EXACT PERIODIC SOLUTIONS FOR A CLASS OF MULTISPEED DISCRETE BOLTZMANN MODELS

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ABSTRACT

Only for multispeed discrete Boltzmann models can we obtain well-defined temperature. Recently different hierarchies of multispeed, multidimensional ($d > 1$) discrete models have been characterized by their $(1+1)$-dimensional Pde along one axis. Here, for the simplest hierarchy with five independent densities and two speeds which are 1 and either $\sqrt{d}$ or $\sqrt{2}$, we construct $(1+1)-$dimensional periodic solutions. The physical corresponding models are the planar square $8v_i, d = 2$ model and two three-dimensional $14v_i, d = 3$ models(one of them being the Cabannes model).

1. INTRODUCTION

For unispeed discrete Boltzmann models (discrete velocities v_i and densities N_i) the energy E and the mass M are propotional so that the temperature $T = 2E/M - V^2$ is ill-defined and cannot be distinguished from the mean velocity V. On the contrary for multispeed discrete Boltzmann models, E and M are independent, all conservation laws are satisfied and we can define and study the temperature.

For the unidimensional $4v_i$, $d = 1$ model with two speeds $1, 2$ and two different masses for the particles, exact solutions have been found[1].

For multidimensional models with $d > 1$ many multispeed models for single gas (same mass for the particles) are known. We can classify these models following the restriction of their Pde along one x-coordinate axis. It was found that different hierarchies exist which , for each class, are defined by the same set of independent densities and a system of Pde differing only by the dimension d of the space[2]. For instance the $8v_i$ square model[3] with the x-axis along either the medians or the diagonals of a square and the two $14v_i$ cubic[4] Cabannes models belong to two different hierarchies and differ only by the two dimensional $d = 2$ or 3 values. Then we can study the classes of d-dependent exact solutions[2] for each hierarchy and find common properties for models which belong to different spatial dimensions. We notice also the connections between discrete kinetic models[5] and their partners in lattice gas models[5].

Height different hierarchies have been found: Class I with two speeds 1 and either $\sqrt{d}$ or $\sqrt{2}$ which include the square $8v_i$ and two $14v_i$ models. ClassII with three speeds $0, 1, \sqrt{2}$ including the square $9v_i$ model. ClassIII with two speeds 2 and either $\sqrt{d}$ or $\sqrt{2}$, still including the square $8v_i$ and two $14v_i$ models. ClassIV with speeds $0, 2, \sqrt{2}$ includes another $9v_i$ model. Class V with $1, \sqrt{2}$ speeds begins with the popular $18v_i$model[5] and ClassVI with $0, 1, \sqrt{2}$ speeds has the $19v_i$ model associated to the $18v_i$ one. Classes VII and VIII have also the $18v_i, 19v_i$ models but with one speed equal to 2 instead of 1.

For all these hierarchies of models, exact shock waves solutions have been found. For the independent densities N_i they are of the type:

$$N_i = n_{0i} + n_i/(1+e^{\gamma\eta}),\ \ \eta = x - \varsigma t$$

Up to now only for Class I, other exact solutions have been found.They are $(1+1)$-dimensional solutions which are the superposition of two similarity waves, either real or complex conjugate:

$$N_i = n_{0i} + n_{1i}/D_1 + n_{2i}/D_2,\ D_i = 1 + d_i e^{\gamma_i \eta_i},\ \ \eta_i = x - \varsigma_i$$

$$N_i = n_{0i} + n_i/\Delta + n_i^*/\Delta^*, \Delta = 1 + \delta e^{\rho t + i\gamma x},\ \ \delta = \text{real const.} \qquad (1,1)$$

Periodic solutions were first obtained for the unispeed Broadwell model[6] and such solutions exist also for one $14v_i$ Cabannes model[7]. Here we present periodic solutions for the Class I hierarchy which include both the planar $8v_i, d=2$ model and two three dimensional $d=3$, $14v_i$ models. For Class I we notice also that $(1+1)$-dimensional shock waves have recently been found[8].

2. CLASS I HIERARCHY OF MODELS

For Class I thete exist 5 independent densities $(N_1, M_4), M_2, (N_2, M_1)$ with coordinates $-1, 0, 1$ along the x-axis and two collision terms C, D

$$C = \bar{c}(N_2M_4 - N_1M_1), D = \bar{d}d_c(M_1M_4 - M_2^2), \bar{d} = 2/d, d_c > 0 \text{ arbitrary} \qquad (2.1a)$$

for two d-dependent subclasses (i) and (ii). We write down the system of five nonlinear equations which include three linear ones equivalent to the mass,momentum and energy conservation laws.We define $p_\pm = \partial_t \pm \partial_x$:

$$p_-N_1 = -p_+N_2 = C, M_{2t} = D, p_+M_1 = -(d-1)D + d_*C, p_-M_4 = -(d-1)D - d_*C$$

$$p_-N_1 + p_+N_2 = p_+M_1 + p_-M_4 + 2(D-1)M_{2t} = p_+M_1 - p_-M_4 - 2d_*p_-N_1 = 0 \quad (2.1b)$$

We define the macroscopic quantities: mass$M$, momentum J, mean velocity $V = J/M$ and energy$E$ from which we can construct nontrivial temperature $T = 2E/M - V^2$:

$$M = M_1 + M_4 + 2(d-1)M_2 + d_*(N_1 + N_2),\ \ J = M_1 - M_4 + d_*(N_2 - N_1)$$

$$E = (M_1 + M_4)/2 + (d-1)M_2 + d_*d_{**}(N_1 + N_2), (i) d_{**} = d/2, (ii) d_{**} = 1 \qquad (2.2)$$

For the subclass (i) we have $(2^d + 2d)v_i$,the speeds are $1, \sqrt{d}$, and $d_* = 2^{d-1}$,$\bar{c} = \sqrt{(d+3)}/2$. For the subclass (ii): $2(3d-2)v_i$,the speeds are $1, \sqrt{2}$, and $2(d-1) = d_*$,$2\bar{c} = \sqrt{5}$. For the d-dimensional coordinates let us write $(x_1 = x, x_2, ..., x_d)$ for the velocities of the different models. The three physical models (see fig.1) $d \leq 3$ are:

$$d = 2, 8v_i:\ N_i\ (\pm1, \pm1),\ M_i\ (\pm1, 0), (0, \pm1) \text{both (i) and (ii)}$$

$$d = 3, 14v_i, (i): N_i\ (\pm1, \pm1, \pm1),\ M_i(\pm1, 0, 0), (0, \pm1, 0), (0, 0, \pm1) \text{ Cabannes model}$$

$$d = 3, 14v_i, (ii): N_i\ (\pm1, \pm1, 0), (\pm1, 0, \pm1),\ M_i(\pm1, 0, 0), (0, \pm1, 0), (0, 0, \pm1)$$

For the $d = 2, d = 3(i), d = 3(ii)$ models we see that N_1 (and N_2) are the densities associated respectively to $2, 4, 4$ velocities in the space. For both models M_1 and M_4 correspond to only one velocity while M_2 is associated with respectively $2, 4, 4$ velocities. For these models three types of collisions can occur: between particles of speed 1, between particles of speed $\sqrt{d}$ (or speed$\sqrt{2}$ for subclass (ii)) and collisions $(1, \sqrt{d}) \longleftrightarrow (1, \sqrt{d})$. However the second subclass (ii) with speeds $1, \sqrt{2}$ is more interesting because we can add another velocity ($d = 2, 9v_i$ and $d = 3, 15v_i$) with speed 0 and then mixed collisions are possible: $(1, 1) \longleftrightarrow (0, \sqrt{2})$, (Class II hierarchy).

3. PERIODIC SOLUTIONS

3.1*Algebraic Determination of the Solutions for the General d case*

For the 5 independent densities we seek positive periodic solutions of the type(1.1):

$$N_1(x,t) = n_{01} + n_1/\Delta + n_1^*/\Delta^*, \; N_2(x,t) = N_1(-x,t)$$

$$M_1(x,t) = m_{01} + 2Re(m_1/\Delta), M_4(x,t) = M_1(-x,t), \; M_2 = m_{01} + m_2 2Re(1/\Delta) \quad (3.1)$$

with δ, γ and ρ in (1.1) and m_2 in (3.1) real while $n_1 = n_{1R} + in_{1I} = |n_1| \, e^{i\theta}$,$m_1 = m_{1I} + m_{1R}$ in (3.1) are complex. We notice that $N_2 = N_1$, $M_4 = M_1$ at $x = 0$ so that these solutions satisfy a specular reflection boundary condition at a wall $x = 0$.
We substitute (3.1) into (2.1) and both from the three linear relations and two nonlinear ones (with linear terms $p_{-}N_1$,M_{2t}) we find:

$$n_{1R}\rho + n_{1I}\gamma = 0, \; m_{1R}\rho - m_{1I}\gamma + \rho(d-1)m_2 = 0$$

$$m_{1R}^2 - m_{1I}^2 = m_2^2, \; m_{1I}\rho + m_{1R}\gamma = d_*(\rho n_{1I} - \gamma n_{1R})$$

$$\rho n_{1I} - \gamma n_{1R} = -2\overline{c}(n_{1R}m_{1I} + n_{1I}m_{1R}) = 2\overline{c}(n_{01}m_{1I} + m_{01}n_{1I})$$

$$\rho m_2 = \overline{d}d_c(|m_1|^2 - m_2^2) = 2\overline{d}d_c m_{01}(m_2 - m_{1R}) \quad (3.2)$$

For ten real parameters $|n_1|, \theta, m_{1R}, m_{1I}, m_2, \rho, \gamma, n_{01}, m_{01}, d_c$ we have eight real relations leaving two arbitrary parameters chosen to be $n_{01} > 0$ as the scaling parameter and d_c or equivalently $\alpha = \overline{c}/\overline{d}d_c$. We introduce scaled parameters $\overline{m}_{1I} = m_{1I}/m_{1R}$, $\overline{m}_2 = m_2/m_{1R}$ and after some trivial algebra rewrite (3.2):

$$\overline{m}_{1I}^2 = 1 - \overline{m}_2^2 = \alpha(d-1)\overline{m}_2^2 \sin^2\theta, \; \overline{m}_{1I} = -\tan\theta(1 + (d-1)\overline{m}_2)$$

$$|n_1| = -n_{01}(1 - (d-1)\overline{m}_2)/\cos\theta(1 + d\overline{m}_2), \; m_{1R} = d_*|\overline{n}_1|/(\overline{m}_{1I}\sin\theta - \cos\theta)$$

$$1 + \overline{m}_2 + m_{01}/m_{1R} = 0, \; \rho = 2\overline{d}d_c m_{1R}\overline{m}_{1I}^2/\overline{m}_2, \; \gamma = -\rho/\tan\theta \quad (3.3)$$

We define $\eta_1 = \pm 1, \eta_2 = \pm 1, {\eta_i}^2 = 1$ and from the first three (3.3) relations we get:

$$\overline{m}_2 = 1/(1 - d + \eta_1\sqrt{\alpha(d-1)}\cos\theta)$$

$$\cos^2\theta - \eta_1\cos\theta\sqrt{(d-1)/\alpha} - 1/2 + d(d-2)/2\alpha(d-1) = 0 \quad (3.4)$$

At the present stage we only give the algebraic construction of the solutions without discussing the possible d_c or α intervals for which the solutions exist and are positive. From d, α given we first obtain $\cos\theta$:

$$2\cos\theta = \eta_1\sqrt{(d-1)/\alpha} + \eta_2\sqrt{2 + (1 + d(2-d))/\alpha(d-1)} \tag{3.5}$$

giving $\overline{m}_2$ with (3.4) and successively $\overline{m}_{1I}$,$|n_1|$,m_{1R}, ρ, γ, m_{01} with (3.3) and $n_{01} > 0$ arbitrary. Then with the scaled parameters we multiply by m_{1R} and reconstruct the original ones m_1, m_2. For the complete positivity discussion we study successively $d = 2$ and 3. We recall[6] that for solutions with $\rho > 0$ it is sufficient for positive N_i that $n_{01} > 0$ as well as $m_{01} > 0$ for positive M_i. If these properties hold then the densities are positive when the time is infinite and for positivity at finite time it is sufficient to choose sufficiently large δ in the complex denominators Δ. We recall also that for the $d = 3$ subclass (i), Cabannes model, then Cabannes and Tiem[7] have previously found periodic solutions.

3.2 *Positive Solutions for the $d = 2, 8v_i$, $d_* = 2$, $d_c = \sqrt{5}/2\alpha$ Planar model*

Theorem 1: For the d=2 case, the sufficient positivity conditions $\rho > 0$, $m_{01} > 0$ are satisfied if: $\eta_1 = -1$, $\eta_2 = 1$ and $0 < \alpha < 4$ or equivalently for the cross-section $d_c > \sqrt{5}/8$.
The solution $(3.3-4-5)$ can be written down analytically:$\gamma\tan = -\rho$ and

$$0 < \cos\theta = (-1 + \sqrt{2\alpha+1}/(2\sqrt{\alpha}) < 1, \ |n_1| = 2n_{01}\sqrt{\alpha}/(3 - \sqrt{2\alpha+1}) > 0 \tag{3.6}$$

$$m_{01} = |n_1|(-1 + \sqrt{2\alpha+1}/(\cos\theta + \sqrt{\alpha}) > 0, \ m_1 = -2|n_1|(1 + \sqrt{\alpha}e^{-i\theta})/(\sqrt{\alpha} + \cos\theta)$$

$$m_2 = 2|n_1|/(\sqrt{\alpha} + \cos\theta), \ \rho = 4\overline{d}d_c\sqrt{\alpha}(\sqrt{2\alpha+1} - 1 + \alpha)|n_1|/(\sqrt{2\alpha+1} - 1 + 2\alpha)$$

The condition $\alpha < 4$ arises from $|n_1| > 0$.

3.3 *Positive Solutions for the two $d = 3, 14v_i$, $d_* = 4$, Three Dimensional models*

Theorem 2: For the two d=3 case, the sufficient positivity conditions $\rho > 0, m_{01} > 0$ are satisfied if: $\eta_1 = -1$, $\eta_2 = 1$ and $(3 - \sqrt{3})/2 < \alpha < 9/2$ or equivalent conditions for the cross-sections with $d_c = 3\sqrt{6}/4\alpha$ for the (i) model and $d_c = 3\sqrt{5}/4\alpha$ for the (ii) model.

In the (i) case we have $\overline{c}/\overline{d} = 3\sqrt{6}/4$ and $\overline{c}/\overline{d} = 3\sqrt{5}/4$ in the (ii) one.The solutions can still be written down analytically.

$$-1 < \cos\theta = (-1 + \sqrt{\alpha - 1/2})/\sqrt{2\alpha} < 1, \ |n_1| = n_{01}\sqrt{2\alpha}/(2 - \sqrt{\alpha - 1/2}) \tag{3.7}$$

$$m_2 = 2\sqrt{2}|n_1|/(\sqrt{\alpha} + \sqrt{2}\cos\theta), \ m_1 = -4(\sqrt{2} + \sqrt{\alpha}e^{-i\theta})/(\sqrt{2}\cos\theta + \sqrt{\alpha})$$

$$m_{01} = 2\sqrt{2\alpha(\alpha - 1/2)}|n_1|/(\alpha - 1 + \sqrt{\alpha - 1/2}) > 0$$

$$\rho = \overline{d}d_c|n_1|8\sqrt{2\alpha}\alpha\sin^2\theta/(\alpha - 1 + \sqrt{\alpha - 1/2}) > 0, \ \gamma\tan\theta = -\rho$$

The conditions on α arise from $\cos\theta$ real, $\rho > 0$, $m_{01} > 0$ and $|n_1| > 0$.

3.4 *Macroscopic Quantities Associated to the $d \leq 3$ models*

$$d = 2 : M = M_1 + M_4 + 2(M_2 + N_1 + N_2),\ J = M_1 - M_4 + 2(N_2 - N_1)$$

$$E = (M_1 + M_4)/2 + M_2 + 2(N_1 + N_2)$$

$$d = 3\ (i)\text{and}\ (ii) : M = M_1 + M_4 + 4(M_2 + N_1 + N_2), J = M_1 - M_4 + 4(N_2 - N_1)$$

$$(i)E = (M_1 + M_4)/2 + 2M_2 + 6(N_1 + N_2),\ (ii)E = (M_1 + M_4)/2 + 2M_2 + 4(N_1 + N_2)$$

3.5 *Numerical Calculations (fig.2)*

We present the curves of the temperature for the $d = 2$ and $d = 3$ subclass (ii) models corresponding to the same arbitrary parameters values: $n_{01} = 1$ and $d_c = 1$. In order that the microscopic densities N_i and M_i be non negative we choose $\delta = 7.8$ for the $d = 2$ model (see fig.3) and $\delta = 4.5$ for the $d = 3$ one. The temperatures are plotted for $\gamma x/2\pi$ varying in the interval $(0, 1)$ and we notice the symmetry with respect to the 0.5 value.

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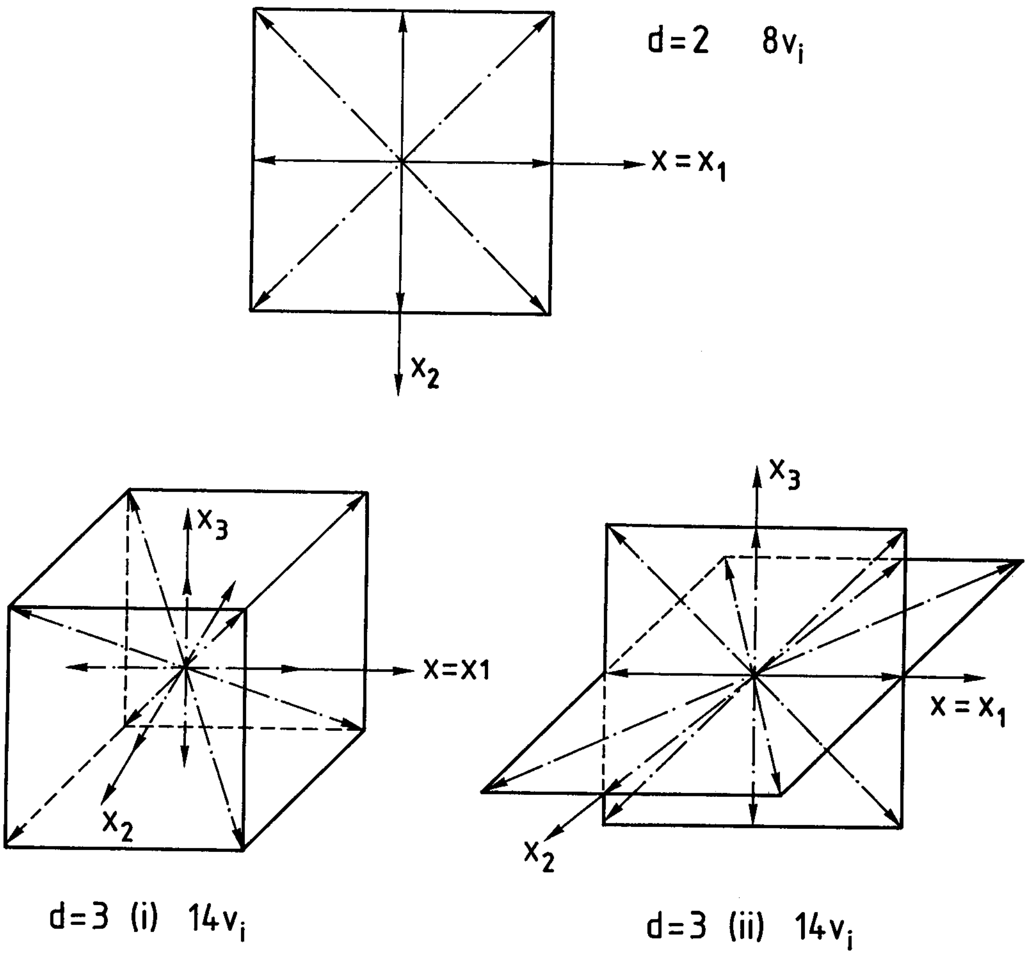

Fig. 1 – $d \leqslant 3$ class I models

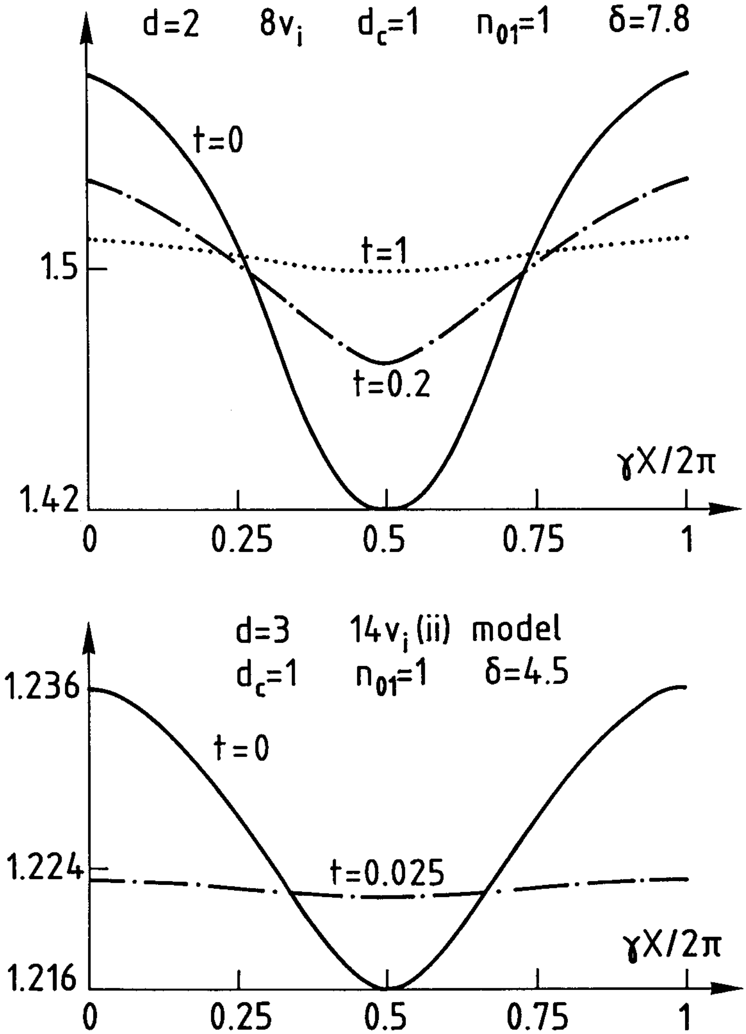

Fig. 2 - Temperature for periodic solutions of class I

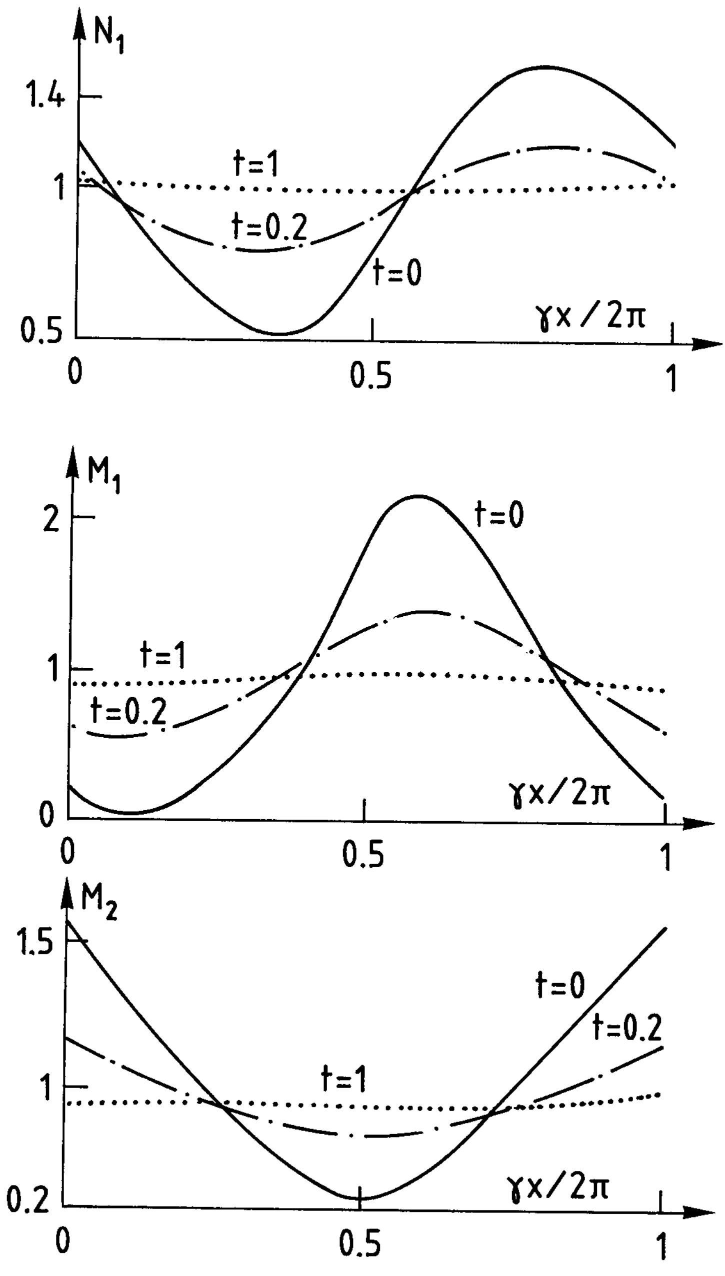

Fig. 3 – Densities for periodic solutions of the d=2 $8v_i$ model

$d_c=1 \quad n_{01}=1 \quad \delta=7.8$