1. Introduction

Let \( \{\varphi_n(x)\} \) \( (n = 0, 1, 2, \ldots) \) be an orthonormal system of functions on the interval \([a, b]\). We consider the orthogonal series

\[
\sum_{n=0}^{\infty} c_n \varphi_n(x), \quad \sum_{n=0}^{\infty} c_n^2 < \infty
\]

and the function \( f(x) \in L_2(a, b) \) defined by the de la Vallee-Poussin theorem for this series. We write \( T_n(x, f) \) for the de la Vallee-Poussin means of (1), i.e.,

\[
T_n(x, f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(x, f),
\]

where

\[
S_k(x, f) = \sum_{i=0}^{k} c_i \varphi_i(x).
\]

Tandori [1] and Leindler [2] have proved the following theorem concerning the rate of approximation of \( f(x) \) almost everywhere in \([a, b]\) by de la Vallee-Poussin sums.

**THEOREM A (K. Tandori).** Let the condition

\[
\sum_{n=0}^{\infty} c_n^2 w^n(n) < \infty,
\]

where \( w(n) \) is a positive nondecreasing sequence, imply that the series (1) is \((C, 1)\) summable almost everywhere in \([a, b]\) to a function \( L_2(a, b) \). Then if the positive convex sequence \([1/\mu (n)]\) \( \downarrow 0 \), and, moreover,

\[
\mu(n) = o\left( \frac{1}{\mu(n)} \right), \quad \sum_{n=0}^{\infty} c_n^2 w^n(n) \mu^n(n) < \infty,
\]

we have \( f(x) - T_n(x, f) = o\left( 1/\mu(n) \right) \) almost everywhere in \([a, b]\) and \( \mu(n) \left| f(x) - \tau_n(x, f) \right| \leq F(x) \), \( F(x) \in L_2(a, b) \).

**THEOREM B. (L. Leindler).** Let \( \mu(n) \) be a positive nondecreasing sequence, such that

\[
\sum_{n=0}^{\infty} \mu^2(2^n) = O(\mu^2(2^n)).
\]

If \( \sum_{n=0}^{\infty} c_n^2 \mu^2(n) < \infty \), then

\[
f(x) - \tau_n(x, f) = o\left( \frac{1}{\mu(n)} \right)
\]

almost everywhere in \([a, b]\).

A comparison of the above theorems shows that if the elements of the sequence \( \mu(n) \) are made to increase more rapidly, then the order of approximation \( o(1/\mu(n)) \) is ensured by "weaker" conditions. In this

connection the question arises of the existence of a sequence \( \lambda(n) \) such that the order of approximation \( o(1/\mu(n)) \) can follow from the convergence of the series

\[
\sum_{n=0}^{\infty} c_n^t x^n(n) \quad \text{and} \quad \frac{1}{\mu(n)} = o\left(\frac{1}{\lambda(n)}\right).
\]

In the present note we give a positive answer to this question and show that, under certain conditions, the ratio \( \lambda(n)/\mu(n) \) tends to zero arbitrarily rapidly. Hence, Theorem B can be strengthened.

The position is completely different if the condition \( \sum_{n=0}^{\infty} c_n^t x^n(n) < \infty \) is replaced by the equivalent condition

\[
\sum_{n=0}^{\infty} E_n^t(f) v^n(n) < \infty,
\]

where

\[
E_n(f) = \inf_{x} \sum_{k=0}^{n} a_k \psi_k(x) \left| f(x) - S_n(x, f) \right|^2 dx = \left( \sum_{k=n+1}^{\infty} \psi_k^2 \right)^{1/2}.
\]

For a definite order of growth of the sequence \( v(n) \), the ratio \( v(n)/\mu(n) \) is a small quantity of the first order with respect to \( 1/n \) and this is in a sense a final result.

2. The Fundamental Lemma

We first prove an auxiliary result concerning numerical sequences.

**Lemma.** Let the positive sequence \( l(n) \) satisfy a relation

\[
\sum_{k=0}^{n-1} l(k) = l(n) \left[ a_1(n+1) + a_2 n + a_3 + \beta(n) \right],
\]

where

\[
\alpha(n) \uparrow \infty, \quad \frac{\alpha(n)}{n+1} \downarrow 0, \quad \beta(n) \to 0 \quad \text{as} \quad n \to \infty;
\]

\[
a_1 > 0; \quad a_2 > 0, \quad \text{if} \quad a_1 = 0;
\]

\[
a_2 > 0 \quad \text{if} \quad a_1 = a_2 = 0; \quad \beta(n) \geq 0 \quad \text{if} \quad a_1 = a_2 = a_3 = 0. \quad \text{There is a number} \quad N \quad \text{such that}, \quad \text{for all} \quad n \geq N, \quad \text{we have}^*
\]

\[
\sum_{k=0}^{n-1} \frac{1}{l(k)} \sim \begin{cases} \frac{n+1}{l(n)}, & a_1 > 0, \\ \frac{\alpha(n)}{l(n)}, & a_1 = 0, a_2 > 0, \\ \frac{1}{l(n)}, & a_1 = a_2 = 0, a_3 > 0. \end{cases}
\]

**Proof.** Writing

\[
Q(m) = a_2 \alpha(m) + a_3 + \beta(m),
\]

we find that (2) yields

\[
l(m) = l(m+1)[a_1(m+2) + Q(m+1)] - l(m)[a_1(m+1) + Q(m)]
\]

or

\[
d_m = \frac{l(m)}{l(m+1)} = 1 - \frac{(1-a_2) - (Q(m+1) - Q(m))}{a_1(m+1) + Q(m) + 1} \quad (4)
\]

We multiply the equations (4) for \( m = n, n+1, \ldots, n+k-1 \):

\[
\frac{l(n)}{l(n+k)} = d_n d_{n+1} \cdots d_{n+k-1}. \quad (5)
\]

\* \( A \sim B \) means that there are positive constants \( C_1 < C_2 \) such that \( C_1 B \leq A \leq C_2 B \).
We have

\[ l(n) \sum_{k=n}^{2n} \frac{1}{I(k)} = 1 + \sum_{k=n}^{2n} d_n d_{n+1} \cdots d_{n+k-1}. \]

Since \( \alpha(m)/(m+1) \uparrow \), it follows that \( \alpha(m+1)/(m+2) < \alpha(m)/(m+1) \) or

\[ \alpha(m+1) - \alpha(m) < \frac{\alpha(m)}{m+1} \cdot \tag{6} \]

Hence (3) and (6) imply

\[ Q(m+1) - Q(m) = a_1 [a(m+1) - \alpha(m)] + \beta(m+1) - \beta(m) \to 0, \]

\[ \frac{Q(m+1)}{m+1} \to 0, \ m \to \infty, \]

and so there is a number \( m_0 \) such that, for all \( m \geq m_0 \), we have

\[ |Q(m+1) - Q(m)| \leq \frac{1-a_1}{2}, \ a_1 = 1, \tag{7} \]

\[ a_1(m+1) + Q(m) + 1 \geq \frac{1}{2} a_1(m+1), \ a_1 > 0, \tag{8} \]

\[ \frac{1}{2} a_2 \alpha(m) < Q(m) + 1 < \frac{3}{2} a_2 \alpha(m), \ a_2 \neq 0, \tag{8'} \]

\[ Q(m) + 1 < \frac{3}{2} a_3, \ a_3 = 0, \ a_3 > 0. \tag{8''} \]

Now consider the four cases separately:

1) \( a_1 = 0, a_2 > 0 \). It follows from (4), (7), and (8') that

\[ 1 - \frac{3}{a_2} \frac{1}{\alpha(m)} < d_m < 1 - \frac{1}{3a_2} \frac{1}{\alpha(m)}, \ m \geq m_0. \]

Since \( \alpha(m) \downarrow \), we have

\[ \left(1 - \frac{3}{a_2} \frac{1}{\alpha(n)}\right) \leq d_n d_{n+1} \cdots d_{n+k-1} < \left[1 - \frac{1}{3a_2} \frac{1}{\alpha(2n)}\right]^k. \]

for \( k \leq n \) and \( n \geq m_0 \). Hence [see (5)],

\[ l(n) \sum_{k=n}^{2n} \frac{1}{I(k)} < 1 + \sum_{k=1}^{n} \left[1 - \frac{1}{3a_2} \frac{1}{\alpha(2n)}\right]^k < 3a_2 \alpha(2n) < 6a_2 \alpha(n), \ n \geq m_0. \]

On the other hand,

\[ l(n) \sum_{k=n}^{2n} \frac{1}{I(k)} > 1 + \sum_{k=1}^{n} \left[1 - \frac{3}{a_2} \frac{1}{\alpha(n)}\right]^k = \frac{a_2}{3} \alpha(n) \left[1 - \left(1 - \frac{3}{a_2} \frac{1}{\alpha(n)}\right)^n\right], \ n \geq m_0. \]

But

\[ \lim_{n \to \infty} \left[1 - \frac{3}{a_2} \frac{1}{\alpha(n)}\right]^n = 0. \]

Hence, for \( n \geq n_0 \) we have \( 1 - \left[1 - \frac{3}{a_2} \frac{1}{\alpha(n)}\right]^n > \frac{1}{2} \) and if \( n \geq N = \max(m_0, n_0) \), we have

\[ l(n) \sum_{k=n}^{2n} \frac{1}{I(k)} > \frac{1}{6} a_2 \alpha(n). \]

2) \( a_1 > 0 \). It follows from (4), (7), and (8) that

\[ d_n < 1, \ a_1 < 1. \]

\[ \frac{1}{1 + \frac{3}{m+1}}, \ a_1 > 1. \]
and
\[
d_m = \begin{cases} 
1 - \frac{3}{a_2(n + 1)}, & a_1 < 1, \\
1, & a_1 > 1, \\
1 - \frac{1}{m + 1}, & a_1 = 1.
\end{cases}
\]

Hence,
\[
l(n) \sum_{k=1}^{m} \frac{1}{n(n + 1)} < \begin{cases} 
(n + 1, a_1 < 1,
1 + \sum_{k=1}^{n} \left(1 + \frac{3}{n + 1}\right)^k < e^3(n + 1), a_1 > 1.
\end{cases}
\]

and
\[
l(n) \sum_{k=1}^{m} \frac{1}{n(n + 1)} > \begin{cases} 
1 + \sum_{k=1}^{n} \left(1 - \frac{3}{a_2(n + 1)}\right)^k > \frac{1}{2} e^{-\frac{3}{a_1(n + 1)}}, a_1 < 1, \\
1 + \sum_{k=1}^{n} \left(1 - \frac{1}{n + 1}\right)^k > \frac{1}{2} e^{-\frac{1}{n + 1}}, a_1 = 1.
\end{cases}
\]

3) \(a_1 = a_2 = 0, a_3 > 0\). It follows from (7) and (8') that
\[
d_m < 1 - \frac{1}{3a_3}, \quad m > m_0.
\]

Hence,
\[
1 < l(n) \sum_{k=1}^{m} \frac{1}{n(n + 1)} < 1 + \sum_{k=1}^{n} \left(1 - \frac{1}{3a_3}\right)^k < 3d_3.
\]

4) \(a_1 = a_2 = a_3 = 0, \beta(n) \geq 0\). In this case,
\[
d_m = \frac{l(m)}{l(m + 1)} < \beta(m + 1), \quad \frac{1}{1 + \beta(m)} < \beta(m + 1).
\]

Hence \(d_m \to 0\) when \(m \to \infty\). Let
\[
d_n^* = \max(d_n, d_{n+1}, \ldots, d_{m-1}), \quad d_n^* \to 0, \quad n \to \infty.
\]

We have
\[
1 < l(n) \sum_{k=1}^{m} \frac{1}{n(n + 1)} < 1 + \sum_{k=1}^{n} (d_n^*)^k < \frac{1}{1 - d_n^*} < 2, \quad n > n_0
\]
and the lemma is proved.

We now give some examples of sequences \(l(n)\) for which the conditions of the lemma are satisfied.

1) \(l(n) = (n + 1)^\gamma, \gamma > 1\). We use Euler's summation formula ([3], pp. 353-354):
\[
\sum_{n=1}^{m-1} \psi(k) = \int \psi(t) \, dt + \sum_{k=1}^{m-1} \frac{1}{k!} B_k \left[ \psi^{(k-1)}(0) - \psi^{(k-1)}(n) \right] + r_m.
\]

where \(r_m = \frac{1}{m!} B_m \sum_{k=0}^{m-1} \psi^{(m)}(k + 1), 0 < \theta < 1, m \) is even, and the \(B_k\) are the Bernoulli numbers. Writing \(m = 2\) and using \(B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}\), we obtain
\[
\sum_{k=0}^{n-1} (k + 1)^\gamma = (n + 1)^\gamma \left[ \frac{n + 1}{\gamma + 1} + \left( \frac{1}{2} - \frac{1}{\gamma + 1} \right) (n + 1)^\gamma - \frac{1}{2} + (n + 1)^{-\gamma} r_n \right],
\]
\[
r_n = \frac{\gamma (\gamma - 1)}{2} \sum_{k=0}^{m-1} (k + 1 + \theta)^{-\gamma} = O((n + 1)^{-1}).
\]
Conditions (3) obviously hold.

2) \( q(n) = q(n+1)^{-1} \gamma (n+1)^{-\gamma}, q > 1, 0 < \gamma < 1. \) From (9),

\[
\sum_{k=0}^{n-1} q^{(k+1)^{-\gamma}} (k+1)^{-\gamma} = q^{(n+1)^{-\gamma}} (n+1)^{-\gamma} \left[ (n+1)^{-1} \frac{1}{(1-\gamma) \ln q} + r_s \right]
\]

It follows from

\[
r_s = \frac{1}{12} \sum_{k=0}^{n-1} q^{(k+1)^{\gamma}} \left[ (1-\gamma)^2 \ln q (k+1+\gamma - 3\gamma (1-\gamma) \ln q (k+1+\gamma) - \gamma (\gamma+1)(k+1+\gamma)^{\gamma-1} \right]
\]

that

\[
\beta(n) = q^{(n+1)^{\gamma}} (n+1)^{\gamma} \left[ q \left( \frac{1}{2} \frac{1}{(1-\gamma) \ln q} \right) + r_s \right] \rightarrow 0, \quad n \rightarrow \infty
\]

and conditions (2) and (3) are satisfied for \( a_1 = 0. \)

3) \( l(n) = q^n, q > 1. \) The relation (2) holds for

\[
a_1 = a_2 = 0, \quad a_3 = \frac{4}{1-q}, \quad \beta(n) = \frac{4}{1-q} q^n.
\]

4) \( l(n) = (n+1)(n+1)^{\gamma}. \) In this case the relation \( \sum_{k=0}^{n-1} (k+1)(k+1) = (n+1)^{\gamma} - 1 \) implies that (2) and (3) hold for \( a_1 = a_2 = a_3 = 0, \) and \( \beta(n) = \frac{1}{n+1} \left[ 1 - \frac{1}{(n+1)!} \right]. \)

3. Estimates for \( f(x) - \tau_n(x, f). \)

We now consider the main theorems. As above, let \( f(x) \) be defined by the series (1) by means of the Riesz-Fisher theorem.

**THEOREM 1.** If \( \sum_{n=0}^{\infty} \nu^q(n) < \infty, \) where \( \nu(n) \) is any sequence, then

\[
f(x) - \tau_n(x, f) = o_x \left( \frac{1}{n+1} \left[ \sum_{l=m}^{\infty} \frac{1}{\nu^q(l)} \right]^{1/2} \right)
\]

almost everywhere in \([a, b]\) and there is a function \( F(x) \in L_2(a, b) \) such that

\[
(n+1) \left[ \sum_{l=m}^{\infty} \frac{1}{\nu^q(l)} \right]^{1/2} |f(x) - \tau_n(x, f)| \leq F(x).
\]

**Proof.** If

\[
\sum_{n=0}^{\infty} \nu^q(n) \int_a^b |f(x) - S_n(x, f)|^q dx < \infty.
\]

Levi's theorem implies that the series

\[
\sum_{n=0}^{\infty} \nu^q(n) |f(x) - S_n(x, f)|^q
\]

converges almost everywhere in \([a, b]\). Hence,

\[
\sum_{l=m}^{\infty} |f(x) - S_k(x, f)|^q = o_x (1).
\]

Moreover, Schwartz's inequality yields

\[
\sum_{l=m}^{\infty} |f(x) - S_k(x, f)|^q \leq \left[ \sum_{l=m}^{\infty} |f(x) - S_k(x, f)| \nu^q(l) \right]^{1/2} \times \left[ \sum_{l=m}^{\infty} \frac{1}{\nu^q(l)} \right]^{1/2} = o_x \left( \left[ \sum_{l=m}^{\infty} \frac{1}{\nu^q(l)} \right]^{1/2} \right)
\]

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almost everywhere in \([a, b]\). Hence,
\[
|f(x) - \tau_n(x, f)| \leq \frac{1}{n+1} \sum_{k=0}^{n} |f(x) - S_k(x, f)| \cdot \frac{1}{\left| \sum_{k=0}^{n} \frac{1}{\sqrt{|f(x)|}} \right|^{1/2}}.
\]

almost everywhere in \([a, b]\). We write
\[
F^2(x) = \sum_{n=0}^{\infty} \nu^2(n) |f(x) - S_n(x, f)|^2.
\]

It is plain that \(F(x) \in L^2(a, b)\). Applying Theorem 1 to those sequences \(I(n) = \nu^2(n)\) for which the lemma holds, we easily obtain:

**Theorem 2.** If \(\sum_{n=0}^{\infty} E_n(n) \nu^2(n) < \infty\) and \(\nu^2(n)\) satisfies conditions (2) and (3), we have
\[
f(x) - \tau_n(x, f) = o_x \left( \frac{1}{\sqrt{\nu(n)}} \right), \quad n \geq N
\]
after everywhere in \([a, b]\), and there is a function \(F(x) \in L^2(a, b)\) such that
\[
\mu(n) |f(x) - \tau_n(x, f)| \leq F(x),
\]
where
\[
\mu(n) = \begin{cases} 
\nu(n) \sqrt{n+1}, & a_1 > 0, \\
\nu(n+1) \sqrt{a(n)}, & a_1 = 0, a_2 > 0, \\
\nu(n+1), & a_1 = a_2 = 0, a_3 > 0.
\end{cases}
\]

**Theorem 3.** Let the conditions of the lemma hold, and let
\[
\lambda^2(n) = \sum_{k=0}^{n-1} \nu^2(k) = \nu^2(n) \left[ a_1(n+1) + a_2 a(n) + a_3 + \beta(n) \right].
\]

Then the convergence of the series \(\sum_{n=1}^{\infty} c_n^2 \lambda^2(n)\) implies that
\[
f(x) - \tau_n(x, f) = o_x \left( \frac{1}{\mu(n)} \right), \quad n \geq N
\]
after everywhere in \([a, b]\), and
\[
\mu(n) |f(x) - \tau_n(x, f)| \leq F(x) \in L^2(a, b),
\]
where
\[
\mu(n) = \begin{cases} 
\lambda(n), & a_1 \neq 0, \\
\lambda(n) \frac{n+1}{a(n)}, & a_1 = 0, a_2 > 0, \\
\lambda(n), & a_1 = a_2 = 0, a_3 > 0, \\
\lambda(n) \frac{n+1}{\sqrt{\beta(n)}}, & a_1 = a_2 = a_3 = 0, \beta(n) > 0.
\end{cases}
\]

**Proof.** The assertion of the theorem follows directly from Theorem 2 and the relation
\[
\sum_{n=0}^{\infty} E_n(n) \nu^2(n) = \sum_{n=0}^{\infty} \nu^2(n) \sum_{k=0}^{\infty} c_k^2 \sum_{n=0}^{\infty} \lambda^2(n)
= \sum_{n=0}^{\infty} c_k^2 \lambda^2(k) [a_1(k+1) + a_2 \lambda(k) + a_3 + \beta(k)] = \sum_{n=1}^{\infty} c_n^2 \lambda^2(n).
\]

Some special cases of Theorem 3 are interesting.

**Corollary 1.** If \(\sum_{n=1}^{\infty} c_n^2 (n+1)^{\gamma} < \infty, \gamma > 0\), we have
\[
f(x) - \tau_n(x, f) = o_x ((n+1)^{\gamma})
\]
almost everywhere in \([a, b]\) and

\[(n + 1)^\gamma | f(x) - \tau_n(x, f) | \leq F(x) \in L_2(a, b).\]

**COROLLARY 2.** If \(\sum_{n=1}^{\infty} c_n^2 (n+1)^{\gamma} < \infty, q > 1, 0 < 2\gamma < 1,\) we have

\[f(x) - \tau_n(x, f) = o_x((q^{n+1})\cdot(n+1)^{-\gamma}) \]

almost everywhere in \([a, b]\), and

\[g^{(n+1)^{\gamma}}\cdot(n+1)^{\gamma} f(x) - \tau_n(x, f) | \leq F(x) \in L_2(a, b).\]

**COROLLARY 3.** If \(\sum_{n=1}^{\infty} c_n \cdot n^2 < \infty, q > 1,\) then

\[f(x) - \tau_n(x, f) = o_x(q^{-n}(n+1)^{-\gamma}) \]

almost everywhere in \([a, b]\), and

\[g^n(n+1)^{\gamma} | f(x) - \tau_n(x, f) | \leq F(x) \in L_2(a, b).\]

**COROLLARY 4.** If \(\sum_{n=1}^{\infty} c_n^2 (n+1)! < \infty,\) then

\[f(x) - \tau_n(x, f) = o_x((n+1)! \cdot (n+1)^{-\gamma}) \]

almost everywhere in \([a, b]\), and

\[\sqrt{(n+1)! \cdot (n+1)^{3/2}} f(x) - \tau_n(x, f) | \leq F(x) \in L_2(a, b).\]

4. **The Nature of the Variation of \(\lambda(n)/\mu(n)\) and \(\nu(n)/\mu(n)\)**

Theorem 3 implies that the ratio \(\lambda(n)/\mu(n)\) depends on the rate of growth of the sequence \(\lambda(n)\), and that this rate of growth must be faster when \(\lambda(n)/\mu(n)\) tends to zero more rapidly, when \(n \to \infty\) (see also Corollaries 1–4). Since \(\beta(n)\) can be arbitrarily small when \(a_1 = a_2 = a_3 = 0\), the same conclusion will hold for \(\nu(n)/\mu(n)\). Hence, the result of Theorem B is not final, and in some cases it can be strengthened (at least Corollaries 2–4 show that stronger results can hold).

It would clearly be interesting to obtain more information concerning the ratio \(\lambda(n)/\mu(n)\) for other methods of summation of orthogonal series. We now consider Theorem 2. When \(\sum_{n=1}^{n-1} \nu^k(k) = \nu^k(n) (a_3 + \beta(n)), a_3 > 0, \beta(n) \to 0,\) we have \(\nu(n)/\mu(n) = 1/n + 1.\) Further growth of the sequence \(\nu(n)\) corresponding to \(a_3 = 0\) does not alter the value of \(\nu(n)/\mu(n)\). We will prove that this result is in a sense best-possible, i.e., that if \(\nu(n) \uparrow\) and

\[\sum_{n=0}^{n} \nu^k(k) = O(\nu^2(n)), \tag{10}\]

then for any sequence \(\varepsilon_n > 0\) tending to zero and for any orthogonal system \(\{\varphi_k(x)\}\), there is an orthogonal series \(\sum_{k=0}^{\infty} c_k \varphi_k(x)\) converging to \(f(x) \in L_2(a, b),\) such that \(\sum_{n=0}^{\infty} E_n(f) \nu^2(n) < \infty\); but neither the condition

\[f(x) - \tau_n(x, f) = o_x \left( \frac{\varepsilon_n}{(n+1) \nu(n)} \right) \tag{11}\]

(almost everywhere in \([a, b]\)) nor the condition

\[\frac{1}{\varepsilon_n} (n + 1) \nu(n) | f(x) - \tau_n(x, f) | \leq F(x) \in L_2(a, b) \tag{12}\]

(uniformly) holds.

To prove this result, we first prove that

\[\| f(x) - \tau_n(x, f) \|_n = \frac{1}{n+1} \left[ \sum_{k=0}^{n} (2k+1) E_{n+k}(f) \right]^{1/6} \tag{13}\]
for every function \( f(x) \) defined by the series (1).

We write the de la Vallee-Poussin sums in the form

\[
\tau_n(x, f) = \sum_{k=0}^{2n} a_k^{(m)} c_k(x),
\]

\[
a_k^{(m)} = \begin{cases} 
1, & 0 \leq k \leq n, \\
\frac{2n+1-k}{n+1}, & n+1 \leq k \leq 2n, \\
0, & k \geq 2n+1.
\end{cases}
\]

Then

\[
\| f(x) - \tau_n(x, f) \|_2^2 = \sum_{k=0}^{2n} \left(1 - \frac{2n+1-k}{n+1} \right) a_k^{(m)} e_k^2 + \sum_{k=2n+1}^{\infty} e_k^2.
\]

The application to the first sum of Abel's transformation in the form

\[
\sum_{k=0}^{n} v_k u_k = \sum_{k=m+1}^{n} (v_k - v_{k-1}) \sum_{i=m+1}^{\infty} u_i + v_{m+1} \sum_{i=m+1}^{\infty} u_i - v_n \sum_{i=m+1}^{\infty} u_i
\]

([4], p. 78) and some simple transformations, yield (13).

We now construct the required orthogonal series. Let \( n_k \) be an increasing sequence of positive integers such that (a) \( e_{n_k} \to 0 \) and (b) \( \sum_{k=1}^{\infty} e_k^2 < \infty \). We write

\[
E_{n_k} = E_{n_k}/\nu(n_k) (k = 1, 2, \ldots), \quad E_n = E_{n_k}, \quad 0 \leq n \leq n_k, \\
E_n = E_{n_k}, \quad n_{k-1} + 1 \leq n \leq n_k \quad (k = 2, 3, \ldots).
\]

The sequence \( E_n \) is nonincreasing and \( E_n \to 0 \); hence for any orthogonal system there exists a series

\[
\sum_{k=1}^{\infty} c_k q_k(x),
\]

converging to \( f(x) \) in \( L_2(a, b) \) and \( E_n(f)_{L_2} = E_n \). Clearly, \( \sum_{k=1}^{\infty} E_k^2 v_k^2(n_k) < \infty \). In fact, (10), (a), and (b) imply

\[
\sum_{k=1}^{\infty} E_k^2 v_k^2(n_k) = \sum_{k=1}^{n_k} E_k^2 v_k^2(n_k) + \sum_{k=n_k+1}^{\infty} E_k^2 v_k^2(n_k)
\]

\[
= E_{n_k}^2 \sum_{k=1}^{n_k} v_k^2(n_k) + \sum_{k=n_k+1}^{\infty} E_k^2 \sum_{k=n_k+1}^{\infty} v_k^2(n_k)
\]

\[= O(1) \left[ E_{n_k}^2 v_k^2(n_k) + \sum_{k=n_k+1}^{\infty} E_k^2 v_k^2(n_k) \right] = O(1) \sum_{k=1}^{\infty} e_k^2 < \infty.
\]

Further, since

\[
\| f(x) - \tau_n(x, f) \|_2 \geq E_n/n + 1
\]

[see (13)], we have

\[
\| f(x) - \tau_{n_k}(x, f) \|_2 \geq \frac{E_{n_k}}{n_k+1} = \frac{\varepsilon_{n_k}}{(n_k+1)\nu(n_k)}
\]

and so

\[
\lim_{n \to \infty} \frac{\nu(n)(n+1)}{e_n} \| f(x) - \tau_n(x, f) \|_2 \geq 1.
\]

(14)

If (12) and (13) held simultaneously, Lebesgue's theorem would imply

\[
\lim_{n \to \infty} \frac{\nu(n)(n+1)}{e_n} \| f(x) - \tau_n(x, f) \|_2 = 0,
\]

but this contradicts (14) and the proof is complete.
LITERATURE CITED