Coherence and Quantum Groups

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Quantum groups have shown in recent years to be an exceptionally promising and rich structure whereby one can expect a growing wealth of new results in statistical mechanics and quantum field theory [1]. Stemmed out of the algebraic structure dictated by integrability conditions for a class of integrable systems, quantum groups can be intuitively thought of as the deformation (determined by the quantum Yang-Baxter equations and the R-matrix they involve) of the universal enveloping algebra of some given Lie algebra $\mathcal{L}$ of dynamical variables yet preserving the associativity properties of $\mathcal{L}$ (we consider here quantum groups as synonymous of Hopf algebras, disregarding more general definitions).

On a different side, an increasing deal of attention has been devoted to the notion of coherence in optics [2]. In particular a strong interest has been devoted to squeezed states those, constructed as generalized coherent states for suitable algebras, may describe a wide class of systems characterized by reduced quantum fluctuations.

Recently, and unexpectedly, the two fast developing fields of research have been connected and quantum groups have been introduced in the context of quantum optics. [3].

An interesting role in the description of the physical features of this new model – which represents situations in which the interaction between atom and radiation field is intensity dependent – is played by a set of states recently introduced by [4].

In the present note the two concepts of quantum groups and squeezing are discussed together taking into account the ideas of ref. [4] and it is shown that they can already be bridged in the simple setting provided by a subtle q-deformation of the usual coherent states.

So, to reach our goal we have to refrase the usual approach to coherent states in such a way that it can be "quantized" i.e. we have to stress the Lie algebra (or superalgebra, as we shell see) aspects of the usual creation and annihilation operators. The Fock space, from this point of view, is simply a representation of this structure.

When the "right" algebraic aspects has been identified, we know, at least in principle, from the general properties of quantum groups how to perform the job: the representation (i.e. the Fock space) will remain unmodified while the generators change in function of a parameter (usually called "q", for "quantum"). This means that the quantum operators are elements of the universal enveloping algebra of the original ones; to be more precise they are the same of the corresponding non quantum ones, but multiplied for integer functions of the Cartan subalgebra and of the Casimir operators.

The first and simpler idea would be to deform the group $H(1)$:

$$[a,\bar{a}] = H, \quad [N,a] = -a, \quad [N,\bar{a}] = \bar{a}, \quad [H,\cdot\,] = 0 \;. \tag{1}$$

The quantum groups are new mathematical objects and their general theory is, quite far from complete, but semisimple quantum groups are well known and, so, the best way for our job is to generalize to quantum groups the concept of contraction [5]. It can be shown that the contaction is more or less the same with the basic difference that the "q" parameter also is involved in the singular transformation.

Conforming to this idea, we define the following transformation on the four generators of $U(2)_q \equiv SU(2)_q \otimes U(1)$ and on the parameter $z = \log q$.

$$\begin{pmatrix} a_q \\ \bar{a}_q \\ N \\ H \\ w \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & 0 \\ 0 & 0 & -1 & \varepsilon^{-2} & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon^{-2} \end{pmatrix} \begin{pmatrix} J_+^q \\ J_-^q \\ J_3 \\ K \\ z \end{pmatrix} \tag{2}$$

where K is the $U(1)$ generator; explicit and simple calculations show that this is exactly the singular transformation – beside the part related to z – giving (1) from $u(2)$.

The transformation (2) is applied to $U(2)_q$ (letting $e^z = q$) i.e. to the structure that have the algebraic relations:

$$[J_3^q, J_\pm] = \pm J_\pm, \qquad [J_+^q, J_-^q] = [2J_3^q]_q \equiv \frac{\mathrm{sh}(zJ_3^q)}{\mathrm{sh}(z/2)} \qquad [K, \cdot\,] = 0,$$

the coalgebra:

$$\begin{aligned} \Delta J_\pm &= e^{-zJ_3^q/2} \otimes J_\pm + J_\pm \otimes e^{zJ_3^q/2} , \\ \Delta J_3^q &= \mathbf{1} \otimes J_3^q + J_3^q \otimes \mathbf{1} , \\ \Delta K &= \mathbf{1} \otimes K + K \otimes \mathbf{1} , \end{aligned}$$

and counit and antipode:

$$\gamma(J_\pm) = -e^{zJ_3^q/2}\, J_\pm\, e^{-zJ_3^q/2} , \qquad \gamma(J_3^q) = -J_3^q , \qquad \gamma(K) = -K$$
$$\epsilon(\mathbf{1}) = 1, \qquad \epsilon(J_\pm) = \epsilon(J_3^q) = \epsilon(K) = 0 ,$$

we get taking the limit $\varepsilon \to 0$:

$$[a_q, \bar{a}_q] = \frac{\mathrm{sh}(wH/2)}{w/2}, \qquad [N, a_q] = -a_q, \qquad [N, \bar{a}_q] = \bar{a}_q, \qquad [H, \cdot\,] = 0 .$$

end

$$\Delta a_q = e^{-wH/4} \otimes a_q + a_q \otimes e^{wH/4} , \qquad \Delta \bar{a}_q = e^{-wH/4} \otimes \bar{a}_q + \bar{a}_q \otimes e^{wH/4} , \tag{3}$$

$$\Delta N = \mathbf{1} \otimes N + N \otimes \mathbf{1} , \qquad \Delta H = \mathbf{1} \otimes H + H \otimes \mathbf{1} .$$

Counit and antipode are:

$$\gamma(a_q) = -a_q\ , \qquad \gamma(\bar{a}_q) = -\bar{a}_q\ , \qquad \gamma(N) = -N\ , \qquad \gamma(H) = -H$$
$$\epsilon(\mathbf{1}) = 1, \qquad \epsilon(a_q) = \epsilon(\bar{a}_q) = \epsilon(N) = \epsilon(H) = 0\ .$$

We note, in passing, that the contraction is a very powerful technique and it can be used also to derive the universal R-matrix of the quantum algebra we obtained (let us call it $H(1)_q$) from the one of $U(2)_q$. Explicit calculations give:

$$R = e^{-(w/2)(H\otimes N + N\otimes H)} \sum_{k=0}^{\infty} \frac{1}{k!} \left(w^{1/2} e^{wH/4}\ a_q\right)^k \otimes \left(w^{1/2} e^{-wH/4}\ \bar{a}_q\right)^k .$$

In this way, the quantified of the algebra (1) has been studied in all aspects. This is very interesting, particularly because it is the first example of non semisimple quantum group studied in all details but cannot help in building deformed coherent states: at the algebra level $H(1)_q$ is completely equivalent to an $H(1)$ in which only the Planck constant has rescaled as: $H' = \dfrac{\mathrm{sh}(wH/2)}{w/2}$. So, disregarding this irrelevant rescaling, the q-oscillators of $H(1)_q$ are the same of the ones of $H(1)$.

The play is different if instead of the algebra (1) we quantize $osp(1|2)$. With a suitable choice of the generators it can be written:

$$\{a, \bar{a}\} = 4M, \qquad [M, a] = -\frac{1}{2}a, \qquad [M, \bar{a}] = \frac{1}{2}\bar{a}.$$

This superalgebra looks unfamiliar but, as a matter of fact, is nothing else that the 3 generators structure of the creation annihilation operators, as can be easy seen defining the number operator as $N \equiv 4M - \frac{1}{2}$:

$$\{a, \bar{a}\} = 2N + 1, \qquad [N, a] = -a, \qquad [N, \bar{a}] = \bar{a}.$$

There is no room to discuss here the details of this quantization [6]. It results:

$$\{a_q, \bar{a}_q\} = \frac{\mathrm{sh}(uM)}{\mathrm{sh}u/4}, \qquad [M, a_q] = -\frac{1}{2}a_q, \qquad [M, \bar{a}_q] = \frac{1}{2}\bar{a}_q \tag{4}$$

The coproduct is easily obtained following [6] and is indeed different from (3):

$$\Delta a_q = e^{-uM/2} \otimes a_q + a_q \otimes e^{uM/2}\ , \qquad \Delta \bar{a}_q = e^{-uM/2} \otimes \bar{a}_q + \bar{a}_q \otimes e^{uM/2}\ ,$$

$$\Delta M = \mathbf{1} \otimes M + M \otimes \mathbf{1}\ , \quad .$$

while counit and antipode are:

$$\gamma(a_q) = -e^{-u/4}\, a_q\ , \quad \gamma(\bar{a}_q) = -e^{u/4}\, \bar{a}_q\ , \quad \gamma(M) = -M\ ,$$
$$\epsilon(\mathbf{1}) = 1, \quad \epsilon(a_q) = \epsilon(\bar{a}_q) = \epsilon(M) = 0\ .$$

These relations turn $\mathcal{S} \equiv osp_q(1|2)$ (sometimes called also $B_q(0|1)$) into a $\mathbf{Z}_2$–graded Hopf algebra. In showing this one has to take into account the graded multiplication and comultiplication on $\mathcal{S}\otimes\mathcal{S}$ [7]. For instance, if b and c are homogeneous elements in $\mathcal{S}$, then the product on $\mathcal{S}\otimes\mathcal{S}$ is defined as:

$$(a\otimes b)(c\otimes d) = (-1)^{p(b)p(c)}(ac\otimes bd)\ ,$$

where $p(b), p(c) \in \mathbf{Z}_2$ are the degrees of b and c respectively.

It is amusing that we have found the bosonic operators into a graded algebra i.e. in a structure more suitable for describing the fermion ones, but this structure admitting the q-expansion has many unexpected features. On this point we limit ourselves to quote that it allows to generalize the well known Jordan-Schwinger realization of $SU(2)$ to $SU(2)_q$: the generators of $SU(2)_q$ are written as bilinear in the creation and annihilation operators of $osp_q(1|2)$ exactly as the generators of $SU(2)$ are written as bilinear in the usual creation annihilation operators [8]:

$$J_+^q = a_q^1\bar{a}_q^2, \quad J_-^q = a_q^2\bar{a}_q^1, \quad J_3^q = \frac{1}{2}(a_q^1\bar{a}_q^1 - a_q^2\bar{a}_q^2).$$

Therefore, because $osp_q(1|2)$ is a promising q-deformed structure at all levels, including algebra, we can attempt to rewrite the usual theory of coherent states in its scheme.

As a first step, we have to equip first the realization of $\mathcal{S}$ with the notion of hermiticity that has been lost in the quantification.

Indeed, as stressed in general, the Fock space is not modified in the quantum expansion and so $\mathcal{S}$ and consequently N can be identified with the customary Fock space and its number operator but a_q and $\bar{a}_q$ are different from a and $\bar{a}$ end are no more hermitian conjugate: conjugation of (4) leads indeed to

$$a_q^\dagger = \bar{a}_{q^*}\,[\chi_q(N)]^{-1} \quad , \quad \bar{a}_q^\dagger = \chi_q(N) a_{q^*} \quad ,$$

where $\chi_q(N)$ is a function to be determined by suitable self-consistency conditions. A convenient, natural assumption to fix it is that $a_q, a_q^\dagger$ are canonically conjugate with each other, *i.e.* that in the corresponding sector $\mathcal{S}$ be identical with the usual Weyl-Heisenberg algebra, namely $[\,a_q, a_q^\dagger\,] = 1$. This condition induces the choice

$$\chi_q(N) = \frac{([\,N+1\,]_{q^*}[\,N+1\,]_q)^{\frac{1}{2}}}{N+1} \quad ,$$

where $[\,n\,]_q \equiv \dfrac{q^{\frac{1}{2}n} - q^{-\frac{1}{2}n}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \dfrac{\sinh(\frac{1}{2}nz)}{\sinh(\frac{1}{2}z)}$, with $z \equiv \ln q$. One has as well $a_{q^*} = a_q\left(\dfrac{[\,N\,]_{q^*}}{[\,N\,]_q}\right)^{\frac{1}{2}}$ and $\bar{a}_{q^*} = \left(\dfrac{[\,N\,]_{q^*}}{[\,N\,]_q}\right)^{\frac{1}{2}}\bar{a}_q$.

Notice that as $q \to 1$, everything collapse on the customary bosonic realization of the Weyl-Heisenberg algebra.

On the states

$$a_q|n> = \left(\frac{[n]_q}{[n]_{q^*}}\right)^{\frac{1}{4}} n^{\frac{1}{2}}|n-1> \quad ,$$
$$\bar{a}_q|n> = \left(\frac{[n+1]_q}{[n+1]_{q^*}}\right)^{\frac{1}{4}} \left(\frac{[n+1]_q[n+1]_{q^*}}{n+1}\right)^{\frac{1}{2}} |n+1> \quad ,$$

whence $\bar{a}_q a_q|n>= [n]_q|n>$, $a_q^\dagger a_q|n>= n|n>$. Note that $a_q^\dagger a_q = a^\dagger a = N$, because a_q end $a_q^\dagger$ differ from a and a_q only for a phase. $|0>$ is the highest weight vector of $\mathcal{S}$: $a|0>= 0$.

We can at this point define the *quantum* analog of position (Q_q) and momentum (P_q) operators :

$$P_q \equiv i\left(\frac{m\hbar\omega}{2}\right)^{\frac{1}{2}}(\bar{a}_q - \bar{a}_q^\dagger) \quad ,$$
$$Q_q \equiv \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(\bar{a}_q + \bar{a}_q^\dagger) \quad . \tag{5}$$

Q_q and P_q are hermitian and have commutation relation

$$[Q_q, P_q] = i\hbar_q \equiv i\hbar(1 + \frac{1}{4}\,\mathrm{Re}\,(z^2)N(N+1) + \mathcal{O}(z^4)) \quad . \tag{6}$$

and give rise to a *quantum* version of the quantized harmonic oscillator, with hamiltonian

$$H_q \equiv \frac{1}{2m}{P_q}^2 + \frac{1}{2}m\omega^2{Q_q}^2 = \frac{1}{2}\hbar\omega\left(\bar{a}_q^\dagger\bar{a}_q + \bar{a}_q\bar{a}_q^\dagger\right)$$
$$= \frac{1}{2}\hbar\omega\left(1 - \frac{1}{2}\mathrm{Re}\,(z^2)(n+1)^2 + \mathcal{O}(z^4)\right) \tag{7}$$

Note also that eq. (7) suggests a possible experimental test of the q-effects through the spectrum of the quantum harmonic oscillator for large N.

We are now ready to define now the coherent states $\{|\alpha;q> |\alpha, q \in \mathbb{C}\}$ by

$$a_q|\alpha;q>= \alpha|\alpha;q> \quad ,$$

it is straightforward to check that

$$|\alpha;q>= \mathcal{N}(|\alpha|)\exp_q(\alpha\bar{a}_q)|0> \quad ,$$

where $\exp_q(x) \equiv \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!}$ is the *quantum* version of the exponential function ($[n+1]_q! \equiv [n+1]_q[n]_q!$), and $\mathcal{N}(|\alpha|)$ is a normalization factor which with the above choice turns out to be independent of q : $\mathcal{N}(|\alpha|) = \exp(-\frac{1}{2}|\alpha|^2)$.

Then also,

$$|\alpha; q> = \exp(-\frac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(\frac{[n]_{q^*}!}{[n]_q!}\right)^{\frac{1}{4}} |n> \quad .$$

It is worth noticing here that : *a)* in the limit $q \to 1$, $|\alpha; q>$ turns into a customary Glauber coherent state, *b)* the definitions adopted give for $|\alpha; q>$ a form which is somewhat different from that of ref.[4] (who, incidentally, give their definition for $q \in \mathbb{R}$); a difference with interesting consequences.

In order to show that the coherent states above defined $\{|\alpha; q>\}$ are squeezed, i.e. that their uncertainty in Q_q or in P_q or in both is smaller of $\frac{1}{2}\hbar$ (or $\frac{1}{2}\hbar_q$, we don't discuss the details here) we have now to evaluate the variances of Q_q and P_q. This can be done numerically. For $|q| \simeq 1$ and $|\alpha| \sim 1$ there are plenty of interesting situations with astonishing squeezing especially when q is near to the unit circle.

To conclude let us stress the fundamental result of the report: the quantum group coherent states seem to be, in general, the natural candidate to describe squeezed quantum states of matter.

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