Introduction

Flow fields with vorticity concentrated in a distinct region have been observed in many fluid dynamic, astrophysical and geophysical problems. Extensive collections of photographs of such flow fields and references to experimental and theoretical investigations can be found in two recent books by van Dyke (1982) and Lugt (1983). Many of those flow fields are very complex, involving body forces, energy sources, compressibility and real gas effects. We shall mention only a few classical examples. When we first studied fluid dynamics we saw photographs of vortical flows in the textbook of Prandtl and Tietjens (1957). They showed the formation and shedding of eddies or vortices in the wake of circular cylinders, blunt bodies and airfoils, the formation of von Kármán vortex streets, the effect of rotation of a circular cylinder on the eddies shedding off its surface, the trapping of vortices in a concave corner, the roll up of an interface of a separated flow into eddies, and the organized eddies in a turbulent flow. Those phenomena have been re-examined theoretically, experimentally and numerically in recent years, see van Dyke (1982) and Lugt (1983), and references therein.

Vortical flow can frequently be observed or experienced in our daily life. We often see smoke rings generated from orifices, a pair of vortex filaments trailing a commercial airplane and encounter the gusty flow around the corner of a tall building. Vortical flows have also been of great practical interest to aeronautical engineers. They have analyzed, for example, the trailing wakes of aircrafts during landing and take off for the sake of flight safety and studied the generation of leading edge vortices and their stability for high performance configurations. Several special conferences or workshops dealing with these two problems were held in recent years, see for example, Olsen et al (1971), Wendt (1982), Young (1983) and Staufenbiel (1985).

A comprehensive theoretical study of inviscid vortical flows can be found in Chapters III and VI of the treatise by Lamb (1932) and the effects of viscosity in his Chapter XI. Classical theories, or results which can be found in those three chapters, will often be quoted here without reference. For the method of matched asymptotics and that of multiple scales our standing references are van Dyke (1975), Schneider (1978) and Kevorkian and Cole (1981).

Now we shall identify the class of problems which we will focus on and define the terminology and symbols to be used in here.

We consider the flow field to be incompressible and viscous, unless stated otherwise. The velocity field $\mathbf{v}(t, \mathbf{x})$ is composed of the velocity induced by the vorticity field $\Omega(t, \mathbf{x})$ and of a background potential flow. Note that all vector quantities will be denoted by boldface Roman letters except for the vorticity vector whose symbol is Ω. All unit vectors will be denoted by symbols with a hat accent. For example, we use $\mathbf{x}$ to denote the position vector with Cartesian coordinates x_i, $i = 1,2,3$, $\mathbf{v}$ the velocity with components v_i, Ω the vorticity with components ω_i and $\hat{\imath}$, $i = 1,2,3$ the unit vectors along the Cartesian coordinate axes.

In **Chapter 1** we study general properties of a viscous vortical flow with a typical length scale ℓ and velocity scale U. Unless specified otherwise, the time is scaled by ℓ/U, vorticity and velocity gradients by U/ℓ and the kinematic viscosity ν by $U\ell$. From the last scaling, we define the Reynolds number of the flow field by

$$R_e = U\ell/\nu \ . \tag{1}$$

For an incompressible fluid we use its density ρ as the density scale, i.e., we set $\rho = 1$. The typical mass scale is therefore $\rho\,\ell^3$. We assume that the initial vorticity distribution is centered near the origin of a suitable reference frame and decays rapidly at large distance, so that at any finite time $t \geq 0$ and for some sufficiently large N we have

$$|\Omega|\ell/U = o\big([\ell/|\mathbf{x}|]^N\big) \quad \text{as} \quad |\mathbf{x}|/\ell \to \infty \ . \tag{2}$$

Condition (2) is certainly fulfilled if the vorticity distribution is of bounded support or decays exponentially. The latter is a realistic condition for a vortical flow and implies that condition (2) holds for all N. Note that we do not distinguish between the order of magnitude of ℓ , the effective size L of the vorticity distribution and a typical vorticity decay length ℓ_d at this point.

In **Sec.1.1** we state the governing equations for the vortical flow induced by any initial vorticity field in free space. In doing so, we assume that there is no background flow and that the distance from the vorticity field to a boundary or body is much larger than ℓ. Some general results for such flow fields are then presented in **Sec. 1.2**. We recount, in particular, the consistency and time invariance conditions on the moments of vorticity due to Truesdell (1951, 1954) and Moreau (1948, 1949), since these conditions are usually not reported in textbooks on Fluid Dynamics. Also, we provide the corresponding results for axi-symmetric and two-dimensional flows. The two-dimensional relations will be employed in **Sec. 2.2** to identify the physical meaning of an optimum similarity solution for a viscous vortex, called an optimum Lamb vortex, and in **Sec.3.2** to generate rules for merging of viscous vortices into one single approximating vortex.

The conditions of Truesdell and Moreau were employed by Ting (1983) to derive an asymptotic description of the far field behavior of a vortical flow. In the far field the vector potential of velocity, $\mathbf{A}$, is represented by a series in inverse powers of $|\mathbf{x}|$,

$$\mathbf{A}(t,\mathbf{x}) = \sum_{n=0}^{m} A^{(n)}(t,\mathbf{x}) + O(|\mathbf{x}|^{-m-2}) \ , \tag{3}$$

where $\mathbf{A}^{(n)}$ is proportional to $|\mathbf{x}|^{-n-1}$. As we shall see later, $\mathbf{A}^{(n)}$ is defined because of (2) when $m+3 < N$. Using Truesdell's consistency conditions, we show that the series for the vector potential begins with $n = 1$ instead of $n = 0$ and that for each $n \geq 1$, $\mathbf{A}^{(n)}$ is defined by a linear combination of $n(n+2)$ instead of all the $3(n+2)(n+1)/2$ nth moments. Using Moreau's conditions, we show, in addition, that the first term $\mathbf{A}^{(1)}$ represents three time invariant doublets and that the second term $\mathbf{A}^{(2)}$ represents eight quadrupoles, three of which are time invariant. This far field behavior will be employed in **Sec.3.1** to provide higher order approximate boundary conditions on a finite computational domain for the numerical simulation of a viscous vortical flow.

In **Sec.1.3** we show that for $n > 2$ the nth term $\mathbf{A}^{(n)}$ depends on only $4n$ linear combinations of the nth moments of vorticity and that only $2n+1$ of them contribute to the far field velocity of order $O(|\mathbf{x}|^{-n-2})$. The latter can be expressed as the gradient of a scalar potential, $\nabla\phi^{(n)}$, for which we provide an explicit expression in terms of $2n+1$ linear combinations of the nth moments of vorticity, see Klein and Ting (1990).

In **Sec.1.4** we identify the solution for an incompressible vortical flow as the zeroth order solution of a compressible flow at a low Mach number, $M \ll 1$. The leading unsteady far field velocity of the incompressible vortical flow is matched with the near field solution of acoustic quadrupoles. The far field contribution of the global compressibility effect of the next order solution of the vortical flow, $O(M^2)$, is matched to the near field solution of an acoustic monopole. This global dilatation effect is related directly to the rate of energy dissipation of the incompressible vortical flow. Thus we obtain the formulae for the leading acoustic field induced by the vortical flow (Crow 1970, Möhring 1978, Obermeier 1985 and Ting and Miksis 1990). These results are applied to turbulent flows and their applications to the mean flow are discussed. The formulae are also employed in **Sec.2.3** to compute the sound generation due to the motion of slender vortex filaments and due to the evolution of their core structures.

In **Chapter 2** we study a special class of vortical flows which are characterized by multiple length scales. This includes, in particular, flows induced by a finite number of vortex filaments submerged in a background potential flow with length scale ℓ and velocity scale U. We call it a vortex filament or in short a filament or a vortex, when the bulk of vorticity is concentrated in a slender tube-like region. We introduce a reference line, called the center line $\mathcal{C}$, which essentially is the line of maximum vorticity in cross sections normal to the vortex tube. This curve is described by a vector function $\mathbf{X}(t,s)$ of time t and of a tangential parameter s. In general, the effective core size of the filament, or the size of the cross section of the tube, δ, is also a function of s and t. By slenderness, we mean that the core size is much smaller than a typical radius of curvature of $\mathcal{C}$ which is assumed to be on the order of the length scale ℓ. Thus, the flow field has two distinct length scales given by a typical core size δ^* and an overall length scale ℓ of either the outer flow or the filament geometry. Their ratio defines a small parameter,

$$\delta^*/\ell = \epsilon \ll 1 \ . \tag{4}$$

Now we can specify the order of magnitude of other dimensionless quantities, namely the velocities scaled by U, the lengths scaled by ℓ, the time t scaled by ℓ/U etc., in terms of the small parameter ϵ as it approaches zero. For example, the statements that the radius of curvature R of $\mathcal{C}$ is of the order of ℓ and that it is one order larger than its effective core size are equivalent to

$$R(t,s)/\ell = O(1) \tag{5}$$

and

$$R(t,s)/\ell = O(\epsilon^{-1}\delta(t,s)/\ell)) \ . \tag{6}$$

Based on this, we replace the qualitative characterization of a vortex filament, saying that the bulk of vorticity is concentrated in a slender tube-like region of effective core size δ, by the following two properties. We require that 1) the effective core size δ remains on the order of δ^* and 2) the vorticity at a point $\mathbf{x}$ decays rapidly as the distance r between $\mathbf{x}$ and $\mathcal{C}$ becomes large relative to δ^*. The second condition means that

$$|\Omega(t,\mathbf{x})|\ell/U = o([\delta^*/r]^N) \qquad \text{as} \qquad r/\delta^* \to \infty \ , \tag{7}$$

for some sufficiently large N and it implies that the vorticity decay length ℓ_d is on the order of the core size,

$$\ell_d = O(\delta^*) \quad \text{or} \quad \ell_d/\ell = O(\epsilon) \ll 1 \ . \tag{8}$$

This filament structure, described by (4-8), will survive the effect of viscous diffusion at least for a finite time, only if the Reynolds number based on the length ℓ is large, i. e., $R_e \gg 1$. The order of magnitude of $1/Re$ relative to ϵ will be specified later in (13). Under these circumstances, the flow field far away from $\mathcal{C}$ is basically irrotational and we can define the strength of the vortex filament by the circulation Γ along a circuit around $\mathcal{C}$ with the minimum distance between $\mathcal{C}$ and the circuit much larger than δ^*. We assume that the strength Γ is finite in the sense that ,

$$\Gamma/(U\ell) = O(1) \ . \tag{9}$$

From (4) and (9), we see that the velocity and the vorticity (or the velocity gradients) near $\mathcal{C}$ are large, of the order of ϵ^{-1} and ϵ^{-2} respectively, which means

$$|\mathbf{v}|/U = O(\Gamma/[U\delta^*]) = O(\epsilon^{-1}) \tag{10a}$$

and

$$|\Omega|\ell/U = O(\Gamma\ell/[\delta^{*2}U]) = O(\epsilon^{-2}) \qquad \text{for} \qquad r/\delta^* = O(1) \ . \tag{10b}$$

In contrast, the velocity at a distance far away from $\mathcal{C}$ (relative to δ^*) is of order one, so that

$$|\mathbf{v}|/U = O(1) \qquad \text{for} \qquad r/\ell = O(1) \ , \tag{11}$$

while the vorticity vanishes in the sense of (7).

In general, there are other typical lengths in the flow field, e. g., the length of a filament, the distance between two filaments, the distance from a filament to a rigid surface, etc. We consider all those typical lengths to be $O(\ell)$ unless stated otherwise. In case that the flow field is composed of vortex filament(s) without a background flow, we can use (5) and (9) to define the length scale ℓ by a typical radius of curvature of $\mathcal{C}$ and the velocity scale U by $|\Gamma|/\ell$.

In the classical inviscid theory, the effective size δ is reduced to zero and the filament becomes a vortex line along C. The velocity $\mathbf{Q}$ at a point $\mathbf{x}$ induced by the vortex line C is defined by the Biot-Savart integral along C,

$$\mathbf{Q}(t,\mathbf{x}) = \frac{\Gamma}{4\pi} \int_C \frac{\mathbf{X}' - \mathbf{x}}{|\mathbf{X}' - \mathbf{x}|^3} \times d\mathbf{X}' \,. \tag{12}$$

From this equation it is evident that the inviscid theory suffers from the two serious defects that 1) the fluid velocity on the vortex line, $\mathbf{x} = \mathbf{X}$, becomes infinite and that 2) the velocity of the vortex line itself is undefined.

Often such flow fields are modelled by an irrotational flow outside a slender tube-like surface, and a rotational flow with a prescribed vorticity distribution inside of it. In that situation the velocity defined by the Biot-Savart formula will be finite everywhere and the velocity of the center line of the vortex tube will be defined and depend on the vorticity distribution. It is important to note, however, that the vorticity distribution inside the tube, no matter how slender it is, cannot be arbitrarily assigned. The reason is that the rotational flow has to fulfill the continuity and Euler equations inside the tube and on the interface its velocity and pressure have to match with those of the potential flow outside in order to satisfy the kinematic interface conditions. Whether we consider the flow in the vortical core to be viscous or not, in either case we may carry out a matched asymptotic analysis using the slenderness ratio ϵ as the small expansion parameter to derive an approximate description of the core flow. Since the velocity gradients in the the vortical core are large of order ϵ^{-2} according to (10b), we opt to retain viscous effects in the inner solution by assuming the distinguished limit

$$Re = \frac{U\ell}{\nu} = O(\epsilon^{-2}) \qquad \text{or} \qquad \bar{\nu} = \frac{1}{\epsilon^2 Re} = \frac{\nu}{U\ell\epsilon^2} = O(1), \tag{13}$$

while the Reynolds number based on the typical core size is

$$Re^* = \frac{U\delta^*}{\nu} = O(\epsilon^{-1}). \tag{14}$$

Thus, the asymptotic solution of the Navier-tokes equations will be valid for any fixed value of $\bar{\nu}$ and will be free of an artificial interface confining the highly rotational flow.

In **Sec.2.1** we describe in detail the deficiencies of the classical inviscid theory. It is well known that in the inviscid theory of two-dimensional flow, a point vortex is assumed to move with the local velocity of the background flow (without the vortex). For a real fluid, a point vortex will diffuse immediately into a vortex with a Gaussian vorticity distribution, called a Lamb vortex. We will show that the inviscid theory is a good approximation only on the normal time scale, ℓ/U, and may not be applicable on a much smaller time scale. To bring home this point, we first reproduce the solution of the oscillatory motion of a small spinning disc in a uniform stream, as described by Milne-Thompson (1973). The period and the amplitude of the oscillation decrease towards zero as the disc radius a, scaled by U/Γ, vanishes, where Γ denotes the circulation around the disc. We then perform a perturbation analysis for the oscillatory motion of a small rotating blob of fluid, called a Rankine vortex, in an inviscid uniform stream. We identify the correspondence between the motions of a spinning disc and the center of a Rankine vortex and show the similarity of the effect of their size on the amplitude and period of their trajectories. In both cases, the respective centers of disc and vortex are drifting along with the uniform stream on the normal time scale. These two simple examples illustrate the necessity for, and also the physical meaning of, the two-time matched asymptotic analysis of a viscous vortex to be presented in **Sec.2.2**. In the last subsection **2.1.2** we explain the defects of the inviscid theory for a curved vortex line in space. We derive the singular behavior of the flow field near the vortex line defined by the Biot-Savart formula (12). These singular terms will be identified with the far field behavior of an inner solution obtained through a matched asymptotic analysis in **Sec.2.2** and **Sec.2.3** for two and three-dimensional problems, respectively.

In **Sec.2.2** we present the matched asymptotic solutions of the N-S equations of Ting and Tung (1965) for a two-dimensional vortex submerged in a background potential flow. By comparing the one-time solution, in the normal time t, and the two-time solution, in t and a short time variable τ, we show that the average of the two-time solution over the short time is equivalent to the one-time solution to leading order and to first order. In particular, the trajectory of the vortex center defined by the two-time solution and its average path differ from the trajectory defined by the one-time solution by merely $O(\epsilon^2)$. The latter is independent of the core structure and differs from the classical inviscid result by the same order. The leading order core structure, which is axi-symmetric in the inner variables, depends only on the normal time and its solution is defined in terms of the initial data. It is not necessarily a similarity profile. The corresponding optimum similarity solution, or the solution for an optimum Lamb vortex, is defined by the condition that it is the best one term truncation of a series solution in terms of powers of $1/t$ for $t \to \infty$. By applying the consistency conditions, to be discussed in **Sec.1.2**, the optimum Lamb vortex turns out to be the one which not only has the same strength and center but also the same second polar moment as that of the non-similar solution for $t \geq 0$.

Similar to the two-dimensional solutions, the correspondence between the two-time and one-time asymptotic solutions of a circular vortex ring submerged in a background axi-symmetric potential flow were obtained by Tung and Ting (1966) and Ting (1971). There, however, the velocity of the center line of the vortex ring does depend on the core structure. Similar correspondence for a general three-dimensional problem is to be expected albeit the two-time analysis will be extremely tedious and has not yet been attempted. In **Sec.2.3** we present the one-time asymptotic analysis of Callegari and Ting (1978) for a curved slender vortex filament in space. We emphasize the essential differences between the three and two-dimensional problems, namely, 1) the dependence of the velocity of the center line on the core structure, 2) the compatibility conditions between large axial and circumferential velocity components in the vortical core and 3) the evolution of the core structure due to both stretching of the filament and viscous diffusion. The general results may then be applied to the case studied by Ting (1971), where the axial velocity in the core is only order one, and to the axi-symmetric case of Tung and Ting (1966).

It is well known that the leading order asymptotic solution in terms of a small expansion parameter $\epsilon \ll 1$, is often accurate enough for practical purposes, even when ϵ is not very small. In order to establish such a practical region of validity for our asymptotic description of a vortex filament, the solution was employed to study the interaction of vortex filaments by Liu , Tavantzis and Ting (1986). Numerical examples were presented for filaments which were initially slender and far apart from each other relative to a typical size of their vortical cores. The motion and deformation of the filaments were compared with a finite difference solution of the N-S equations with the same initial data. It was found that the asymptotic solution remains in good agreement with the finite difference solution even when the ratio between the typical core size δ and the reference length ℓ is almost 1/2. Here the reference length is either the minimum half distance between two filaments or the minimum radius of curvature of the filament centerline. The numerical comparison, which will be reported in the last section of **Chapter 2**, suggests that the practical region of validity for the asymptotic solution is $\epsilon < 1/2$. The methods used for the numerical solution of the N-S equations, suitable for the study of viscous vortical flows with rapidly decaying vorticity, will be described in **Chapter 3**.

In **Chapter 3**, we present efficient numerical schemes for the simulation of viscous vortical flows, i.e., for solving the initial value problem formulated in **Sec.1.2.** In particular, we study the merging or intersection of vortex filaments. We assume that the initial data are either specified or provided by the matched asymptotic solutions when these solutions are still valid, i.e., when ϵ is within its region of validity, say $\epsilon \leq 1/4$. A numerical scheme is considered to be efficient when the finite computational domain $\mathcal{D}$ is as small as possible under the side condition that the error in the boundary conditions imposed along $\partial\mathcal{D}$ is of the same order as that of the finite difference is within $\mathcal{D}$. Thus, it is essential to specify approximate boundary conditions along $\partial\mathcal{D}$ which are consistent with the unbounded domain problem and to assess their accuracy.

In **Sec. 3.1**, we explain the basic concepts in the choice of a computational domain, $\mathcal{D}$, and the formulation of boundary conditions tailored for a specified type of problem. We explain in **Sec. 3.1.1** our classification of different types of merging problems, such as global merging of filament(s) or local merging of small segments of filament(s).

In **Sec. 3.1.2** we consider global merging problems for which the decay length ℓ_d can be of the order of the size, $L = O(\ell)$, of the initial vorticity distribution. For example, global merging is taking place when the core radius δ of a filament becomes comparable to its radius of curvature. In this case the size of the computational domain $\mathcal{D}$ has to be much larger than L, say $4L$, and then the volume of $\mathcal{D}$ is about 64 times the volume of the effective vorticity distribution. Approximate boundary data along $\partial\mathcal{D}$ are obtained from the far field behavior of the vortical field described in **Sec.1.2**.

In **Sec. 3.1.3** we consider global merging problems for which the decay length ℓ_d is much smaller than the size ℓ. For example, in a nearly head on collision of two slender filaments, the core size becomes comparable to the distance between the filaments but is still much smaller than their radii of curvature. In this case, the size of the computational domain may be only several core sizes larger than L and the approximate boundary data are obtained by a suitable adaptation of the far field expansion of the vortical field.

In **Sec.3.1.4** we consider the case of local merging, which does not occur in two-dimensional or axi-symmetric problems. For example, when two slender filaments intersect each other at a finite number of non-overlapping local regions, then small segments of the filaments merge with each other in each of these regions. In this case, the computational domain for a local region has to cover only the merged segments and thus may be much smaller than the overall size L of a filament. The numerical solution then must be supplemented by the asymptotic solution for the motion and diffusion of the filament(s) outside the computational domains.

In **Sec.3.2** and **3.3** we study the merging of vortices in two and three dimensions. The numerical schemes formulated in **Sec. 3.1** are employed to study the merging of two-dimensional vortices, coaxial vortex rings and vortex filaments in **Sec. 3.2.1, 3.3.1** and **3.3.2** respectively.

Recall that integral invariants for the moments of a two-dimensional vorticity distribution are discussed in **Sec.1.2.4** and that they are employed to identify the physical meaning of the optimum similarity solution for a viscous vortex. Using those results, we present in **Sec.3.2.2**, rules for merging of two-dimensional viscous vortices into a single one with an optimum similarity core structure, i. e., an optimum Lamb vortex (Ting (1986)). In continuation of these physical concepts, an approximate solution to an initial value problem for a two-dimensional viscous vortical flow was formulated by Ting (1986) and Ting and Liu (1986). The vorticity field is represented by a finite number of Lamb vortices which may overlap each other. The velocities of the vortex centers are defined by the condition that the N-S equations are satisfied approximately such that the integral of the square of the error is a minimum. For a vortex which is not merging or overlapping with the others, the velocity of the vortex center defined by the minimum principle agrees with the single time asymptotic solution and hence with the classical inviscid theory. Numerical examples for the merging of vortices based on the minimum principle are presented and are shown to be in good agreement with the corresponding finite difference solutions.

In the last Chapter, we mention additional topics of vortical flows to demonstrate that there are many interesting and challenging problems which can be analyzed by modifications or extensions of the method of matched asymptotic expansions and/or the numerical methods discussed in **Chapters 2** and **3**. The topics are assembled according to the types of modifications required into three categories : 1) vortex filaments with multiple axial length scales, 2) vortices in a background rotational flow with order one vorticity and 3) the interaction of a filament with a moving surface. They are described in **Sec. 4.1, 4.2** and **4.3** respectively.