Resonances in nonlinear Klein-Gordon kink scattering by impurities

Yuri S. Kivshar,[1,2] *Angel Sánchez*[1] *and Luis Vázquez*[1]

[1] Departamento de Física Teórica I, Facultad de Ciencias Físicas Universidad Complutense, E-28040 Madrid, Spain
[2] On leave from: Institute for Low Temperature Physics and Engineering, 47 Lenin Avenue, 310164 Kharkov, U.S.S.R

Abstract. The scattering of topological kinks by point-like impurities is numerically studied in the framework of the sine-Gordon and ϕ^4 models. In the first case, we show that previous approximate analytical results are indeed applicable. Thus, for low velocities, the reflection coefficient depends oscillatorily on the distance between impurities, i.e., resonant scattering takes place. On the other hand, this effect disappears for higher velocities. This result is found to hold also for the ϕ^4 model nonlinearly coupled to the impurities, whereas the linear coupling for the same model gives rise to a different behaviour.

1 Introduction.

In recent years, it has become clear the importance of the interplay between disorder and nonlinearity in many physical contexts, which is at the root of many novel and striking phenomena and their corresponding unsolved mathematical problems [1,2]. Among these, the question as to whether solitons, supported by nonlinearity, and Anderson localization effects influence each other, changing the transmission properties of systems, deserves careful study in view of its practical consequences. As a first step in this direction, the scattering properties of solitons in a system with few impurities must be understood. Even more, as there are three main types of solitons (see [3,4] and references therein), their scattering features are different, and a separate analysis is required. The scattering of non-topological solitons by one and two impurities has been already studied numerically by Li *et al.* [5]: both kink-like and envelope solitons were considered in the simple model of a nonlinear atomic chain with nearest-neighbor interparticle interactions. A part of the results obtained there [5] has been explained analytically [6] by means of the perturbation theory for solitons based on the inverse scattering transform (IST), [7] (see [8] for a comprehensive review of this technique). On the contrary, the current knowledge about the scattering properties of the other kind of solitons, topological kinks similar to those beared by Nonlinear Klein-Gordon (NKG) equations (like the sine-Gordon [sG] or ϕ^4 ones), is quite limited.

As it was previously pointed out [6,7], interference phenomena are the most remarkable effects to study wave properties of solitons. Interference may arise when the characteristic distance between impurities becomes commensurable with the characteristic length of the linear waves emitted by the soliton. As a consequence, the simplest way to observe interference is to consider a system with two point impurities [6,7]. Solitons emit a rather wide spectrum of linear waves, and the problem is far more difficult than the scattering of a monochromatic plane wave. As a matter of fact, the above mentioned condition that the two characteristic lengths must be commensurate has to be valid for *an averaged* spectral structure of the soliton emission. When a soliton has an internal frequency (like, e.g., an envelope soliton), resonant scattering is naturally expected [6,7], while for topological kinks this phenomenon does not seem to be possible. However, resonant scattering has been predicted [6] for a slowly moving sG kink when the spectral density of the emission generated by it has a narrow maximum: interference effects should appear as oscillations of the soliton reflection coefficient dependence on the distance between impurities. It is not at all a trivial matter to ask if this prediction is actually useful, for if the kink is too slow, it may be pinned or reflected by attractive or repulsive impurities, respectively (see [8] and references therein). As these possible effects were neglected in the analytical calculation [7] of the reflection coefficient, it is crucial to determine whether they become as relevant as to inhibit the interference effects by forbidding kink propagation. This is the problem we address here, namely the resonant scattering of sG and ϕ^4 kinks: we study it numerically as to compare the so obtained results to the analytical predictions, thus establishing their validity range, if any.

2 Model, predictions and numerical procedure

The model we deal with is an inhomogeneous NKG system which, in dimensionless units, is described by the equation

$$\phi_{tt} - \phi_{xx} + V'(\phi) + \epsilon[\delta(x) + \delta(x - D)]\, V_{imp}(\phi) = 0; \tag{1}$$

In particular, we will consider the following choices for potentials and perturbations:

$$V'(\phi) = V_{imp}(\phi) = \sin\phi, \tag{2}$$

$$V'(\phi) = -\phi + \phi^3,\ V_{imp}(\phi) = V'(\phi), \tag{3}$$

$$V'(\phi) = -\phi + \phi^3,\ V_{imp}(\phi) = -\phi, \tag{4}$$

Equation (2) corresponds to a sG model with two point-like impurities, whereas equations (3) and (4) are similarly perturbed ϕ^4 models in which the impurities are coupled to the wavefield either nonlinearly, equation (3), or linearly, equation (4). All of these systems are very well-known in the unperturbed case, i.e., $\epsilon 00$, and their properties have been widely described, but the perturbed problems are rather difficult and cannot be solved exactly. Nevertheless, recalling that the homogeneous sG equation is integrable, some theoretical analysis of the problem (2) is possible through perturbation theory for solitons based on IST. This analysis has been recently carried out by Kivshar *et al.* [6], who were interested in the influence of the parameter D on the scattering properties of sG kinks, or more precisely, in their reflection coefficient. This coefficient, R, may be defined as $R \equiv E_{em}^{(-)}/E_k$, where $E_k = 8/\sqrt{1-v^2}$ is the energy of a sG kink with velocity v far away from the inhomogeneous region, and $E_{em}^{(-)}$ is the energy the kink emits backwards as radiation, due to its interaction with an impurity. The emitted energy $E_{em}^{(-)}$, and, consequently, the reflection coefficient R can be calculated analytically by means of the aforementioned IST perturbation theory, the main restriction of it being the necessary assumption that the kink velocity does not change during the scattering (the so called Born approximation). By this means, the emitted spectral density can be shown [6] (see similar computations in [7,8]) to be given by

$$\varepsilon_2(k) = 4\varepsilon_1(k)\cos^2\left\{\frac{D}{2v}[kv - \omega(k)]\right\}, \tag{5}$$

$$\varepsilon_1(k) = \frac{\pi\epsilon^2}{8v^6}(1-v^2)^2\,\frac{[k-\omega(k)]^2}{\cosh^2[\pi\sqrt{1-v^2}\omega(k)/2v]}, \tag{6}$$

where $\omega(k) \equiv \sqrt{1+k^2}$, and $n = 1,2$ stand for the case when one and two impurities are present in the system, respectively. Having these expressions in mind, it is straightforward to obtain the corresponding reflection coefficients, which turn out to be

$$R_n = \frac{1}{8}\sqrt{1-v^2}\int_0^\infty dk\,\varepsilon_n(-k), \tag{7}$$

where $n = 1,2$ stands for the case with one or two impurities, respectively.

It can be seen from equations (5) and (6) that for small v, $v^2 << 1$, the spectral density $\varepsilon_1(k)$ has a single maximum at $k = 0$, with a quite narrow peak of width of order $2v/\pi \sim v$. As a consequence, such maximum will provide the main contribution to the emitted energy and should give rise to a resonant dependence: it is possible to obtain from equations (5)–(7) an approximate estimation for the value $R_2/2R_1$ when $v^2 << 1$, which turns out to oscillate as

$$\frac{R_2}{2R_1} \simeq 1 + \frac{1}{(1+D^2/\pi^2)^{1/4}}\cos\left[\frac{D}{v} - \frac{1}{4}\tan^{-1}\left(\frac{D}{\pi}\right)\right]. \tag{8}$$

On the other hand, if the speed is large, there are two maxima at $\pm k_m$, $k_m = 2v/\pi\sqrt{1-v^2} \simeq (1-v^2)^{-1/2}$, and the function $\varepsilon_1(k)$ is not exponentially small in the region $|k| < k_m$. Hence, after averaging over all wave numbers there is no leading contribution, the oscillatory dependence dissapears, and resonant scattering is not to be expected.

It must be noticed that, of course, equation (8) makes sense only from a theoretical viewpoint, because due to total reflection the kink velocity can not be less than a certain threshold, $v_{thr} \equiv \sqrt{\epsilon/2}$, below which the kink is reflected by the impurity. This is the fundamental reason for the necessity of numerical simulations: to see whether velocities over the threshold still are well accountd for by the perturbative prediction. The v_{thr} value can be obtained thinking of the kink as a point-like particle moving in an effective potential originated by the impurities (see, e.g., [8]). Finally, let us insist that this result applies only to sG systems; the non-integrability of the ϕ^4 system does not allow to use the same technique, though it is possible to obtain some results which we will describe elsewhere [13].

The numerical procedure we use to simulate the kink scattering is the finite difference scheme of Strauss and Vázquez [9] for nonlinear Klein-Gordon equations. This scheme has been succesfully employed to study a number of different perturbed nonlinear Klein-Gordon problems (for instance, see [10,11]; see also references therein). Moreover, its most important property is that it exactly conserves the system energy in the unperturbed evolution, which is relevant to the accuracy of the computation we intend to do; notice that all we must evaluate is the energy content at the left, between and at the right of the impurities. Further details on the scheme can be found in the literature [9,11].

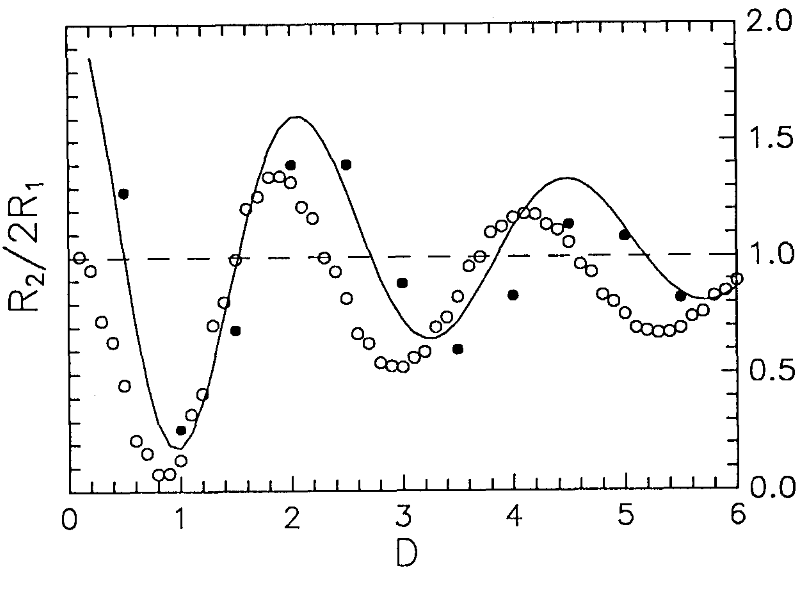

Fig. 1. Reflection coefficient for a $v = 0.4$ sG kink vs distance D between impurities.

3 Results and discussion

The numerical results for two values of the initial kink velocity in the sG model (2) are shown in figures 1 and 2. We plot the ratio $R_2/2R_1$ which not only can be directly compared to the prediction (8), but also is a suitable quantity to search for interference effects. Indeed, when the impurities are far from each other, i.e., when $D \to \infty$, the reflection coefficient has to coincide with $2R_1$, and the above mentioned quotient must go to unity; any difference from unity may be treated as coming from interference. In both figures, the full lines correspond to the analytically computed dependence given by Eqs.(5)–(7). In addition, for $v = 0.4$

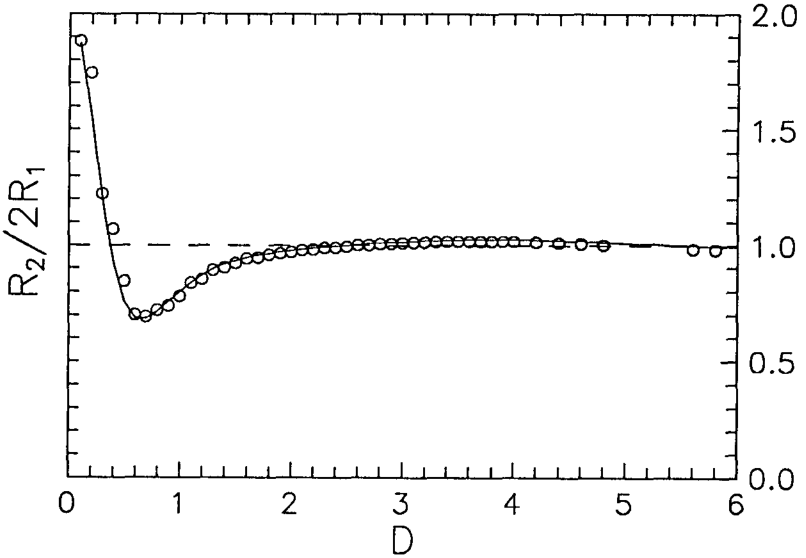

Fig. 2. Reflection coefficient for a $v = 0.9$ sG kink vs distance D between impurities.

we performed some simulations for attractive delta-functions, $\epsilon = -0.1$, because predictions do not depend on the sign of ϵ, cf. equation (6; these are shown as full circles. It comes out that the agreement between analytical and numerical curves is fairly good. Therefore, in view of our previous considerations, we conclude that the numerical simulations establish that the resonant scattering of the sG kink is stipulated by the spectral properties of its emission, according to the above discussed perturbative predictions.

The main differences arise at $v = 0.4$ (figure 1); in particular, the asymptotic behaviors of the analytical and the numerical curves are not the same. This disagreement, that in principle should not be expected, can be explained in a natural way. Recall that the analytical results for two impurities were obtained under the assumption that the kink *does not change* its velocity during the scattering (Born approximation). However, as a matter of fact, after the first scattering the kink loses some part of its kinetic energy, so that it interacts with the second impurity at a *smaller* velocity, say $v - \Delta v$. Hence, the ratio $R_2/2R_1$ does not go to 1 when the distance between deltas, go to infinity; rather well, it verifies

$$\frac{R_2}{2R_1} \to \frac{[E_{em}^{(-)}(v) + E_{em}^{(-)}(v - \Delta v)]}{2E_{em}^{(-)}(v)}, \tag{9}$$

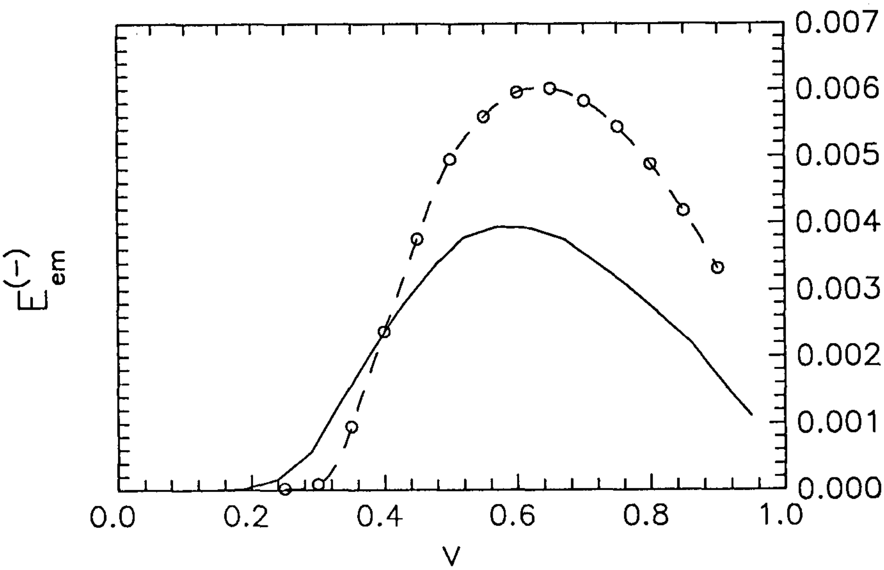

Fig. 3. Emitted energy by a sG kink vs kink velocity.

$E^{(-)}_{em}$ being the energy reflected by a single impurity. To understand the difference between $E^{(-)}_{em}(v)$ and $E^{(-)}_{em}(v - \Delta v)$, we have analyzed the emitted energy for a single impurity versus the kink velocity. Figure 3 presents both numerical and analytical dependences.[1] From this plot, it turns out that when the initial velocity is $v = 0.4$, $E^{(-)}_{em}(v) > E^{(-)}_{em}(v - \Delta v)$, so that the asymptotics of $R_2/2R_1$ computed numerically is always *smaller* than the analytical predictions. This ceases to be true when v becomes larger than $v_{cr} \simeq 0.6$ because in that range $E^{(-)}_{em}(v)$ decreases and hence $E^{(-)}_{em}(v - \Delta v) > E^{(-)}_{em}(v)$. Thus, in figure 2, the numerical asymptotics is slightly above the line $R_2/2R_1 = 1$, the difference being small due to the short time that this kink takes to cross the distance D between, much shorter than in the other case. The remaining, little discrepancy for attractive impurities is due to the fact that, in this case, the linearized sG model supports the so-called impurity mode (see, e.g., [12]), which is excited by the kink, giving an additional contribution to the radiated energy as computed in the region $x < 0$. Detailed analysis of the impurity mode excitation during the kink scattering and also a quasi-resonant behavior originated from an energy-exchange mechanism between the kink and this impurity mode will be presented elsewhere [13,14].

After this proof of the validity of the approximate perturbative results (which, besides, is also a new checking of the Strauss-Vázquez procedure) we have ca-

[1] The analytical results, that are below the numerical ones, assumed that the velocity of the kink does not change, but, in fact, the change in velocity will produce an additional emission. This effect can explain the amount of emission observed in the simulations, larger than the predicted.

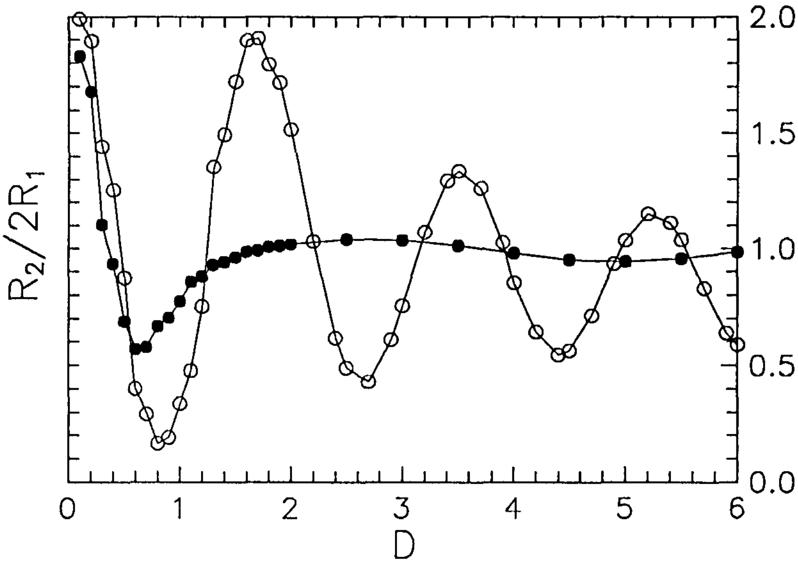

Fig. 4. Reflection coefficient for ϕ^4 kinks with nonlinear coupling [model (3)] vs distance D between impurities.

rried out identical simulations for the two ϕ^4 models. In the one with nonlinear coupling, everything is very much like the sG system, as is shown in figure 4: resonant scattering arises again for slow kinks, and attractive impurities exhibit also the signature of their mode (described in [15]). This strongly suggests that the radiation emission by the ϕ^4 kink is quite the same as that of the sG kink. On the contrary, the phenomena that take place for the ϕ^4 model linearly coupled to the impurities (figure 5) are different, and cannot be simply explained in terms of the spectral content of the emission. Let us stress that the reflection coefficient is never very small, and then we cannot speak of resonant scattering. Moreover, kinks with $v = 0.4$ are reflected by the joint action of the impurities if $D < 1.4$. Finally, we have observed that the role played by the localized impurity mode is much more important in this system, seemingly because of the linear nature of the coupling; notice that now the kink tails interact with the impurities and hence they are not a ground state of the model anymore. Further research on this model is needed to describe properly these cooperative scattering effects.

Acknowledgments

This work has been supported in part by the Dirección General de Investigación Científica y Técnica (D.G.I.C. y T.) of Spain, through project MAT90-0544. The stay of Yu. S. Kivshar in Madrid has been supported by the Universidad Complutense de Madrid through the Programa de Sabáticos Complutense.

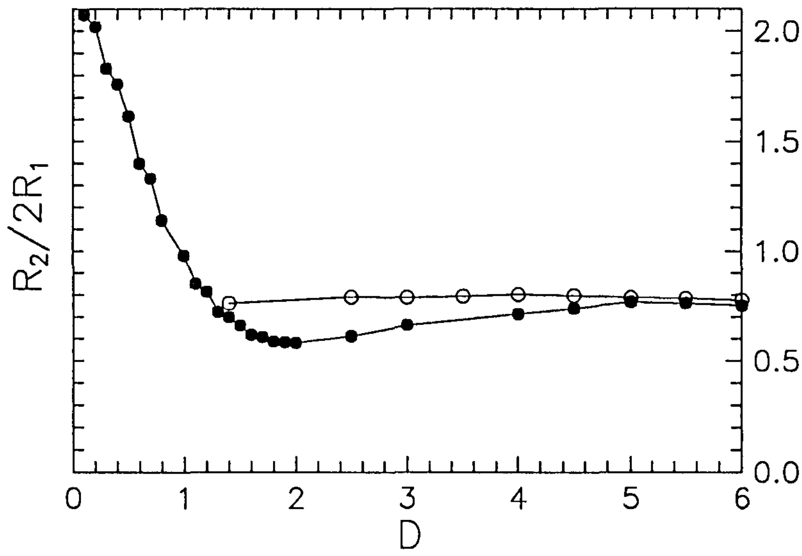

Fig. 5. Reflection coefficient for ϕ^4 kinks with linear coupling [model (4)] vs distance D between impurities.

References

1. Bishop, A. R., Campbell, D. K., Pnevmatikos, St. (eds.): *Disorder and Nonlinearity* (Springer Proceedings in Physics no.39, Springer, Berlin-Heidelberg, 1989).
2. Abdullaev, F. Kh., Bishop, A. R., Pnevmatikos, St. (eds.): *Nonlinearity with Disorder* (Springer Proceedings in Physics, Springer, Berlin-Heidelberg, in press, 1991).
3. Kivshar, Yu. S.: in Reference [2].
4. Sánchez, A. S., Vázquez, L.: Int. J. Mod. Phys. B, in press (1991).
5. Qiming Li, Pnevmatikos, St., Economou, E. N., Soukoulis, C. M.: Phys. Rev. B **37**, 3534 (1988).
6. Kivshar, Yu. S., Kosevich, A. M.,Chubykalo, O. A.: to be published.
7. Kivshar, Yu. S., Kosevich, A. M.,Chubykalo, O. A.: Phys. Lett. A **125**, 35 (1987).
8. Kivshar, Yu. S., Malomed, B. A.: Rev. Mod. Phys. **61**, 763 (1989).
9. Strauss, W., Vázquez, L.: J. Comp. Phys. **28**, 271 (1978).
10. Sánchez, A., Vázquez, L.: Phys. Lett. A **152**, 184 (1991).
11. Sánchez, A., Vázquez, L., Konotop, V. V.: Phys. Rev. A **43**, in press (1991).
12. Braun, O. M., Kivshar, Yu. S.: Phys. Rev. B **43**, 1060 (1991).
13. Kivshar, Yu. S., Sánchez, A., Vázquez, L.: to be published.
14. Kivshar, Yu. S., Zhang Fei, Vázquez, L.: to be published.
15. Fraggis, T., Pnevmatikos, St., Economou, E. N.: Phys. Lett. A **142**, 361 (1989).

This article was processed using the LaTeX macro package with ICM style