathcal{D}_\nu(x)$ denotes the parabolic cylinder function [12].
The extreme values of P_s are given by $x^2 = b \pm \sqrt{b^2 + a - 1/D}$. The phase diagram is shown in Fig. 8.

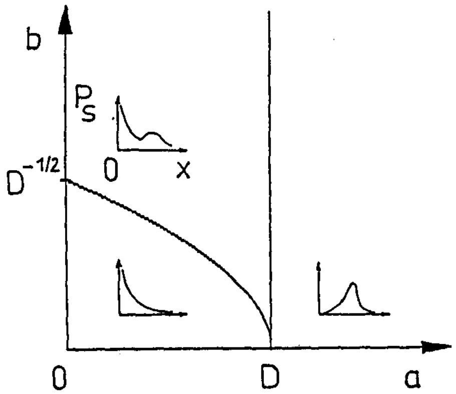

Fig.8.Phase diagram

The bifurcation patterns are similar to those of Figs. 4a, c and d. In Fig. 9 we show the moments and variance as functions of a.

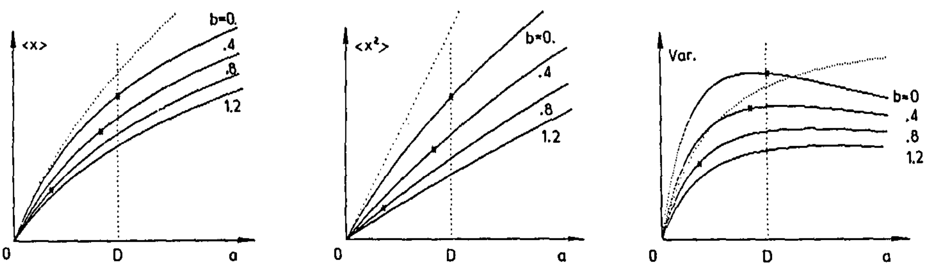

Fig.9. Moments for model (3.1). The dotted lines correspond to the Stratonovich model.

3.2. The Case of DMP

After a certain time the system is trapped between the boundaries x_{++} and x_{+-}, where $x_{\sigma\sigma'} = b+\sigma\sqrt{b^2+a+\sigma'\Delta}$. The shape of the support is shown in Fig.10.

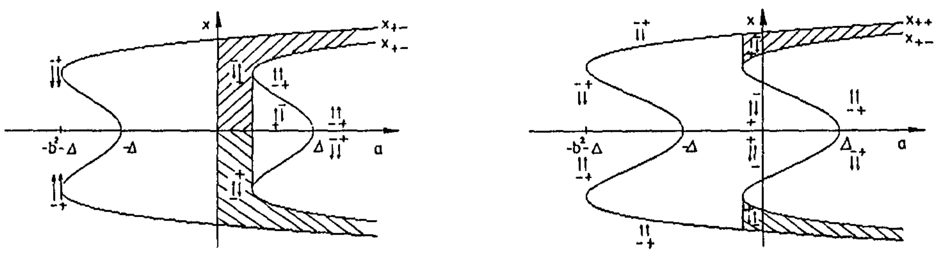

Fig.10. Support of the stationary probability density (3.4) for $b^2>\Delta$ (a) and $b^2<\Delta$ (b).

The stationary probability density is given by

$$P_s(x|x_0 > 0) \sim x^{-1} \prod_{\substack{\sigma \pm 1 \\ \sigma' \pm 1}} x^{-2\lambda_\sigma} \cdot \left(x^4 - 2x^2_{\sigma\sigma'} x^2 + 2bx^2_{\sigma\sigma'} + a + \sigma'\Delta \right)^{\varkappa} \qquad (3.4)$$

with the exponents $2\lambda_\sigma = \alpha/(a+\sigma\Delta)$ and $\varkappa = \alpha/4(a+\sigma'\Delta+bx^2_{\sigma\sigma'})-1$. The extreme values of P_s obey a quartic equation in $y = x^2$

$$9y^4 - 28by^3 + (20b^2 - 18a - 2\alpha)y^2 + (4\alpha b + 12ab)y + 2\alpha a + a^2 - \Delta^2 = 0$$

The bifurcation of the extreme values occurs at $a_c = \min\left(\Delta - b^2, -\alpha + \sqrt{\alpha^2 + \Delta^2}\right)$. Due to the four parameters of the model the phase diagramm is complicated again. In analogy to the situation in Section 2 different bifurcation patterns may be obtained changing the parameters in an appropriate way. The moments can be expressed by series expansions. A detailed discussion is however feasible only for fixing special values of the parameters.

References

[1] Kitahara,K.; Horsthemke,W.; Lefever,R.: Phys.Lett. 70A (1979) 377

[2] Kitahara,K.; Horsthemke,W.; Lefever,R.; Inaba,Y.: Prog.Theor.Phys. 64 (1980) 1233

[3] Klyatskin,V.I.: Izv.VUZ Radiofiz. (UdSSR) 20 (1977) 562

[4] Pawula,R.F.: IEEE Trans.Inf.Theory 13 (1967) 33

[5] Pawula,R.F.: Internat.J.Control 25 (1977) 238

[6] Pomeau,Y.: J.Stat.Phys. 24 (1981) 189

[7] Sancho,J.M.; San Miguel,M.: Prog.Theor.Phys.69 (1983) 1085

[8] Behn,U.;Schiele,K.;Teubel,A.:Wiss.Z.KMU Math.-Naturwiss.R. 34 (1985) 6

[9] Teubel,A.; Behn,U.; Kühnel,A.: Z.Phys.B Condensed Matter 71 (1988) 393

[10] Behn,U.; Schiele,K.: Dubna-Preprint EE 17-84-300 (1984)

[11] Stratonovich,R.L.: *Topics in the theory of random noise.* Gordon and Breach New York 1967 Vol.2

[12] Abramowitz,M.;Stegun,I.A. (eds) :*Pocketbook of mathematical functions.* Thun, Frankfurt/Main Deutsch 1984