RESONANT KINK–IMPURITY INTERACTIONS

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ABSTRACT We demonstrate that the impurity mode plays an important role in the kink–impurity interactions, and a kink may be totally *reflected by an attractive impurity* if its initial velocity lies in some resonance "windows". This effect is quite similar to the resonance phenomena in kink-antikink collisions in some nonlinear Klein-Gordon equations, and it can be explained by a resonant energy exchange mechanism. Taking the sine-Gordon and the ϕ^4 models as examples, we find a number of resonance windows by numerical simulations, and develop a collective-coordinate approach to describe the interactions analytically.

1 Introduction

It is well known that nonlinearity may drastically change transport properties of disordered systems when it contributes to create soliton pulses [1,2]. As a first step to understand the soliton transmission through disordered media, one has to study the soliton scattering by a single impurity. The kink-impurity interactions have been explained, for a long time, by the well-known model in which a kink moving in an inhomogeneous medium is considered as an extended classical particle obeying Newton's Law of Motion (see Refs. [3]-[5] and references therein). In particular, it was shown by Malomed [6] that a sine-Gordon kink may be trapped by an attractive impurity due to radiative losses, and a threshold velocity was found analytically. However, the previous theoretical studies [3]-[6] totally ignored the fact that the underlying nonlinear system supports a localized impurity mode which may be *excited* due to the kink scattering. Recently, the importance of the impurity mode have been noticed in our papers [7]-[9]. We have found that a kink may be totally reflected by an attractive impurity due to resonance energy exchange between the kink translational mode and the impurity mode. This effect is quite similar to the resonance phenomena in kink-antikink collisions in some nonlinear Klein-Gordon equations [10]–[13]; and it cannot be predicted by the previous theoretical approach, which took into account only radiative losses [3]-[6].

In the present paper we briefly review our recent numerical and analytical results related to the kink-impurity interactions in two well-know kink-bearing systems, namely the sine-Gordon (SG) and the ϕ^4 models. In section 2 we describe the resonance phenomena in kink-impurity interactions in the SG model, and develop a collective-coordinate approach to explain the phenomena analytically. Section 3 is devoted to the similar problem in the ϕ^4 model. We draw conclusions in section 4.

2 Kink-impurity interactions in the SG model

Firstly, we consider the sine-Gordon system including a local inhomogeneity

$$u_{tt} - u_{xx} + [1 - \epsilon\delta(x)] \sin u = 0, \tag{1}$$

where $\delta(x)$ is the Dirac δ-function. When the perturbation is absent ($\epsilon = 0$), the SG model (1) supports a topological soliton, the so-called kink which is given by

$$u_k(x,t) = u_k(z) = 4 \tan^{-1} \exp(\sigma z), \tag{2}$$

where $z = (x - X)/\sqrt{1 - V^2}, X = Vt + X_0$ is the kink coordinate, $\dot{X} = V$ is its velocity, and $\sigma = \pm 1$ is the kink polarity (without loss of generality we assume that $\sigma = +1$).

To describe the motion of the kink (2) in the presence of a localized inhomogeneity, the so-called adiabatic perturbation theory for solitons was usually used [3]–[5]. In the framework of this perturbation theory, the kink coordinate X is considered as a collective variable, and its evolution is described by a simple motion equation of a classical particle with mass $m = 8$ placed in the effective potential

$$U(X) = -2\epsilon / \cosh^2 X. \tag{3}$$

For $\epsilon > 0$, the impurity in Eq.(1) gives rise to an attractive potential $U(X)$ to the kink. Since the particle (kink) conserves its energy, it can not be trapped by the potential well if it has a non-zero velocity at infinity. However, according to the result of Malomed [6], the kink may be trapped by an attractive impurity due to radiative losses, and there exists a threshold velocity $V_{thr}(\epsilon)$, such that if the kink initial velocity is larger than $V_{thr}(\epsilon)$, it will pass through the impurity and escape to infinity, otherwise it will be trapped by the impurity [6]. In this consideration the reflection of the kink is *impossible.*

We have studied the kink-impurity interactions by numerical simulations [7]. We use a conservative scheme [14] to integrate Eq.(1), and carry out the simulations in the spatial interval $(-40, 40)$ with discrete stepsizes $\Delta x = 2\Delta t = 0.04$. When handling the Dirac δ–function, we take its value equal to $1/\Delta x$ at $x = 0$, and zero otherwise. The initial conditions are always taken as a kink centered at $X = -6$, moving toward the impurity with a given velocity $V_i > 0$. We have made intensive numerical simulations of the problem for $\epsilon > 0$ (attractive impurity), and here we will describe the results for the case $\epsilon = 0.7$ in detail.

In the numerical simulations, we find that there are three different regions of initial kink incoming velocity, namely, region of pass, of capture, and of reflection (see Fig.1); and a critical velocity $V_c \approx 0.2678$ (for $\epsilon = 0.7$) exists, such that if the incoming velocity of the kink is larger than V_c, the kink will pass the impurity inelastically and escape to the positive direction, losing part of its kinetic energy through radiation and excitation of an impurity mode. In this case, there is a linear relationship between the squares of the kink initial velocity V_i and its final velocity V_f : $V_f^2 = \alpha(V_i^2 - V_c^2)$, $\alpha \approx 0.887$ being constant.

If the incoming velocity of the kink is smaller than V_c , the kink cannot escape to infinity from the impurity after the first interaction, but will stop at a certain distance and return back, due to the attracting force of the impurity, to interact with the impurity again. For most of the velocities, the kink will lose energy again in the second interaction and finally get trapped by the impurity (see Fig. 1). However, for some special incoming velocities, the kink may escape to the *negative* infinity after the second interaction, i.e., the kink may be totally *reflected* by the impurity (see Fig. 1 and Fig. 2). This effect is quite similar to the resonance phenomena in kink-antikink collisions[10]–[13]. The reflection is possible only if the kink initial velocity is situated in some resonance windows. By numerical simulation, we have found eleven such windows. The detailed results are presented in Table I.

In order to understand the resonance structure, we define the center of the kink, $X(t)$, as the point at which the field function $u(x,t)$ is equal to π. We define T_{12} as the time between the first and the second interaction. It is clear that the attractive potential caused by the impurity falls off exponentially, so that using the same arguments as in Ref.[10] [see Eq.(3.6)], we obtain an approximate formula to estimate $T_{12}(V)$

$$T_{12}(V) = \frac{a}{\sqrt{V_c^2 - V^2}} + b \tag{4}$$

where V is the kink initial velocity, a and b are two constants. For $\epsilon = 0.7$, the parameters are empirically determined by numerical data: $a \approx 3.31893, b \approx 1.93$. We have found that formula (4) is very accurate for the velocities over the interval $(0.10, 0.267)$.

On the other hand, as we have observed that the first kink-impurity interaction always results in exciting the impurity mode, and the resonant reflection of the kink after the second interaction is just a reverse process, i.e., to extinguish the impurity mode (see Fig.2), when the timing is right, to restore enough of the lost kinetic energy and to escape from the impurity to infinity. Favorable timing in this case means that the occasion of the second interaction coincides with the passage of the impurity oscillation through some phase angle characteristics of the impurity mode extinction. Thus, the condition for restoration of the kink kinetic energy after the second interaction ought to be of the form

$$T_{12}(V) = nT + \tau. \tag{5}$$

where T_{12} is the time between the first and the second interaction, T is the period of the impurity mode oscillation, τ is an offset phase, and n is an integer. From numerical data we find that $\tau \approx 2.3$ (for the case $\epsilon = 0.7$)

Combining Eqs.(4) and (5), we may obtain a formula to predict the centers of the resonance windows,

$$V_n^2 = V_c^2 - \frac{11.0153}{(nT + 0.3)^2}, \quad n = 2, 3, \ldots.. \tag{6}$$

Similar formulas have been derived for kink-antikink collisions [10]–[13]. In Table I, we show the centers of the resonance windows predicted by Eq.(6), where $T = 2\pi/\Omega \approx 6.707$, Ω is determined by Eq.(8) with $\epsilon = 0.7$. Numerical $T_{12}(V_n)$ is defined as the time between the first and the second interactions. Note that $T_{12}(V_{n+1}) - T_{12}(V_n) \approx 6.7$ is just another expression of the resonance condition (5). From Table I we see that formula (6) can give very good predictions of the resonance windows.

To analyze the kink-impurity interactions analytically, first, we note that the nonlinear system (1) supports a localized mode. By linearizing Eq.(1) in small u, the shape of the impurity mode can be found analytically

$$u_{im}(x,t) = a(t)e^{-\epsilon|x|/2}, \tag{7}$$

where $a(t) = a_0\cos(\Omega t + \theta_0)$, Ω is the frequency of the impurity mode,

$$\Omega = \sqrt{1-\epsilon^2/4} \tag{8}$$

and θ_0 is an initial phase. As a matter of fact, the impurity mode (7) can be considered as a small-amplitude breather trapped by the impurity, with energy [8]

$$E_{im} = \frac{1}{2}\int_{-\infty}^{\infty}[(\frac{\partial u_{im}}{\partial t})^2 + (\frac{\partial u_{im}}{\partial x})^2 + (1-\epsilon\delta(x))u_{im}^2] = \Omega^2 a_0^2/\epsilon. \tag{9}$$

Now we analyze the kink-impurity interactions by collective-coordinate method taking into account two dynamical variables, namely the kink coordinate $X(t)$ [see Eq. (2)] and the amplitude of the impurity mode oscillation $a(t)$ [see Eq. (7)]. Substituting the ansatz

$$u = u_k + u_{im} = 4\tan^{-1}\exp[x - X(t)] + a(t)e^{-\epsilon|x|/2} \tag{10}$$

into the Lagrangian of the system,

$$L = \int_{-\infty}^{\infty}[\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - (1-\epsilon\delta(x))(1-\cos u)], \tag{11}$$

and assuming a and ϵ are small enough so that the higher-order terms can be neglected, we may derive the following (reduced) effective Lagrangian

$$L_{eff} = 4\dot{X}^2 + \frac{1}{\epsilon}(\dot{a}^2 - \Omega^2 a^2) - U(X) - aF(X), \tag{12}$$

where $U(X)$ is given by Eq.(3), and $F(X) = -2\epsilon\tanh X/\cosh X$. The equations of motion for the two dynamical variables are

$$\begin{cases} 8\ddot{X} + U'(X) + aF'(X) = 0, \\ \ddot{a} + \Omega^2 a + (\epsilon/2)F(X) = 0. \end{cases} \tag{13}$$

The system (13) describes a particle (kink) with coordinate $X(t)$ and mass 8 placed in an attractive potential $U(X)$ ($\epsilon > 0$), and "weakly" coupled with a harmonic oscillator $a(t)$ (the impurity mode). Here we say "weakly" because the coupling term $aF(X)$ is of order $O(\epsilon)$ and it falls off exponentially. The system (13) is a generalization of the well-known equation $8\ddot{X} = -U'(X)$ describing the kink-impurity interactions in the adiabatic approximation (see,e.g., Ref. [5]).

We find that the dynamical system (13) can describe all features of the kink-impurity interactions. Firstly, it may be used to calculate the threshold velocity of kink capture. By using an energy transfer argument, we have found the threshold analytically [4]

$$V_{thr} = \frac{\pi\epsilon}{\sqrt{2}} \frac{\sinh[\Omega Z(V_{thr})/2V_{thr}]}{\cosh(\Omega\pi/2V_{thr})}, \tag{14}$$

where $Z(V) = \cos^{-1}[(2V^2 - \epsilon)/(2V^2 + \epsilon)]$. For a given $\epsilon > 0$, this equation can be solved by Newton iteration to obtain the threshold velocity $V_{thr}(\epsilon)$. Comparing the analytical results with the direct numerical simulations of Eq.(1), we find that the perturbation theory used in Ref.[6] is valid only for very small ϵ, ($\epsilon \leq 0.05$), while formula (14) gives good estimations of $V_{thr}(\epsilon)$ for ϵ over the region (0.2, 0.7).

Furthermore, Eqs. (13) can be easily solved numerically. We perform the numerical simulations under initial conditions $X(0) = -7$, $\dot{X}(0) = V_i > 0$, $a(0) = 0$, $\dot{a}(0) = 0$. We find that, for a given $\epsilon > 0$, there exists a threshold velocity $V_{thr}(\epsilon)$ such that if the initial velocity of the particle is larger than $V_{thr}(\epsilon)$, then the particle will pass the potential well $U(X)$ and escape to $+\infty$, with final velocity $V_f < V_i$ because part of its kinetic energy is transferred to the oscillator.

Below the threshold velocity, the behaviour of the particle is very interesting. More precisely, if the initial velocity of the particle is smaller than the threshold velocity, the particle can not escape to $+\infty$ after the first interaction with the oscillator, but it will return to interact with the oscillator again. Usually the particle can be trapped by the potential well, which corresponds to a kink trapped by the attractive impurity. However, for some special initial velocities, the second interaction may cause the particle to escape to $-\infty$ with final velocity $V_f < 0$. This resonance phenomenon can be explained by the mechanism of resonant energy exchange between the particle and the oscillator. The resonant reflection of the particle by the potential well corresponds to the reflection of the kink by the attractive impurity. Therefore, the collective-coordinate approach can give a qualitative explanation of the resonance effects in the kink-impurity interactions in the SG model (1).

3 Kink-impurity interactions in the ϕ^4 model

Now let's consider the kink-impurity interactions in the ϕ^4 model

$$\phi_{tt} - \phi_{xx} + [1 - \epsilon\delta(x)](-\phi + \phi^3) = 0 \tag{15}$$

The inelastic interaction of a kink with an attractive impurity ($\epsilon > 0$) was briefly discussed by Belova and Kudryavtsev [15]. They analyzed the problem by collective-coordinate approach using two dynamical variables: the kink translational mode and its internal mode. By numerical simulation of the collective-coordinate dynamical system, they predicted that the kink may be reflected by the impurity due to energy exchange with its internal mode. However, they totally ignored the impurity mode. As we have observed that, although the SG kink does not have internal mode, it still can be reflected by an attractive impurity due to resonant energy exchange between the kink translational mode and the impurity mode, so we have reason to believe that the impurity mode may also play an important role in the ϕ^4 kink-impurity interactions.

We have studied the kink-impurity interactions in the ϕ^4 model [9], and extended the previous work in three directions. Firstly, by numerical simulation we have confirmed the previous claim that a kink can be reflected by an attractive impurity, meanwhile, we have observed more resonance windows of kink reflection. For example, at $\epsilon = 0.5$ we have found seven resonance windows below the threshold velocity of kink capture. Secondly, and the most importantly, we have observed that both the the impurity mode and the kink internal mode take part in the interactions, and the resonance window structure cannot be predicted by supposing that there exists only one localized mode. In particular, we have found that due to the joint effect of the impurity and the kink internal mode oscillation, some resonance windows may disappear. Finally, we have developed a collective–coordinate approach taking into account three dynamical variables: the kink coordinate, the amplitude of the impurity mode and that of the kink internal mode. Our collective–coordinate approach can give a qualitative description of the resonance phenomena in the kink-impurity interactions. The detailed results is reported Ref.[9].

4 Conclusion

We have briefly reviewed our recent numerical and analytical results related to the kink-impurity interactions in the sine-Gordon and the ϕ^4 models. In particular, we have demonstrated that a kink can be totally reflected by an attractive impurity *if* its initial velocity is situated in some well-defined resonance windows. This effect can be explained by a mechanism of resonant energy exchange between the kink translational mode and the impurity mode (for the SG system), as well as the kink internal mode (for the ϕ^4 model).

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TABLE I Resonance windows of the kink-impurity interactions in the SG model

n	V_n predicted by Eq.(6)	Numerical $T_{12}(V_n)$	Resonance Windows
6	0.25498	42.5	(0.2548, 0.25505)
7	0.25842	49.2	(0.25825, 0.2585)
8	0.26064	56.2	(0.2605, 0.2607)
9	0.26215	62.8	(0.26205, 0.26222)
10	0.26323	69.5	(0.26315, 0.26327)
11	0.26403	75.9	(0.26395, 0.26408)
12	0.26463	82.8	(0.26461, 0.264635)
13	0.26510	89.6	(0.26510, 0.26512)
14	0.26547	97.1	(0.26546, 0.26547)
15	0.26577	103.3	(0.26577, 0.26579)
16	0.26602	109.9	(0.26600,0.26602)

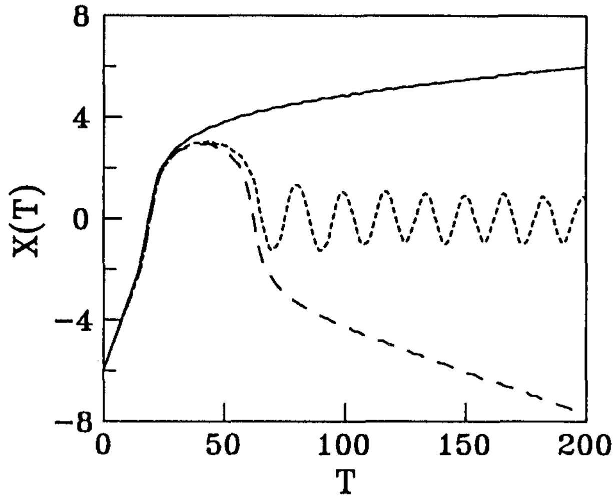

Fig.1 The kink coordinate $X(t)$ vs time for initial velocities V_i situated in three diffferent regions: the region of pass (solid line, $V_i = 0.268$), of capture (dotted line, $V_i = 0.257$), and of reflection (dashed line, $V_i = 0.255$).

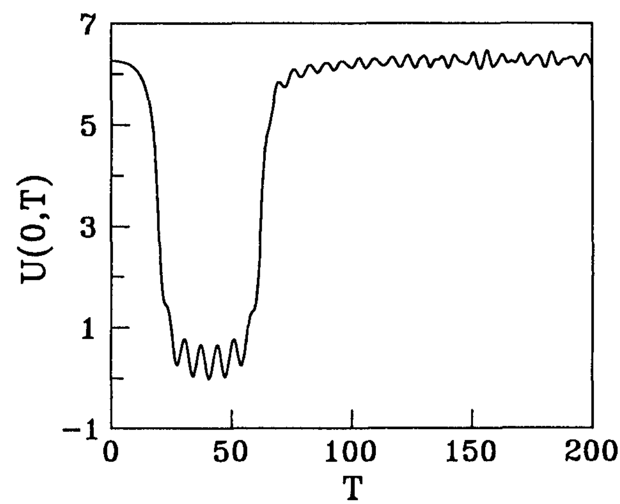

Fig.2 The impurity displacement u(0,t) vs time in the case of resonance ($V_i = 0.255$). Note that between the two interactions there are four small bumps which show the impurity mode oscillation, and after the second interaction the energy of the impurity mode is resonantly transferred back to the kink.