LOCALIZED SELF-SIMILAR STRUCTURES FOR A COUPLED NLS EQUATION: AN APPROXIMATE ANALYSIS

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ABSTRACT

We perform an approximate analysis of some particular self-similar solutions of the (2+1)-dimensional coupled nonlinear Schrödinger equation. These solutions are invariant under a point-symmetry subgroup of the model that involves the Schrödinger conformal symmetry. We use a variational approach to classify them and to determine their approximate structures.

Many physical systems deal with the propagation of two waves that interact nonlinearly. In this paper, we concentrate on the model

$$\begin{aligned} &\sqrt{-1}\,\psi_z^{(1)} + \psi_{xx}^{(1)} + \psi_{yy}^{(1)} + \eta\left[|\psi^{(1)}|^2 + (1+h)|\psi^{(2)}|^2\right]\psi^{(1)} = 0, \\ &\varepsilon\sqrt{-1}\,\psi_z^{(2)} + \psi_{xx}^{(2)} + \psi_{yy}^{(2)} + \eta\left[|\psi^{(2)}|^2 + (1+h)|\psi^{(1)}|^2\right]\psi^{(2)} = 0, \end{aligned} \tag{1}$$

where $\psi^{(i)}(x,y,z)$ are complex functions (throughout the text, $i = 1,2$), h is a real parameter, $\eta = \pm 1$ and $\varepsilon = \pm 1$. The above (2+1)-dimensional coupled nonlinear Schrödinger equation is of particular interest in the field of transverse effects in nonlinear optics (a review and extensive bibliography on the subject can be found in Ref. 1). In fact, it is the basic model that describes the time-independent copropagation[2] ($\varepsilon = 1$) and counterpropagation[3,4] ($\varepsilon = -1$) of two waves in a self-focusing ($\eta = 1$) or self-defocusing ($\eta = -1$) media.

Equation (1) is not completely integrable in the sense that it cannot be solved by inverse scattering techniques. However, it has the property of being invariant under a point-symmetry group for which the corresponding algebra is the direct sum between the 9-dimensional Schrödinger algebra sch(2) and a change of phase generator. The whole set of generators then includes one conformal symmetry C, one dilation D, one rotation J, two Galilean boosts K_x and K_y, three coordinate translations P_x, P_y and P_z and two constant change of phase $M^{(1)}$ and $M^{(2)}$. Thus, equation (1) is very appropriate for the application of the symmetry reduction method.

Here we concentrate on the specific 2-dimensional symmetry subalgebra

$$X_1 = J + a_1 M^{(1)} + a_2 M^{(2)} \ , \ X_2 = C + P_z + b_1 M^{(1)} + b_2 M^{(2)} \ , \tag{2}$$

where a_i and b_i are real parameters and

$$\begin{aligned} &J = y\,\partial_x - x\,\partial_y \ , \qquad M^{(i)} = -\varepsilon^{i-1}\sqrt{-1}\left(\psi^{(i)}\,\partial_{\psi^{(i)}} - c.c.\right), \qquad P_z = \partial_z \ , \\ &C = z\left(z\,\partial_z + x\,\partial_x + y\,\partial_y\right) - \frac{1}{4}\left(x^2 + y^2\right)\left(M^{(1)} - M^{(2)}\right) - \sqrt{-1}\,z\left(M^{(1)} + M^{(2)}\right) . \end{aligned} \tag{3}$$

Following the standard symmetry reduction procedure[5], we calculate the invariants of the subgroups by solving the equation

$$X_i\, Q\!\left(x, y, z, \psi^{(1)}, \psi^{(1)*}, \psi^{(2)}, \psi^{(2)*}\right) = 0 \ , \tag{4}$$

where Q is an auxiliary function. Since subalgebra (2) has generic orbits of codimension 1 in the space of independent variables and 4 in the space of dependent variables, the solution of Eq. (4) leads to five invariants ξ, $f^{(i)}(\xi)$ and $f^{(i)*}(\xi)$ satisfying

$$\begin{aligned} &\psi^{(i)} = \left(1 + z^2\right)^{-1/2} f^{(i)}(\xi)\exp\!\left[\sqrt{-1}\,\varepsilon^{i-1}\left(\frac{1}{4}\,z\,\xi^2 + a_i\,\theta - b_i \arctan z\right)\right], \\ &\xi^2 = r^2\left(1 + z^2\right)^{-1} , \end{aligned} \tag{5}$$

where $r^2 = x^2 + y^2$, $\theta = \arctan\,(y/x)$ and a_i are chosen as integers. Substituting relations (5) in Eq. (1) yields the reduced coupled ordinary differential equations

$$f^{(i)}_{\xi\xi} + \frac{1}{\xi} f^{(i)}_{\xi} + \left(b_i - \frac{1}{4}\,\xi^2 - \frac{a_i^2}{\xi^2}\right) f^{(i)} + \eta\left[\left|f^{(i)}\right|^2 + (1 + h)\left|f^{(3-i)}\right|^2\right] f^{(i)} = 0 \ . \tag{6}$$

The task of solving Eq. (6) exactly is quite difficult. Usually, one restricts the analysis to the determination of conditions under which the reduced equations are of Painlevé type, that is, when none of their solutions have movable critical points. The method is well adapted for single equations since a large classification of second and third order Painlevé type equations exists[6,7]. This is not so easy for coupled systems. In any case, one can show for the uncoupled case $h = -1$, that Eq. (6) does not even have the Painlevé property. For all these reasons, we restrict our analysis of Eq. (6) to the determination of approximate solutions. The method we use is based on a variational principle and is well described elsewhere[8,9]. In the following, we will only summarize the main steps of the calculations.

Equation (6) can be reformulated as a variational problem with the Lagrangian

$$V = V^{(1)} + V^{(2)} - \eta\,(1+h)\,|f^{(1)}|^2\,|f^{(2)}|^2 \quad , \tag{7}$$

where

$$V^{(i)} = \left|f^{(i)}_{\xi}\right|^2 - b_i\,|f^{(i)}|^2 + \frac{1}{4}\,\xi^2\,|f^{(i)}|^2 + \frac{a_i^2}{\xi^2}\,|f^{(i)}|^2 - \frac{1}{2}\,\eta\,|f^{(i)}|^4 . \tag{8}$$

Equation (6) is then derived from the cylindrical Euler equations

$$\frac{\partial}{\partial \xi}\left[\frac{\partial V}{\partial f^{(i)*}_{\xi}}\right] + \frac{1}{\xi}\,\frac{\partial V}{\partial f^{(i)*}_{\xi}} - \frac{\partial V}{\partial f^{(i)*}} = 0 \ . \tag{9}$$

The essence of the variational approach lies in the choice of the most appropriate trial functions that describe, as faithfully as possible, the exact solutions behaviour. On the other hand, since we want to obtain simple analytical results, we have to restrict our choice to a generic one. We found that a good compromise between simplicity and accuracy is given by the trial functions

$$f^{(i)} = A_i\,L^{(i)}\!\left(\frac{\xi}{W_i}\right) \quad , \tag{10}$$

where A_i and W_i are real parameters and

$$L_1^{(i)} = \exp\left[-\zeta_i^2\right] \qquad a_i = 0$$

$$L_2^{(i)} = \zeta_i \exp\left[-\zeta_i^2\right] \qquad a_i = \pm 1$$

$$L_3^{(i)} = \left(1 - 2\zeta_i^2\right) \exp\left[-\zeta_i^2\right] \qquad a_i = 0 \tag{11}$$

$$L_4^{(i)} = \zeta_i^2 \exp\left[-\zeta_i^2\right] \qquad a_i = \pm 2$$

The choice of real functions $L^{(i)}(\zeta_i)$, $\zeta_i = \xi / W_i$, is based on the form of the exact localized solutions of Eq. (6) in the linear limit $\eta = 0$, which are the well known Laguerre-Gauss modes. Relation (11) gives the expression of the first four modes.

Substituting the ansatz (10) in the Lagrangian (7) and integrating the ξ-variable from 0 to infinity yield a reduced Lagrangian denoted $\langle V\rangle$ that is independent of ξ. Thus, solving the corresponding reduced Euler equations

$$\frac{\partial}{\partial \xi}\left[\frac{\partial \langle V\rangle}{\partial y_{i\xi}}\right] + \frac{1}{\xi}\frac{\partial \langle V\rangle}{\partial y_{i\xi}} - \frac{\partial \langle V\rangle}{\partial y_i} = \frac{\partial \langle V\rangle}{\partial y_i} = 0 \ , \quad y_i \equiv A_i \text{ and } W_i \ , \tag{12}$$

lead to the four relations

$$E_i + \frac{(1+h)}{\alpha_{5i}\, W_{3-i}^2}\left[2\, \alpha_6 - W_i \frac{\mathrm{d}\alpha_6}{\mathrm{d}W_i}\right] E_{3-i}$$

$$= \frac{\eta}{\alpha_{5i}}\left[2\, \alpha_{3i} + 2\, a_i^2\, \alpha_{4i} - \frac{1}{2}\, W_i^4\, \alpha_{2i}\right] , \tag{13}$$

and

$$b_i = \frac{1}{\alpha_{1i}\, W_i^2}\left[\alpha_{3i} + a_i^2\, \alpha_{4i} + \frac{1}{4}\, W_i^4 \alpha_{2i} - \eta\, E_i\, \alpha_{5i} - \eta\,(1+h)\frac{E_{3-i}}{W_{3-i}}\, \alpha_6\right] . \tag{14}$$

The constant $E_i = A_i^2\, W_i^2$ are proportional to the energy Σ_i in each wave through

$$\Sigma_i = \int_0^{2\pi} \int_0^{\infty} |\psi^{(i)}|^2 \, r \, dr \, d\theta = 2\pi \int_0^{\infty} |U^{(i)}|^2 \, \xi \, d\xi = 2\pi \, \alpha_{1i} \, E_i \ . \tag{15}$$

The parameters α_{ki} (k=1,...,5) and α_6 are given by

$$\alpha_{1i} = \int_0^{\infty} [L^{(i)}]^2 \, \zeta_i \, d\zeta_i \ , \qquad \alpha_{2i} = \int_0^{\infty} [L^{(i)}]^2 \, \zeta_i^3 \, d\zeta_i \ ,$$

$$\alpha_{3i} = \int_0^{\infty} \left[\frac{dL^{(i)}}{d\zeta_i}\right]^2 \zeta_i \, d\zeta_i \ , \qquad \alpha_{4i} = \int_0^{\infty} [L^{(i)}]^2 \, \zeta_i^{-1} \, d\zeta_i \ , \tag{16}$$

$$\alpha_{5i} = \int_0^{\infty} [L^{(i)}]^4 \, \zeta_i \, d\zeta_i \ , \qquad \alpha_6 = \int_0^{\infty} \left[L^{(1)}\left(\frac{\xi}{W_1}\right)\right]^2 \left[L^{(2)}\left(\frac{\xi}{W_2}\right)\right]^2 \xi \, d\xi$$

and can be evaluated analytically.

Relations (13) and (14) are parametric equations that give E_i and b_i as function of the widths W_1 and W_2 of the approximate localized solutions (10). We solved them for various values of W_1 and W_2.

For instance, figure 1 shows the normalized energy $S = (2 + h)\ \Sigma_i$ in each wave as function of b for two identical beams, i.e. $A_1 = A_2$, $W_1 = W_2$ and $b_1 = b_2 = b$. The curve numbers refer to the mode number in Eq. (11). The "+" and "–" signs refer to a self-focusing medium ($\eta = 1$) and a self-defocusing medium ($\eta = -1$) respectively. The points b = 1,2,3... and $S = 0$ correspond to the linear limit $\eta \to 0$. The case described by curve 1^+ was studied in Ref. 10. The energy values at $b \to -\infty$ are 4π, 16π, 24π, 32π and correspond approximately (within 5%) to the energy of the first self-trapping solutions of Eq. (1) (no z-dependence in the amplitudes)[11,12]. The fundamental self-trapping solution is known to be unstable and to eventually collapse under a self-focusing process (a review and extensive bibliography can be found in Ref. 13). We suspect a similar behaviour for the higher-order self-trapping solutions.

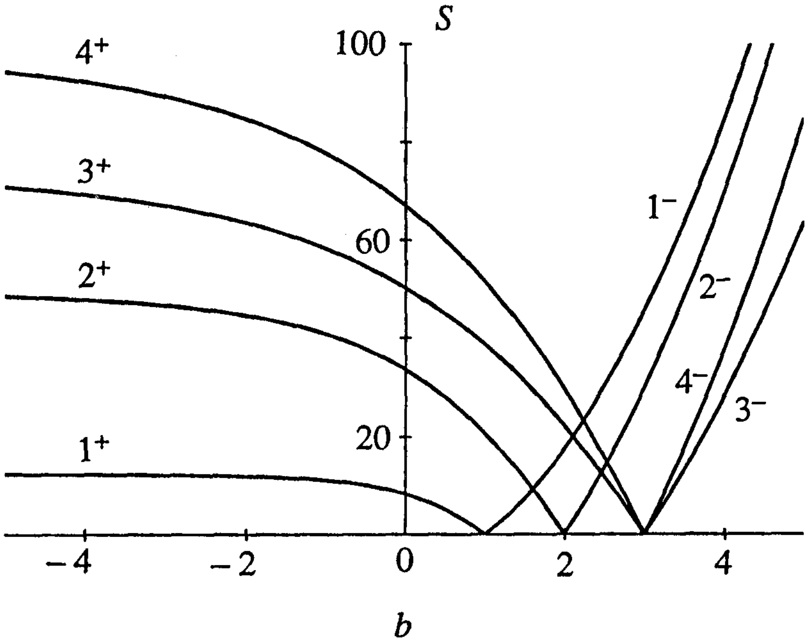

Figure 1: Normalized energy $S = (2 + h)\Sigma_i$ in each wave for two identical modes. Numbers refer to $L_1,\dots, L_4$ and "+" and "−" signs stand for $\eta = 1$ and $\eta = -1$.

In addition to the above identical localized self-similar solutions, there is a large set of solutions where $W_1 \neq W_2$. For example, solid curves on figures 2 show the energies Σ_1 and Σ_2 as function of b_1 and b_2 for two beams in the fundamental mode with $h = 1$ and $\eta = 1$. For comparison, coarse dashed curve give the energy for the case $W_1 = W_2$ and fine dashed curve gives the energy for $h = -1$ (no nonlinear coupling). In figure 2, we have chosen $W_1 = 0.928$ which leads to $0.695 \leq W_2 \leq 1.209$ and provides the possible self-similar solutions centered around $b_1 = b_2 = -2$. The points on the solid curves correspond to $W_2 = 1.209$.

The most significative result of our analysis is the possible coexistence of self-similar solutions having different mode profiles. Our calculations show that this "nonlinear superposition" seems to be always possible within a certain parameter range. Specific examples of a such behaviour will be reported elsewhere.

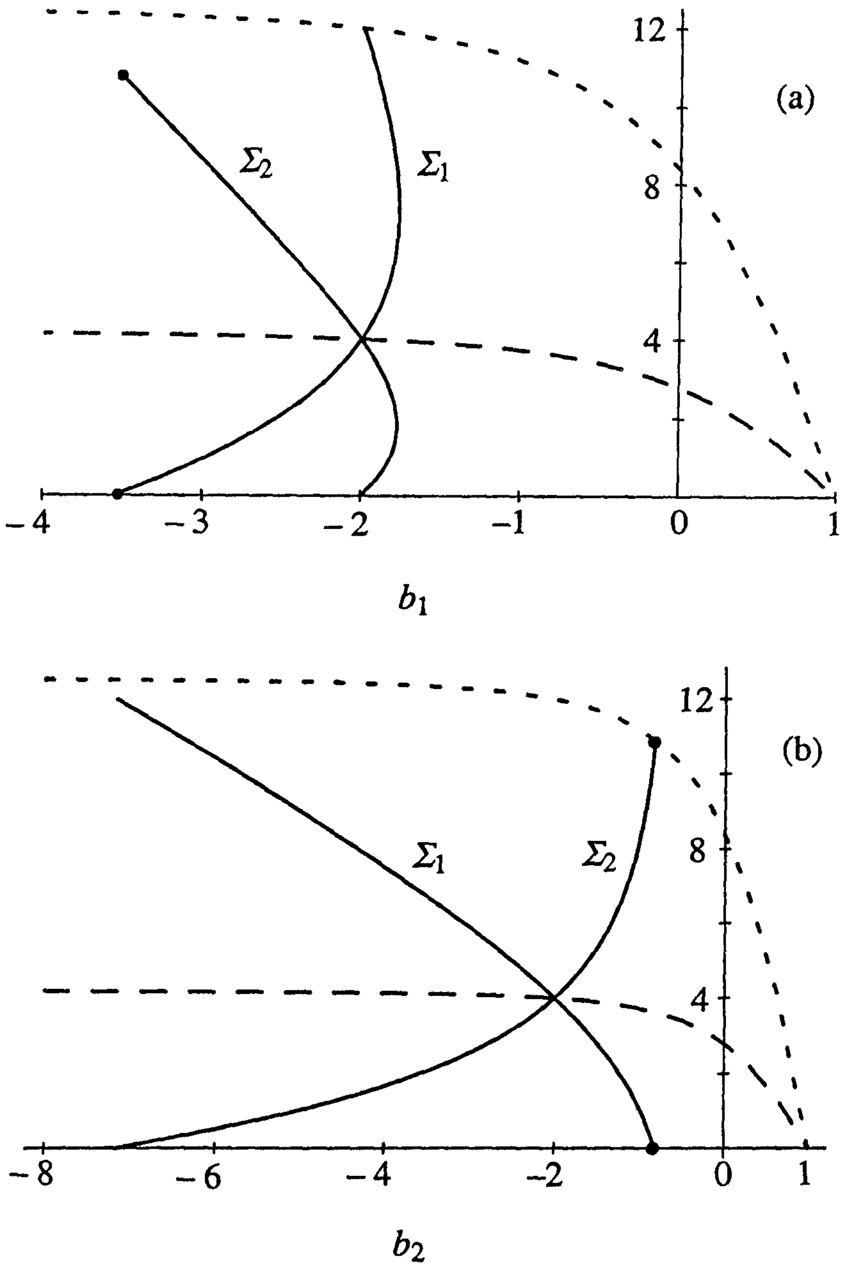

Figure 2: Energies Σ_1 and Σ_2 as function of b_1 (a) and b_2 (b) for two waves in the fundamental mode with $\eta = 1$, $h = 1$, $W_1 = 0.928$ and $0.695 \leq W_2 \leq 1.209$.

In conclusion, one can say that the results of this study are indicative of a large set of self-similar nonlinear coherent structures predicted by the model (1). The coexistence of these self-similar coupled waves can be of interest in the study of various nonlinear systems and in particular for the counterpropagation of two Gaussian optical beams in a Kerr media[4]. In that case, the question of temporal stability of such modes has to be addressed. We plan to go back to that issue in a near future.

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