NUMERICAL RESULTS CONCERNING THE GENERALIZED ZAKHAROV SYSTEM

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Abstract. A generalization of the well known Zakharov system of ion-acoustic waves (Langmuir solitons) has been obtained while studying the coupling between shear-horizontal surface waves and Rayleigh surface waves propagating on top of a structure made of a nonlinear elastic substrate and a superimposed thin elastic film. The generalization consists in a nearly integrable system made of a nonlinear Schrödinger equation (thus including self-interactions) coupled to two wave equations for the secondary acoustic system (Rayleigh mode). Here we present essentially the numerical simulations pertaining to the uncoupled case (pure SH mode) and the coupled case (influence of viscous dissipation in the Rayleigh subsystem, collision of solitons).

1. General problem

The problem considered consists in studying the possible propagation of *surface* solitary waves, eventually solitons, of the surface-wave type (amplitude decreasing in the substrate) in a structure made of a *nonlinear* elastic isotropic *substrate* (half-space $X_2 > 0$) and a superimposed *linear* elastic isotropic *thin film*, the latter being perfectly bonded to the former (Figure 1). The *nonlinearity* originates thus from the substrate while *dispersion* is induced by the film which plays the role of a *wave guide*. In the mathematical description, the thin film is reduced to an *interface* of vanishing thickness which, however, still carries a mass density (hence inertia) and membrane elasticity in agreement with a general continuum approach [1]. A general surface wave problem in this structure involves both an SH (shear horizontal) elastic component (polarized along X_3) and a Rayleigh two-component displacement polarized parallel to the so-called sagittal plane P_s [2]. The complete coupled nonlinear wave problem is a tedious one which is shown to be tractable in several steps. First in the linear approximation an SH *dispersive* surface mode of the type of Murdoch [3], and a classical Rayleigh (nondispersive) mode propagate independently as a consequence of the assumed isotropy of the materials. At the next order, both modes couple through the nonlinearity [4]. However, if the primary signal entered in the system through a transducer is of the SH type and is $O(\varepsilon)$, then the Rayleigh subsystem will develop an $O(\varepsilon^2)$ component. This nonlinear mutual coupling [4] is neglected in the first instance

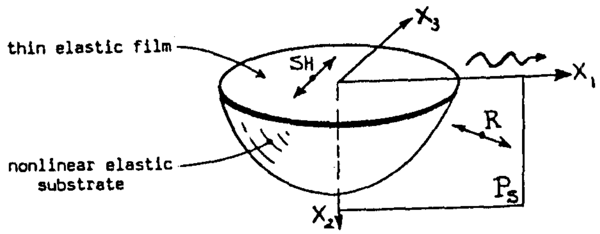


Figure 1 : Setting of the surface elastic-wave problem

and the pure *nonlinear* SH mode is shown to be governed by a single cubic Schrödinger equation *at the interface* for modulated signals with slowly varying envelope [5]. Then the problem accounting for the nonlinear coupling with the Rayleigh components is shown to be reducible to the announced generalized Zakharov system [6] when the main field still is of the SH type. Here essentially numerical simulations are presented, analytical results being found in other publications [5], [7].

2. The pure SH surface-wave problem

In this simplified case, using the boundary condition provided by the theory of material interfaces [1], we find that the initial mechanical (two space dimensions) problem is governed by the following set of equations in nondimensional units [5] :

$$\beta^2 U_{tt} - (U_{xx} + U_{yy}) = \beta^2 \Delta \left\{ \left[U_x \left(U_x^2 + U_y^2 \right) \right]_x + \left[U_y \left(U_x^2 + U_y^2 \right) \right]_y \right\} \text{ for } X_2 = y > 0 \ ,$$

$$\hat{U}_{tt} - \hat{U}_{xx} = U_y \left\{ 1 + \beta^2 \ \Delta \ \left(U_x^2 + U_y^2 \right) \right\} \ , \quad U = \hat{U} \ , \quad \text{at } X_2 = y = 0 \qquad , (2.1)$$

$$U \ (x, \ y \to \infty \ , \ t) = 0 \qquad ,$$

where subscripts indicate space (x and y) and time (t) differentiation, Δ is the *nonlinearity* parameter, and β is the *dispersion* parameter. In the absence of nonlinearity ($\Delta = 0$) the above system yields *Murdoch's linear surface waves* [3] ; in the absence of dispersion (zero left-hand side in $(2.1)_2$), it yields *Mozhaev's nonlinear surface waves* [8]. The full system possesses all good ingredients to exhibit *solitary* waves of the surface wave type (so as to satisfy the last of (2.1)). This is proven analytically by using the Whitham-Newell [9] technique of treatment of nonlinear dispersive small amplitude, almost monochromatic waves [5]. In the process "*wave action*" conservation laws and "dispersive" nonlinear dispersion relations are established for this type of surface waves that could also be approached by using Whitham's averaged Lagrangian technique as modified by Hayes to account for the transverse modal behavior [10]. The analysis [5] is conducted simultaneously in the bulk ($y > 0$) and at the interface ($y = 0$). Combining the two at the interface results in a single nonlinear

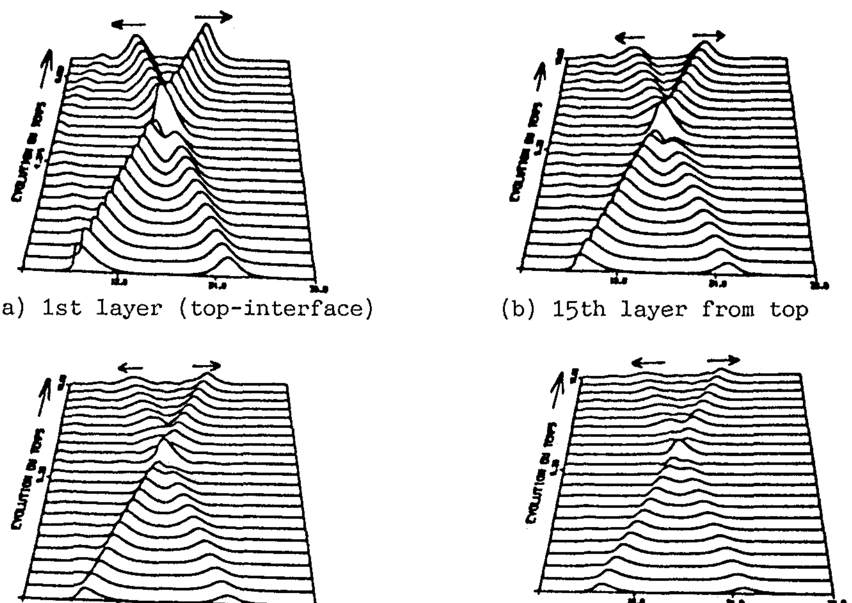
Schrödinger (NS) equation for the envelope of complex amplitude a (in reduced coordinates) :

$$i\ a_t + p\ a_{xx} + q\ |a|^2\ a = 0 \tag{2.2}$$

where p and q are real and depend on the working regime (ω_0 , k_0) along the linear dispersion relation of Murdoch's waves. Explicitly,

$$p\ (\omega_0\ ,\ k_0) = \frac{1}{2}\,\omega_0'' \quad , \quad q\ (\omega_0\ ,\ k_0) = \frac{3}{8}\,\Delta\,\beta^4\ \omega_0\ \frac{\left(\beta^2\ \omega_0^2 - 2\ k_0^2\right)}{\beta^2 + 2\ \left(\omega_0^2 - k_0^2\right)}\ , \tag{2.3}$$

where ω_0'' is the curvature of the linear dispersion relation. The NS equation (2.2) is *exactly integrable* [11] and admits *bright* and *dark* envelope (true) solitons depending on the sign of the product pq. If the nonlinear material making up the substrate is known (e.g., $LiNbO_3$ [4] for which $\Delta > 0$), then this criterion allows one to select the thin film material to guarantee the existence of the desired stable surface solitary wave. In the present case with $\Delta > 0$, $\frac{1}{2} < \beta^2 < 1$ (film of aluminum) and $\beta^2 < \frac{1}{2}$ (film of gold) provide stable bright and dark solitons, respectively [5]. The analytical solutions thus obtained are used as initial-boundary value conditions in direct numerical simulations performed on the original (obviously *non* exactly integrable) two-space-dimension system (2.1). Explicit and implicit numerical finite-difference methods in three-dimensional.



(a) 1st layer (top-interface)

(b) 15th layer from top

(c) 25th layer from top

(d) 35th layer from top

Figure 2 : Collision of unsymmetric envelope- solitons of the Surface-wave type in system (2.1).

Euclidian space-time grids were used for this (see [12] for technical details). The surface waves indeed propagate as *solitary* waves along the $X_1 = x$ direction with a nice exponential decrease with depth (along $X_2 = y$). A lack of accuracy in numerical solutions in depth will show up at the interface sooner or later. A predominance of nonlinear effects over dispersion may yield the formation of *surface shock waves after* a typical steepening [12]. While (2.2) obviously exhibits a true *solitonic* behavior in soliton interactions, the *rather* pure solitonic behavior of system (2.1) for small amplitudes, can only be checked numerically. This is indeed practically the case as shown in Figures 2 which exhibit the interaction of two colliding unequal solitons in the SH system at different depths in the substrate (fifty layers are accounted for in the computation along depth). At this point it should be noted that there is no difficulty to account for the *viscosity* in the pure SH system and the subsequent alteration in (2.2).

3. The coupled SH-Rayleigh problem

In agreement with Section 1, the main displacement field $O(\varepsilon)$ is the SH component, in which case the Rayleigh components are $O(\varepsilon^2)$. Considering slowly-varying envelope solutions for the SH components, a long asymptotic evaluation [6] allows one to show that, after integration along the transverse coordinate y, and appropriate scaling, the whole problem is governed at $y = 0$ by the following system of equations for the complex amplitude a of the SH mode and the real components v and w of elastic displacements along x and y, respectively, parallely to the sagittal plane P_s (Figure 1).

$$i\, a_t + a_{xx} \pm 2\lambda |a|^2 a + 2a(\alpha_L n_1 + \alpha_T n_2) = 0 \quad ,$$

$$(n_1)_{tt} - c_L^2 (n_1)_{xx} - \eta_L (n_1)_{xxt} = -\mu_L (|a|^2)_{xx} \quad , \qquad (3.1)$$

$$(n_2)_{tt} - c_T^2 (n_2)_{xx} - \eta_T (n_2)_{xxt} = -\mu_T (|a|^2)_{xx} \quad ,$$

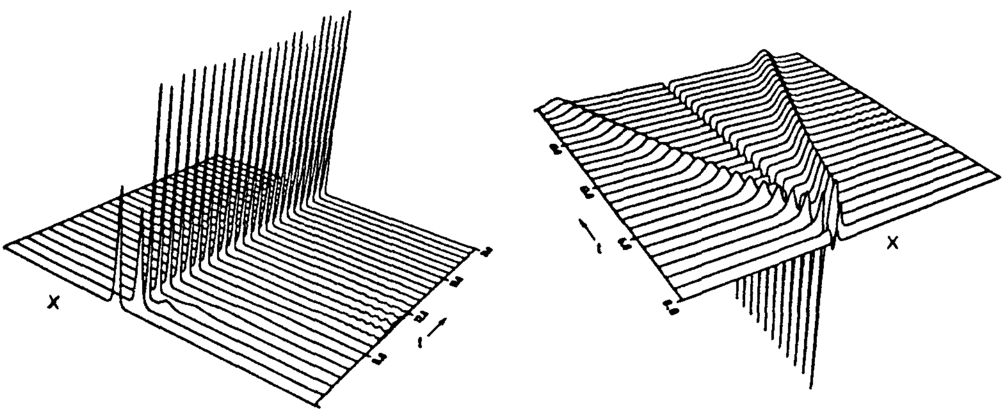
where $n_1 = v_x$, $n_2 = w_x$, and viscosity has been introduced for the Rayleigh components only (on account of the last remark in Section 2). System (3.1), a *nearly integrable system* only, is a system which generalizes the system of Zakharov [13]- for which $\lambda = 0$, $w \equiv 0$, $\alpha_T = \mu_T = 0$ - that appears in ion-acoustic systems in plasmas (Langmuir solitons). This system has been extensively studied analytically. The general system (3.1) obviously is richer and presents many interesting features. Two of these are especially examined below.

4. Dissipation-induced evolution of solitons

We consider the evolution of envelope solitary waves in the (SH) a-system of (3.1) under the influence of dissipation (viscosities η_L and η_T) in the (v , w) Rayleigh systems. The two are coupled through the coupling coefficients μ_L and μ_T. In spite of its appearance, system (3.1) conserves the *number of surface phonons* (or *wave action*)

$$N = \int_{-\infty}^{+\infty} |a|^2 \, dx \tag{4.1}$$

In the analytical treatment $[7]_a$, which applies the *balance-equation analysis* to the *slow* dissipation-induced evolution of the exact one-soliton solution of the Zakharov system (for the sake of simplicity w = 0 , $\mu_T = 0$ in (3.1) ; this system is *not* exactly integrable, three different scenarii of evolution are shown to be possible : (i) adiabatic (slow) transformation of a moving *subsonic* soliton into the stable quiescent one, (ii) complete adiabatic decay of a *transsonic* soliton with a small amplitude, and (iii) coming of the *transsonic* soliton with a large amplitude into a critical state, from which a further adiabatic evolution is not possible. In the latter case a numerical investigation of the further evolution of the soliton is particularly enligthening. In a general case, it is shown that it abruptly splits into the stable quiescent soliton, the slowly decaying small-amplitude transsonic one, and a pair of left and right-traveling acoustic pulses slowly fading under the action of the weak dissipation. This is exhibited in the numerical simulations in Figures 3 and 4. The abrupt splitting seems to be a new type of *inelastic* process for a soliton induced by small perturbations (see the review given in Ref.[14]). This concludes our brief excursion in the evolution of *one* soliton in the *damped* generalized Zakharov system.



(a) $|u|^2$ (b) $-n_1$

Figure 3 : Dissipation-induced evolution of the exact one-soliton solution of Zakharov's system : abrupt split into three pulses in the n_1-system (large wave action, large velocity)

5. Soliton-soliton collision in the generalized Zakharov System.

As seen above the generalized Zakharov system (3.1) in the absence of viscosity in the Rayleigh subsystem admits both *subsonic* and *transsonic* one-soliton solutions. The question naturally arises of the interaction (collision) of such solitons, for instance in the symmetric soliton-soliton collisions. In the analytical study $[7]_b$ the collision-induced emission of acoustic waves (in the Rayleigh subsystem) was treated for soliton velocities much larger than their amplitudes. In particular, it was shown that the acoustic losses are exponentially small unless the solitons' velocities are much larger than the characteristic sound velocity in the Rayleigh subsystem. The numerical simulation of the head-on soliton-soliton collision brings up two basic phenomena : (i) the collision of *subsonic* solitons always leads to their *fusion into a breather*, provided the system is sufficiently far from the integrable limit ; (ii) the collision between *transsonic* solitons gives rise to a multiple production of solitons (both sub- and transsonic solitons are produced), and the *quasi-elastic* character of

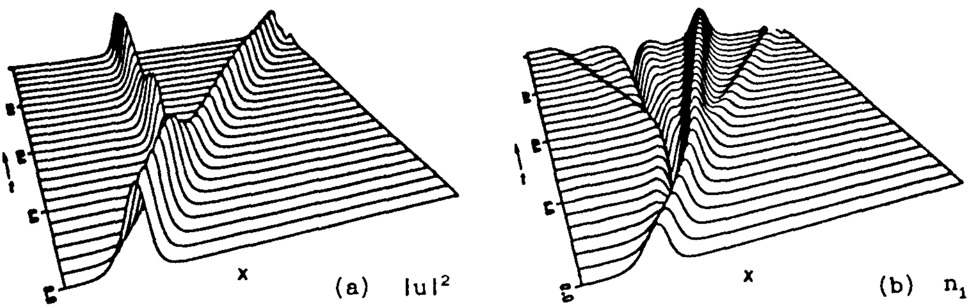


Figure 4 : Dissipation-induced evolution of the exact one-soliton solution of Zakharov's system : rearrangement of the soliton in the intermediate case. (Smaller values of wave action and velocity than in Figure 4).

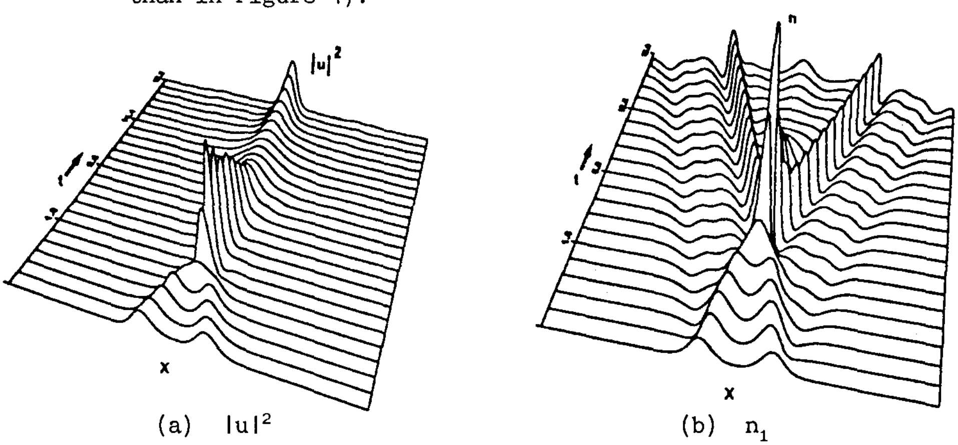
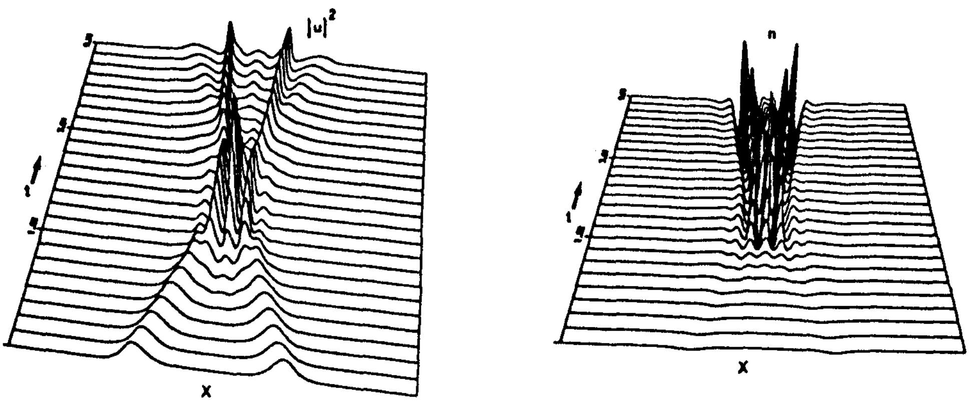


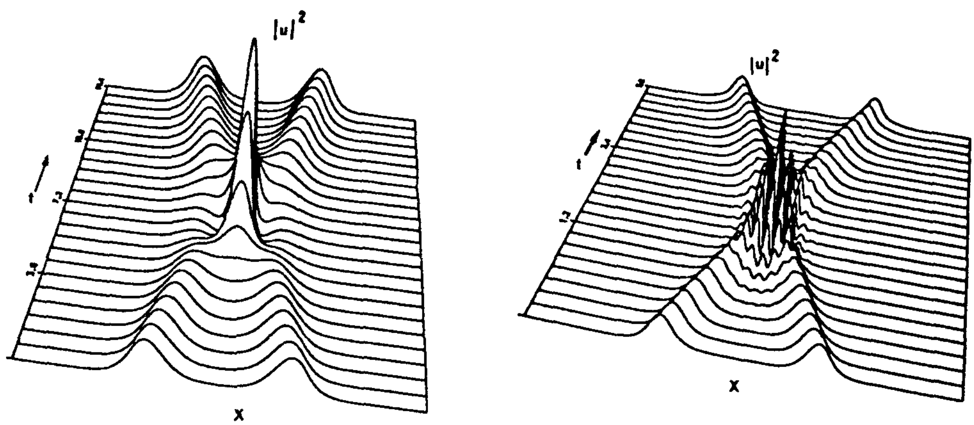
Figure 5 : Collision-induced fusion of *subsonic* solitons into a breather with acoustic emission in the Rayleigh subsystem

the collision is recovered in the limit of large velocities. This is illustrated in Figures 5, 6 and 7.



(a) $|u|^2$ (b) n_1

Figure 6 : Collision of two transsonic solitons at moderate velocities



(a) (b)

Figure 7 : Comparison between the soliton-soliton collision for system (3.1) close to the NS equation (a) and the collision of two *transsonic* solitons at high velocities in the generalized Zakharov system (b).

6. Conclusion

It appears that the initial, purely mechanical, surface-wave problem considered yields, on the one hand, a very interesting physical application which may be of interest in signal processing (we have a mechanical analog of light solitons in optical fibers; compare [15]) and, on the other hand, a class of paradigmatic problems in soliton theory for *nearly integrable* systems made of an exactly integrable equation coupled nonlinearly to d'Alembert equations. The *sine-Gordon-d'Alembert* systems introduced previously by Maugin and Pouget [16] in a different physical context belong to the same class. The *modified-Boussinesq-d'Alembert* system introduced recently by Maugin and Cadet [17] in martensitic alloys appears to be even more difficult to deal with.

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