A universal space for normal bundles of $n$-manifolds

E. H. Brown, Jr., and F. P. Peterson

§1. Introduction

In [3] the authors gave a simple criterion for deciding whether a polynomial in Stiefel–Whitney classes is zero on the normal bundles of all smooth $n$-manifolds. The ideal of relations among Stiefel–Whitney classes for all $n$-manifolds, $I_n \subseteq H^*(BO)$ was defined by

$$I_n = \{ w \in H^*(BO) \mid w(\nu_M) = 0 \text{ for all } M^n \}$$

where $M^n$ denotes a smooth $n$-manifold and $\nu_M$ is its stable normal bundle. Let $\Phi : H^*(BO) \to H^*(MO)$ be the Thom isomorphism and for $w \in H^*(BO)$, define $w Sq^i$ to be $\Phi^{-1}(\chi(Sq^i)\Phi(w))$. It was shown that $I_n$ consists of all $\mathbb{Z}_2$-linear combinations of elements of the form $w Sq^i$ where $2i > n - |w|$ ($|w|$ = dimension of $w$).

In this paper we give a stronger version of this result, namely:

THEOREM 1. There is a space $BO/I_n$ and a map $\pi : BO/I_n \to BO$ such that
(a) If $M$ is a smooth, compact $n$-manifold and $h : M \to BO$ classifies $\nu_M$, then there is a map $\tilde{h} : M \to BO/I_n$ such that $\pi \tilde{h} = h$.
(b) The following sequence is exact.

$$0 \to I_n \subset H^*(BO) \xrightarrow{\pi^*} H^*(BO/I_n) \to 0.$$ 

Theorem 1 shows that $BO/I_n$ is a universal space for normal bundles of $n$-manifolds in that stably, every such bundle is induced from the bundle over $BO/I_n$ and $BO/I_n$ is the space with the smallest cohomology having this property.

Our original result on $I_n$ suggested the possibility of defining higher order characteristic classes, that is, one could form a space $B$ over $BO$ by killing the

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1 During the work on this paper the authors were supported by NSF grant MCS76-08804 A01 and MCS 76-06323.
elements of $I_n$. Then an element of $H^\ast(B)$ might give a "new" characteristic class for $n$-manifolds. For example, with $n = 4$ or $5$, the relation

$$(Sq^2 + w_1 \cup Sq^1 + w_2 U)(v_3) = v_3 Sq^2 = (1 Sq^3) Sq^2 = 0$$

where $v_3$ is the Wu class, gives a class in $H^4(B)$ which is not a polynomial in Stiefel-Whitney classes. Theorem 1 shows that on an $n$-manifold this "new" class will be a polynomial in Stiefel-Whitney classes modulo indeterminacy.

The spaces $BO/I_n$ are also related to the conjecture that any smooth $n$-manifold immerses in $R^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the dyadic expansion of $n$. Since this conjecture is equivalent to the normal bundle map $h : M^n \to BO$ lifting to $BO_{n-\alpha(n)}$ ([9]), the following is a stronger form of the conjecture:

**Conjecture.** $\pi : BO/I_n \to BO$ lifts to $BO_{n-\alpha(n)}$.

Using our proof of Theorem 1, our results in [4] can be restated in the following way which gives some plausibility to the above conjecture.

**Theorem 2.** If $\zeta$ is the stable universal bundle over $BO$, $MO$ is its Thom spectrum, $MO/I_n$ is the Thom spectrum of $\pi^*\zeta$ and $MO(n-\alpha(n))$ is the Thom spectrum of the universal bundle over $BO_{n-\alpha(n)}$, then $MO/I_n$ lifts to $MO(n-\alpha(n))$.

This paper is organized as follows: In §2 we give a detailed outline of the proof of Theorem 1 setting forth most of the notation and describing the various technical problems arising in the construction of $BO/I_n$. Then in Sections 3, 4, 5, and 6 we prove the various lemmas stated in §2. Throughout the remainder of this paper $n$ is a fixed positive integer.

**§2. Outline of the Proof of Theorem 1**

All cohomology will be with $Z_2$ coefficients, $A$ will be the mod two Steenrod algebra and $\chi : A \to A$ will be the canonical anti-automorphism. The semi-tensor product of $A$ and $H^\ast(BO)$ ([6]) will be denoted by $A(BO)$, that is, $A(BO) = A \otimes H^\ast(BO)$ with the algebra structure defined by

$$(a \otimes u)(b \otimes v) = \sum ab_i' \otimes (\chi(b_i')u)v$$

where $b \to \sum b_i' \otimes b_i'$ under the diagonal of $A$. We denote $a \otimes u$ by $a \circ u$. 
By a spectrum $Y$, we will mean a collection of spaces $Y_q$ and maps $g_q : SY_q \to Y_{q+1}$. If $X$ and $Y$ are spectra, a map $f : X \to Y$ of degree $p$ will be a collection of homotopy classes $f_q \in [X_q, Y_{q+p}]$ compatible with the maps $g_q$. If $\xi$ is a real $k$-plane bundle, $T(\xi)$ will denote its Thom spectrum, i.e., $T(\xi)_q = S^{q-k}$ (Thom space of $\xi$). Thus the Thom class is in $H^0(T(\xi))$. If $\xi$ is a vector bundle over $B$, $\Phi : H^\bullet(B) \to H^\bullet(T(\xi))$ will be the Thom isomorphism. We make $H^\bullet(T(\xi))$ into an $A(BO)$ module as follows: Let $h : B \to BO$ classify $\xi$. If $u \in H^\bullet(T(\xi))$, $w \in H^\bullet(BO)$ and $a \in A$, $(a \circ w)u = a(h^*(w)u)$. One easily checks that $\Phi(I_n) \subset H^\bullet(MO)$ is an $A(BO)$ submodule.

We begin by constructing an $A$-free, acyclic resolution of $\Phi(I_n)$. In [3] the following was proved:

**Theorem 2.1.** If $\{u_i\}$ is an $A$ basis for $H^\bullet(MO)$, then $\Phi(I_n)$ is the $A$ module generated by $\{\chi(Sq^j)u_i | 2j > n - |u_i|\}$.

For a partition $\omega = \{j_1, j_2, \ldots, j_h\}$ let $s_\omega \in H^\bullet(BO)$ be the usual class ([17]) associated with the symmetric function $\sum t_1^{j_1}t_2^{j_2}\cdots t_h^{j_h}$. For each partition $\omega$ let $\omega'$ be the partition consisting of odd integers $j$, one for each $j2^r \in \omega$. Let

$$u_\omega = \prod_s s_\omega^{2^r}$$

Since

$$u_\omega = s_\omega + \sum s_\omega'$$

where $\omega'$ has fewer entries than $\omega$ and $\{s_\omega\}$ is a basis for $H^\bullet(BO)$, $\{u_\omega\}$ is also a basis for $H^\bullet(BO)$. Also $\{\Phi(u_\omega) | 2^i - 1 \notin \omega\}$ is an $A$ basis for $H^\bullet(MO)$ since $\{\Phi(s_\omega) | 2^i - 1 \notin \omega\}$ is.

In [2] an $A$-free acyclic resolution of $A/A\{\chi(Sq^i) | i > h\}$ was constructed. Combining these resolutions with 2.1 and the $\Phi(u_\omega)$ basis, we obtain the following resolution of $\Phi(I_n)$.

Let $A$ be the graded free associative algebra over $Z_2$ with unit generated by $\lambda_i$, $i = 0, \pm 1, \pm 2, \ldots, |\lambda_i| = i$, modulo the relations: If $2i < j$

$$\lambda_i\lambda_j = \sum_{s=0}^j \binom{s-1}{2s - (j - 2i)} \lambda_{i+s}\lambda_{j-s}.$$  

If $I = \{i_1, i_2, \ldots, i_t\}$, let $\lambda_I = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_t}$, $l(I) = l$, $t(I) = t$, and $\lambda_{\langle I \rangle} = 1$. We define $I$
to be admissible if $2i \geq n_{i+1}$. As we will see in §3, $\{\lambda_i \mid I \text{ admissible}\}$ is a $Z_2$ basis for $\Lambda$. Let $\{\lambda^i \mid I \text{ admissible}\}$ be the dual basis of $\Lambda^* = \text{Hom}(\Lambda, Z_2)$.

Let $U_i$ be the vector space over $Z_2$ with basis the symbols $\lambda^i u_o$ where $I$ is admissible, $2^i - 1 \in \omega$, $I(I) = l$ and $2(t(I) + 1) > n - |u_o|$. Grade $U_i$ by $|\lambda^i u_o| = |\lambda^i| + |u_o|$. Let $d : A \otimes U_i \rightarrow A \otimes U_{i-1}$ be the $A$ linear map defined by

$$d(1 \otimes \lambda^i u_o) = \sum \lambda^i(\lambda_j \lambda_j) \chi(Sq^i) \otimes \lambda^j u_o$$

where the sum ranges over all $j$ and admissible $J$. Note by 2.2, if $\lambda^i(\lambda_j \lambda_j) \neq 0$, $\lambda(I) \geq l(I)$ and hence $d$ is well defined. Let $\eta : A \otimes U_0 \rightarrow H^*(MO)$ be given by $\eta(\alpha \otimes \lambda^i u_o) = a\Phi(u_o)$.

**PROPOSITION 2.3.** The following sequence is exact:

$$\cdots \rightarrow A \otimes U_i \rightarrow A \otimes U_{i-1} \rightarrow \cdots \rightarrow A \otimes U_0$$

and

$$\Phi(I_n) = \eta(\text{image } (d : A \otimes U_i \rightarrow A \otimes U_{i-1}))$$

We prove 2.3 in §3.

For a graded vector space $V$ over $Z_2$, let $K(V)$ denote the Eilenberg-MacLane spectrum such that $\pi_n(K(V)) = V^*$ and $H^*(K(V)) = A\otimes V$.

**PROPOSITION 2.4.** There is a sequence of $\Omega$-spectra $X_l$, $l = 0, 1, 2, \ldots$ and maps $\alpha : X_{l-1} \rightarrow K(U_l)$ of degree $+1$ such that

(i) $X_0 = K(U_0)$

(ii) $X_l$ is the fibration over $X_{l-1}$ induced by $\alpha_l$ from the contractible fibration over $K(U_l)$.

(iii) If $i : K(U_l) \rightarrow X_l$ is the inclusion of the fibre of $X_l \rightarrow X_{l-1}$, $(\alpha_{l+1})^* = d : A \otimes U_{l+1} \rightarrow A \otimes U_l$.

(iv) If $M$ is a smooth $n$-manifold, $\nu$ is its normal bundle, $g : MO \rightarrow K(U_0)$ realizes $\eta$ and $h : T(\nu) \rightarrow MO$ comes from the classifying map of $\nu$, then any lifting of $gh : T(\nu) \rightarrow X_0$ to $X_{l-1}$ lifts to $X_l$.

Since the $X_l$'s are constructed from an acyclic complex,

$$\lim H^*(X_l) = \text{Coker } (d : A \otimes U_1 \rightarrow A \otimes U_0) \approx H^*(MO)/\Phi(I_n).$$

To construct $BO/I_n$ we essentially construct a tower of spaces

$$B_l \rightarrow B_{l-1} \rightarrow \cdots \rightarrow B_0 = BO.$$
with fibres Eilenberg-MacLane spaces, such that if $T_i = T(\xi_i)$ where $\xi_i \to B_i$ is the pull back of the universal bundle over $BO$, then $T_i = X_i$ in dimensions $\leq n$. We can then, more or less, define $BO/I_0 = \lim B_i$.

We recall how the cohomology of a Thom space of a vector bundle changes, in a stable range, when a cohomology class in the base is killed. Suppose $g : B \to BO$ is a map such that $g_* : \pi_\ast(B) \to \pi_\ast(BO)$ for $2q \leq n$, $V$ is a graded vector space with $V_q = 0$ for $2q \leq n$ and $p : B' \to B$ is the fibration induced by a map $\gamma : B \to K(V)_1$ ($K(V) = \{K(V)_q\}$). Let $T = T(g^*\xi)$ and $T' = T(p^*g^*\xi)$. Viewing $B' \subset B$ as the fibre of $\gamma$, $\gamma$ factors as $B \to B/B' \to K(\gamma)_1$. Let

$$\Psi : (A(BO) \otimes V)^q \to H^{q+1}(T/T')$$

be given by $\Psi(a \otimes u \otimes v) = a(\Phi((\gamma')^*(v_1)))$ where $v_1 \in H^*(K(V)_1)$ is the element corresponding to $v \in V$ and $\Phi$ is the relative Thom isomorphism. In §6 we show that $\Psi$ is an isomorphism for $q \leq n$. (An equivalent form of this was proved in [1].) Combining this with the exact sequence of the pair $(T, T')$ we obtain an exact sequence,

$$\to H^q(T) \to H^q(T') \to (A(BO) \otimes V)^q \to H^{q+1}(T) \to$$

for $q \leq n$.

The cohomology of $X_i$ and $X_{i-1}$ are related by the Serre exact sequence,

$$\to H^q(X_{i-1}) \to H^q(X_i) \to (A \otimes U_i)^q \to H^{q+1}(X_{i-1}) \to .$$

Thus if we have constructed $B_{i-1}$ such that $T_{i-1} = X_{i-1}$ in dimensions $\leq n$ and we wish to construct $B_i$, we should take $B = B_{i-1}$ in the above and choose $V_i$ so that $A(BO) \otimes V_i = A \otimes U_i$ as $A$ modules. Our main algebraic result asserts that this is possible. Let

$$V_i = \{ \lambda \omega \in U_i \mid \omega = \{ \} \quad \text{for} \quad r \geq l \}$$

**Proposition 2.5.** There are $A$ linear isomorphisms $\theta : A \otimes U_l \to A(BO) \otimes V_l$ and $A(BO)$ linear maps $d : A(BO) \otimes V_i \to A(BO) \otimes V_{i-1}$, $l > 1$ and $d : A(BO) \otimes V_1 \to H^*(MO)$ such that the following diagram is commutative:

$$\begin{array}{cccccc}
A \otimes U_l & \xrightarrow{d} & A \otimes U_{l-1} & \to & \cdots & \to A \otimes U_0 \\
\downarrow \theta & & \downarrow \theta & & \downarrow \quad & \\
A(BO) \otimes V_i & \xrightarrow{d} & A(BO) \otimes V_{i-1} & \to & \cdots & \to H^*(MO).
\end{array}$$
Furthermore, if \( u \in V_1 \subset U_1 \), then \( \theta(1 \otimes u) = 1 \otimes u \).

The construction of spaces \( B_t \) can now be made, modulo technical problems, using 2.5. Given \( B_{t-1} \) and \( f_{t-1} : T_{t-1} \to X_{t-1} \), the \( k \)-invariant \( \beta_t : B_{t-1} \to K(V_t) \) is defined by:

\[
\theta \beta_t^*(v_1) = f_{t-1}^* \alpha_t^*(v)
\]

where \( \alpha_t : X_{t-1} \to K(U_t) \) is the \( k \)-invariant for \( X_t \), \( v \in V \) and \( v_1 \in H^*(K(V_t)) \) corresponds to \( v \). If \( M \) is an \( n \)-manifold and \( h : M \to BO \) classifies its normal bundle, 2.4(iv) shows that any lifting of \( h \) to \( B_{t-1} \) lifts to \( B_t \). The \( A(BO) \) linearity of \( d \) allows one (more or less) to construct \( f_t : T_t \to X_t \). Actually, this straightforward procedure is marred by two technical details which we now describe.

Let \( s = \lceil n/2 \rceil \). To form \( B_1 \) from \( BO \), one kills, among other things, the Wu class \( v_{s+1} \), i.e. \( d\lambda^s = \chi(Sq^{s+1}) U = v_{s+1} U \), where the \( U \) is the Thom class. The map \( \Psi \) is zero on

\[
\sum_{j>0} (Sq^j \circ v_{s+1-j}) \otimes \lambda^s \in (A(BO) \otimes V_1)^{2s+1}
\]

As a result, there is a class \( x \in H^{2s+1}(X_1) \) which goes to zero in \( H^{2s+1}(T_1) \). The class \( x \) is killed in going from \( X_1 \) to \( X_2 \). Hence if one were to follow the recipe given by 2.5, one would kill a class in \( B_1 \) which is already zero and thus produce a class in \( H^2(B_2) \) not coming from \( H^2(X_2) \). To avoid this, we omit a basis element from \( V_2 \). This same phenomena occurs in dimension \( 2s + 2 \) so we omit some more elements from \( V_2 \) and \( V_3 \). Namely, let \( \bar{V}_1 \subset V_1 \) be spanned by \( \lambda^t u_0 \in V_1 \) except \( \lambda^{0,0} w_s^2, \lambda^{0,-1} w_{s+1}^2, \lambda^{1,-2} w_{s+2}^2 \) and for \( s \) odd, \( \lambda^{-1,2} w_4^2 w_s^2 \) (\( w_s = u_{(1,1,\ldots,1)} \)).

In §3 we define a certain \( A(BO) \) linear map

\[
r : A(BO) \otimes V_1 \to A(BO) \otimes \bar{V}_1
\]

such that \( r | A(BO) \otimes \bar{V}_1 \) is the identity. We then use \( r \theta \) in place of \( \theta \) in our construction of \( B_t \).

The second difficulty arises in the following fashion. Again suppose we have \( B_{t-1} \) and \( f_{t-1} : T_{t-1} \to X_{t-1} \) and we construct \( B_t \) using \( \bar{V}_t \) instead of \( V_t \) as above. Let \( g_t : T_{t-1}/T_t \to K(U_t) \) be the map such that \( g_t^*(u) = \Psi r \theta(u) \) for \( u \in U_t \). In order to construct \( f_t : T_t \to X_t \) we need commutativity of the diagram

\[
\begin{array}{c}
T_{t-1} \xrightarrow{f_t} T_{t-1}/T_t \\
\downarrow \quad \downarrow g_t \\
X_{t-1} \xrightarrow{\alpha_t} K(U_t).
\end{array}
\]
We can only prove that this diagram commutes in dimensions $\leq 2s + 1$. To correct for this we relabel $B_i$ above, $B_i'$ and we form $B_i$ from $B_i'$ by killing the obstructions to commutativity as follows:

Define $\Delta = \Delta (f_{i-1}) : U_i \to H^* (T_{i-1})$ by

$$\Delta (u) = f_{i-1}^* \alpha_i^* u - \sum x_i f_{i-1}^* \alpha_i^* u_i$$

where $r\theta (u) = \sum x_i u_i$, $x_i \in A (BO)$, $u_i \in \widetilde{V}_i$. Then

$$j^* g_i^* u = j^* \Psi r \theta (u) = j^* \Psi \left( \sum x_i u_i \right) = \sum x_i j^* \phi \left( (\beta_i^*)^* (u_i) \right)$$

$$= \sum x_i \phi (\beta_i^* (u_i)) = \sum x_i f_{i-1}^* \alpha_i^* (u_i) = \Delta (u) + f_{i-1}^* \alpha_i^* (u)$$

Thus $\Delta$ is the deviation from commutativity of our diagram above. Let $W_i = U_i / \ker \Delta$. We kill $\Phi^{-1} (\Delta (W))$ in $B_i'$ to form $B_i$.

To recapitulate, we inductively construct a sequence of spaces $B_i$, stable vector bundles $\zeta_i$ over $B_i$ and maps $f_i : T_i = T(\xi_i) \to X_i$ such that $\Delta (f_i) = 0$. We take $B_0 = BO$, $\zeta_0 = \xi$ the universal bundle and $f_0$ the map such that $f_0^* (u_w) = \Phi (u_w)$ for $u_w \in U_0$. $(X_0 = K(U_0).)$ Referring to 2.5, $f_0^* = \eta$, $\alpha_i^* = d$ and $\Delta (f_0) = \eta d - d \theta = 0$. Suppose $B_{i-1}$, $\zeta_{i-1}$ and $f_{i-1}$ have been defined and $\Delta (f_{i-1}) = 0$. Let $p'_i : B_i' \to B_{i-1}$ be the fibration induced by $\beta_i : B_{i-1} \to K(\widetilde{V}_i)$ where $\beta_i$ is defined by

$$\phi (\beta_i^* (v_i)) = f_{i-1}^* \alpha_i^* (v)$$

for $v \in \widetilde{V}_i \subset U_i$ and $v_i \in H^* (K(\widetilde{V}_i))$ the element corresponding to $v$. Let $\zeta_i = (p'_i)^* \xi_{i-1}$ and $T_i = T(\zeta_i)$.

Viewing $B_i \subset B_{i-1}$ as the fibre of $\beta$, $\beta_i$ factors through $\beta'_i$. $B_{i-1}/B_i' \to K(\widetilde{V}_i)$.

Let $\Psi : A (BO) \otimes \tilde{V}_i \to H^* (T_{i-1}/T'_i)$ be the $A (BO)$ linear map such that $\Psi (v) = \Phi ((\beta'_i)^* (v_i))$ for $v \in \tilde{V}_i$. Let $\theta$ be as in 2.5, $r$ as in 2.6, and let $g'_i : T_{i-1}/T'_i \to K(U_i)$ be defined by $(g'_i)^* (u) = \Psi r \theta (u)$. Since $\Delta (f_{i-1}) = 0$, there is a map $f'_i$ making a commutative diagram

$$\begin{array}{ccccccc}
T_{i-1}/T'_i & \longrightarrow & T_i & \longrightarrow & T_{i-1} & \longrightarrow & T_{i-1}/T'_i \\
\downarrow \xi' & & \downarrow f_i & & \downarrow f_{i-1} & & \downarrow \xi' \\
K(U_i) & \longrightarrow & X_i & \longrightarrow & X_{i-1} & \longrightarrow & K(U_i).
\end{array}$$

Let $\Delta (f'_i) : U_{i+1} \to H^* (T_i)$ be given by $\Delta (f'_i) (u) = (f'_i)^* \alpha_i^* u + \sum x_i (f'_i)^* \alpha_i^* u_i$ where $r\theta u = \sum x_i u_i$. Let $W_{i+1} = U_{i+1}/\ker \Delta (f'_i)$ and let $p : B_i \to B_i'$ be the fibration induced
by $\gamma_i : B_i \to K(W_{i+1})$, where $\Phi(\gamma_i^* u_i) = \Delta(f_i)(u)$ for $u \in W_{i+1}$. Finally let $\zeta_i = p^* \zeta_i$ and $f_i = f_i T(p)$. Then $\Delta(f_i) = T(p)^* \Delta(f_i) = 0$ and the inductive step is complete.

In §5 we prove:

**Lemma 2.7.** If $l \geq 3$ and $q \leq n$, $f_l^* : H^q(X_i) \approx H^q(T(\zeta_i))$. Furthermore, if $M$ is a smooth $n$-manifold and $h : M \to B_0 = BO$ classifies its normal bundle, then any lifting of $h$ to $B_{l-1}$ lifts to $B_l$.

We next examine $H^*(B_l)$ for $l$ large.

**Lemma 2.8.** If $l \geq n$, $V_l^q = U_l^q = 0$ for $q \leq n-1$, $W_l^q = 0$ for $q \leq n$ and

$V_l^{q-1} = U_l^{q-1} = \{\lambda^{(0,0,..,0)} u_\omega \mid u_\omega \in U_0^{q-1}\}$. Furthermore,

$\Phi(\beta_l^*(\lambda^{(0,0,..,0)} u_\omega)) = \delta_l \tilde{u}_\omega$

$\tilde{u}_\omega \in H^* (T_{l-1}; Z_2)$, $u_\omega U \in H^* (T_{l-1})$ is the mod two reduction of $\tilde{u}_\omega$ and $\delta_l$ is the Bockstein associated with $Z_2 \to Z_{2l+1} \to Z_2$.

Thus for $l \geq n$,

$$H^q(B_l) \approx H^q(BO)/I^q_n \quad q < n$$

$$H^n(B_l)/\Phi^{-1}\{\delta_{l+1} \tilde{u}_\omega\} \approx H^n(BO)/I^n_n$$

We form $B_\infty$ from $B_l$, $l \geq n$, by killing classes $\Phi^{-1}(\delta^{l+1} \tilde{u}_\omega) \in H^{n+1}(B_l; Z_n)$ where $Z_\tau$ denotes twisted integer coefficients, twisted by $w_1$, $\Phi : H^*(B_l; Z_\tau) \approx H^*(T(\zeta_l); Z)$ is the Thom isomorphism and $\delta^l$ is the Bockstein associated with $Z \to Z \to Z_\tau$. Let $\tilde{B}_l$ be the two sheeted cover of $B_l$ defined by $w_1$. The classes $\Phi^{-1}(\delta^{l+1} \tilde{u}_\omega)$ may be represented by $Z_2$-equivariant maps $x_\omega : \tilde{B}_l \to K(Z, n)$ where $K(Z, n)$ has the action defined by the nontrivial action of $Z_2$ on $Z$. Let $\tilde{B}_\infty$ be the fibration over $\tilde{B}_l$ induced by

$$x = \prod x_\omega : \tilde{B}_l \to \prod K(Z, n)$$

Since $x$ is $Z_2$-equivariant, $Z_2$ acts freely on $\tilde{B}_\infty$. Let $B_\infty = \tilde{B}_\omega/Z_2$. The map $\beta_\omega \tilde{B}_\omega/Z_2 \to \tilde{B}_l/Z_2 = B_l$ has fibre $\Pi K(Z, n)$. With $Z_2$ coefficients, $\pi_1(B_l)$ acts trivially on the cohomology of the fibre. The Serre spectral sequences, with $Z_2$ coefficients has its usual, nonlocal coefficient form and the usual argument shows
that in dimensions \( \leq n \),

\[
H^k(BO_\infty) = H^k(B) / \langle \Phi^{-1}(\delta^{i+1} u_0) \rangle.
\]

Thus for \( q \leq n \)

\[
0 \to I_n^q \to H^q(BO) \to H^q(B_\infty) \to 0
\]

is exact. Also if \( M \) is an \( n \)-manifold and \( h : M \to B \) is covered by a bundle map \( g : v \to \xi \), \( T(g)^*(\delta^{i+1} u_\infty) = \delta^{i+1} T(g)^*(u_i) = 0 \) since the top homology class of \( T(v) \) is spherical. Therefore, \( h \) lifts to \( B_\infty \).

Finally, assume \( B_\infty \) is a \( CW \) complex and let

\[
BO/I_n = B^n \cup e_1^{n+1} \cup e_2^{n+1} \cdots e_m^{n+1}
\]

where \( e_1^{n+1} \) is attached by \( f_i | S^n, f_i : (D^{n+1}, S^n) \to (B^{n+1}, B^n) \) and \( [f_i] \in \pi_{n+1}(B^{n+1}, B^n) \) give a \( \mathbb{Z}_2 \)-basis for the image of

\[
\pi_{n+1}(B^{n+1}, B^n) \to H_{n+1}(B^{n+1}, B^n) \to H_n(B^n, B^{n-1})
\]

The maps \( f_i \) give an extension of \( B^n \subset B_\infty \), \( f : BO/I_n \to B_\infty \) and

\[
\begin{align*}
f^* : H^q(BO) & \cong H^q(BO/I_n) \quad \text{for} \quad q \leq n \\
H^q(BO/I_n) & = H^q(BO)/I_n = 0 \quad \text{for} \quad q > n
\end{align*}
\]

Also any map of an \( n \)-manifold into \( B_\infty \) is homotopic to a map factoring through \( f \). The proof of Theorem 1 is thus complete, modulo the lemmas and propositions of this section.

§3. Proofs of 2.3, 2.5, and 2.6

Let \( \Lambda^k \) be the \( \mathbb{Z}_2 \)-subspace of \( \Lambda^* \) generated by \( \lambda^l \) with \( l(I) = l \), \( t(I) \geq k \), and \( I \) admissible. Let

\[
d : A \otimes \Lambda^k \to A \otimes \Lambda_{l-1}^k
\]

be defined by

\[
d(1 \otimes \lambda^l) = \sum \lambda^l (\lambda_j \lambda_j) \chi \langle S q^{l+1} \rangle \otimes \lambda^j
\]

(3.1)
where the sum is over all \( j \) and admissible \( J \). Proposition 2.3 follows from 2.1 and 3.2(ii) below:

**PROPOSITION 3.2.**

(i) \( \{ \lambda_I \mid I \text{ admissible} \} \) is a \( \mathbb{Z}_2 \)-basis for \( \Lambda \).

(ii) The following is exact:

\[
\cdots \longrightarrow A \otimes A_0^+ \xrightarrow{d} A \otimes A_{1}^+ \longrightarrow \cdots \longrightarrow A \otimes A_k^+ \xrightarrow{e} A/A\{\chi(Sq^i) \mid i > k\}
\]

where \( e(a \otimes \lambda^j) = \{a\} \).

(iii) If \( I \) and \( J \) are admissible, \( l(I) = l, l(J) = l - 1 \), and \( I_1 = (1, 2, 4, \ldots, 2^{l-1}) \), then

\[
\lambda^{I+J_1}(\lambda_{I+J_2}) = \lambda^I(\lambda_J).
\]

**Proof.** For any sequence \( T = (t_1, t_2, \ldots, t_i) \) and integer \( r \), let \( h'(\lambda_T) = \lambda_{T+R} \). Extending linearly, \( h' \) gives a well defined map \( h' : \Lambda \to \Lambda \) since for any element of \( \Lambda \) of the form \( \alpha = \lambda_{t_1} \beta \lambda_{t_2} \) where \( \beta \) is a relation for \( \Lambda \) as in 2.2, \( h'(\alpha) \) also has this form. Since \( h'h^{-r} \) is the identity, \( h' \) is an isomorphism for all \( r \). Furthermore, \( h'(\lambda_I) \) is admissible if and only if \( \lambda_I \) is admissible.

Let \( \tilde{A} \subset \Lambda \) be the subalgebra generated by \( \lambda_0, \lambda_1, \lambda_2, \ldots \). In [8] it is proved that \( \{ \lambda_I \mid I \text{ admissible} \} \) is a basis for \( \tilde{A} \). For any \( \lambda_I \), \( h'(\lambda_I) \in \tilde{A} \) for \( r \) sufficiently large. Thus \( \{ \lambda_I \mid I \text{ admissible} \} \) is a basis for \( \Lambda \).

In [2], 3.2(ii) was proved for \( k \geq 0 \). From 2.2 one sees that \( \lambda_{-1} \lambda_{-1} = 0 \) and if \( t(J) \geq 0, \lambda_{-1} \lambda_J = 0 \) is a sum involving \( \lambda_J \)'s with \( t(J') > 0 \) and \( \lambda_{t(J')} \lambda_{-1} \). Suppose \( J_1 = (j_1, \ldots, j_m), J_2 = (j_{m+1}, \ldots, j_l) \) and \( J = (j_1, \ldots, j_l) \) are admissible with \( J_1 \) or \( J_2 \) possibly the empty sequence ( ). Define \( \lambda^{J_1} \lambda^{J_2} = \lambda^J \). Suppose \( j_m \geq 0 \) and \( j_{m+1} < -1 \). Then 3.1 yields

\[
d(\lambda^{J_1} \lambda^{J_2}) = (d\lambda^{J_1})\lambda^{-1} \lambda^{J_2} + \lambda^{J_1} \lambda^{J_2}
\]

\[
d(\lambda^{J_1} \lambda^{J_2}) = (d\lambda^{J_1})\lambda^{J_2}.
\]

Let

\[
D(\lambda^{J_1} \lambda^{J_2}) = \lambda^{J_1} \lambda^{-1} \lambda^{J_2}, D(\lambda^{J_1} \lambda^{-1} \lambda^{J_2}) = 0.
\]

Then for \( k < 0, D : A \otimes A_k^+ \to A \otimes A_{k+1}^+ \) satisfies \( dD + Dd = \text{identity} \). Therefore 3.2(ii) holds for \( k < 0 \).
Finally we prove 3.2(iii). Note that if $I$ is admissible, $I + rI_l$ is admissible and if $(h')^*: A^* \to A^*$ is the dual of $h'$, $(h')^* \lambda^I = \lambda^{I - rI_l}$. Therefore

$$
\lambda^I(\lambda_{A_j}) = (h')^*(\lambda^{I + rI_l})(\lambda_{A_j}) \\
= \lambda^{I + rI_l}(h'(\lambda_{A_j})) = \lambda^{I + rI_l}(\lambda_{A_j + 2rI_{l-1}})
$$

Proof of 2.5. Let $C_l = A \otimes U_l$, $D_l = A(BO) \otimes V_l$, $l > 0$, and $D_0 = H^*(MO)$. Denote $a \otimes u \in C_l$ by $au$ and $a \circ v \otimes w \in D_l$, $l > 0$, by $(a \circ v)w$. We filter $C_l$ and $D_l$ as follows: $F_q(C_l)$ is spanned by $a\lambda^I u_l$ with $|u_\omega| \leq q$ and $F_q(D_l)$, $l > 0$, is spanned by all $a \circ v \lambda^I u_l$ with $|u_\omega| + 2^l |v| \leq q$. $F_q(D_0)$ is spanned by all $au_\omega$ where $a \in A$, $u_\omega \in U_0 = \{u_\omega, 1 \leq l \leq \rho \}$ and $|u_\omega| \leq q$.

The chain complex $(C_l, d)$ is a direct sum of chain complexes of the form described in 3.2, indexed by the $u_\omega \in U_0$. Hence $d$ is filtration preserving and:

(3.3) The following is exact.

$$
\cdots \longrightarrow F_q(C_l) \xrightarrow{d} F_q(C_{l-1}) \longrightarrow \cdots \longrightarrow F_q(C_0)
$$

Using induction on $l$ we define a linear maps $\theta: C_l \to D_l$ and $A(BO)$ linear maps $d: D_l \to D_{l-1}$ such that

(i) $\theta$ is an isomorphism and $\theta: C_0 \to D_0$ is given by $\theta(a \otimes u_\omega) = a\Phi(u_\omega) \in H^*(MO)$, $u_\omega \in U_0$.

(ii) $d\theta = \theta d$

(iii) If $u \in V_l \subset U_l$, $\theta(u) = u$

(iv) $\theta(F_q(C_l)) = F_q(D_l)$

(v) Suppose $\lambda^I u_\omega \in U_l$. Let $\alpha$ and $\beta$ be the partitions

$$
\alpha = \bigcup_{r < l} 2^r \omega_r, \quad \beta = \bigcup_{r \geq l} 2^{r-l} \omega_r
$$

Note $u_\omega = u_\alpha u_\beta^\omega$. Then $\theta$ satisfies

$$
\theta(\lambda^I u_\omega) = u_\beta \lambda^{I'} u_\alpha \mod F_{|u_\beta| - 1}(D_l)
$$

where $I' = I + |u_\beta| I_l$.

Note that Proposition 2.5 consists of statements (i), (ii), and (iii) above.

For $l = 0$, $\theta$ is defined by (i) and $d = 0$ on $D_0$.

Suppose $\theta$ and $d$ have been defined on $C_k$ and $D_k$, $k < l$, and satisfy (i)–(v). Define $d = d_D: D_l \to D_{l-1}$ to be the $A(BO)$ linear map such that for $u \in V_l$,
\( d_D(u) = \theta(d_C u) \). We next define \( \theta : C_t \to D_t \). Suppose \( \lambda^t u_\alpha \in U_1 \) and \( u_\alpha = u_\alpha u_\beta^2 \) as in (v). If \( u_\beta = 1 \), \( \lambda^t u_\alpha \in V_1 \) and we define \( \theta(\lambda^t u_\alpha) = \lambda^t u_\alpha \). In this case (i)–(v) are satisfied. Suppose \( u_\beta \neq 1 \). Let

\[
X = \theta(d(\lambda^t u_\alpha)) + u_\beta \theta(d\lambda^t u_\alpha)
\]

where \( I' = I + |u_\beta| I \). By induction, \( d\theta = d\theta \) on \( C_{t-1} \) and hence \( dX = 0 \). We show that \( X \in F_{p-1}(D_t) \) where \( p = |u_\alpha| \). Decompose \( u_\alpha \) into \( u_\alpha, u_\alpha^2 \) as in (v).

\[
\theta(\lambda^t u_\alpha) = \sum \lambda^t(\lambda_i \lambda_j) \chi(Sq^{i+1}) \theta(\lambda^K u_\alpha) = \sum \lambda^t(\lambda_i \lambda_j) \chi(Sq^{i+1}) \circ u_\alpha u_\beta^2 \lambda^K u_\alpha, \text{ mod } F_{p-1}
\]

where \( K' = K + |u_{\alpha_1} u_\beta^2| I_{t-1} \). On the other hand,

\[
u_\beta \theta(d\lambda^t u_\alpha) = \sum \lambda^t(\lambda_i \lambda_j) u_\beta \chi(Sq^{i+1}) \theta(\lambda^t u_\alpha)
\]

In \( A(BO) \),

\[
u_\beta \chi(Sq^{i+1}) = \chi(Sq^{i+2}) \circ u_\beta^2 + \sum_{k < q} \chi(Sq^{i+k+1}) \circ Sq^k u_\beta
\]

where \( q = |u_\beta| \).

\[
\theta(\lambda^t u_\alpha) = u_\alpha \lambda^t u_\alpha, \text{ mod } F_{|u_\alpha|-1}
\]

where \( J' = J + |u_{\alpha_2}| I_{t-1} \). If \( u \lambda^t v \) has filtration less than \( |u_\alpha|-1 \) and \( k < q \), \( Sq^k u_\beta \mu \lambda^t v \) has filtration less than \( p = |u_\alpha| \).

Hence

\[
\theta(\lambda^t u_\alpha) = \sum_{i,j} \lambda^t(\lambda_i \lambda_j) \chi(Sq^{i+1}) \circ u_\alpha u_\beta^2 \lambda^K u_\alpha, \text{ mod } F_{p-1}
\]

In the above sum, replace \( j \) by \( j + q \) and \( J \) by \( K + 2q I_{t-1} \). Then

\[
u_\beta \theta(d\lambda^t u_\alpha) = \sum_{i,K} \lambda^t(\lambda_i K + 2q I_{t-1}) \chi(Sq^{i+1}) \circ u_{\alpha_2} u_\beta^2 \lambda^K u_\alpha, \text{ mod } F_{p-1}
\]

where \( K' = K + |u_{\alpha_2} u_\beta^2| I_{t-1} \). But \( I = I + q I \) and hence by 3.2(iii),

\[
\lambda^t(\lambda_i K + 2q I_{t-1}) = \lambda^t(\lambda_i \lambda_K)
\]

Hence \( X \in F_{p-1}(D_t) \).
By (iv) there is a $Y \in F_{p-1}(C, \mathbb{C})$ such that $\theta(Y) = X$ and by (i) and (ii), $dY = 0$. Hence for $l > 1$, by 3.3, there is a $Z \in F_{p-1}(C, \mathbb{C})$ such that $dZ = Y$. We verify that there is such a $Z$ for $l = 1$ by showing that when $l = 1$, $X \in \Phi(I, \mathbb{C})$. In this case

$$X = \chi(Sq^{i+1})\Phi(u_\alpha u_\beta^2) + u_\beta \chi(Sq^{i+q+1})\Phi(u_\alpha)$$

$$= \sum_{i=0}^{\infty} \chi(Sq^{i+q+1})\Phi((Sq^j u_\alpha)u_\alpha)$$

where $2(i + 1) > n - q$, $q = |u_\beta|$. But then, $2(i + q + 1) > n - |(Sq^i u_\alpha)u_\alpha|$ and hence $X \in \Phi(I, \mathbb{C})$.

We now define $\theta(\lambda^i u_\alpha)$ by induction on $|u_\alpha| = $ filtration degree of $\lambda^i u_\alpha$. For $|u_\alpha| = 0$, $\theta(\lambda^i) = \lambda^i 1$. If $\theta$ is defined on $F_{|u_\alpha|+1}(C, \mathbb{C})$, let

$$\theta(\lambda^i u_\alpha) = u_{p\lambda^i} \lambda^i u_\alpha + \theta(Z)$$

where $Z, \alpha, \beta,$ and $I'$ are as above. Then $d\theta(Z) = \theta(dZ) = \theta(Y) = X$ and

$$d\theta(\lambda^i u_\alpha) = d(u_{p\lambda} \lambda^i u_\alpha) + d\theta(Z)$$

$$= u_{p\lambda} \theta(d(\lambda^i u_\alpha)) + X = \theta(d(\lambda^i u_\alpha))$$

Note that elements of the form $u_{p\lambda^i} \lambda^i u_\alpha$, as above, together with $F_{p-1}(D, \mathbb{C})$ over $A$. Thus $\theta : C \rightarrow D$ is an epimorphism. (It is at this point that we use $\lambda^i$ where $I$ has negative entries. For each $u_{p\lambda^i} \lambda^i u_\alpha \in H^*(BO)V$ we need $\lambda^i u_\alpha u_\beta^2 \in U$ such that $I' = I + |u_\alpha| I, \mathbb{C})$. Elements of the form $\lambda^i u_\alpha u_\beta^2$ are an $A$ basis for $C$, and elements of the form $u_{p\lambda^i} \lambda^i u_\alpha$ are an $A$ basis for $D$. Hence $\theta : C \rightarrow D$ is an isomorphism and the proof of 2.5 is complete.

Proof of 2.6. Let $v_i \in H^*(BO)$ be the Wu classes, that is, $\Phi(v_i) = \chi(Sq^i)\Phi(1)$ where $\Phi : H^*(BO) \rightarrow H^*(MO)$ is the Thom isomorphism.

Lemma 3.4.

$$v_i = \sum s_\omega$$

where the sum ranges over all $\omega$ with entries only of the form $2^j - 1$ and $|s_\omega| = i$.

Proof. We view $H^*(BO) \subset Z[\zeta_1, \zeta_2, \ldots]$ such that $|\zeta_i| = 1, \mathbb{C}$, and $t_1 t_2 \ldots$ as the Thom class. Let $S = Sq^0 + Sq^1 + \cdots$ and $v = v_0 + v_1 + \cdots$. Then

$$\chi(Sq)t_i = \sum t_i^\omega$$
and

\[ v(t_1, t_2, \ldots)(t_1 t_2 \cdots) = \chi(Sq)(t_1 t_2 \cdots) \]

\[ = \prod_i \left( \sum_j t_j^{2i-1} \right)(t_1 t_2 \cdots) = \left( \sum_\omega \omega \right)(t_1 t_2 \cdots) \]

where the sum ranges over \( \omega \) with entries only of the form \( 2^i - 1 \).

Let \( x_1 \) and \( x_2 \in A(BO) \) be given by

\[ x_1 = \sum_{j>0} Sq^j \circ v_{s+1-j}, \quad x_2 = \sum Sq^j \circ v_{s+2-j} \]

Recall \( s = \left[ n/2 \right] \) and \( n \) is the dimension of the manifolds we are considering. Let \( y_i^1 \in D_1 \) be defined by

\[ y_1^1 = x_1 \lambda^s, \quad y_2^1 = x_2 \lambda^s, \quad y_3^1 = v_{s+1} \lambda^{s+1} + v_{s+2} \lambda^s + x_2 \lambda^s \]

**Lemma 3.5.** There are elements \( y_i^2 \in D_2 \) such that \( dy_i^2 = y_i^1 \) and

\[ y_1^2 = \lambda^{0,0} v_s^2 \mod F_{2s-1} \]
\[ y_2^2 = \lambda^{0,-1} v_{s+1}^2 \mod F_{2s+1} \]
\[ y_3^2 = \lambda^{-1,-2} v_{s+2} \mod F_{2s+3} \]

If \( s \) is odd, there is an element \( y_3^3 \) such that \( y_3^3 = (Sq^1 + w_4)y_2^3 \) and

\[ y_3^3 = \lambda^{-1,-2,-4} w_1^4 v_{s+2}^2 \mod F_{2s+7} \]

**Proof.** We first show that \( dy_i^1 = 0 \), \( d : D_1 \to D_0 = H^*(MO) \). Let \( U \in H^0(MO) \) be the Thom class.

\[ dy_1^1 = x_1 d \lambda^s = \sum Sq^i (v_{s+1-j} \chi(Sq^{s+1}) U) + v_{s+1} \chi(Sq^{s+1}) U \]
\[ = (Sq^{s+1} v_{s+1}) U + v_{s+1}^2 U = 0 \]

\[ dy_2^1 = \sum Sq^i (v_{s+2-j} \chi(Sq^{s+1}) U) \]
\[ = \sum Sq^i (v_{s+1} \chi(Sq^{s+2-j}) U) = (Sq^{s+2} v_{s+1}) U = 0 \]

\[ dy_3^1 = v_{s+1} \chi(Sq^{s+2}) U + v_{s+2} \chi(Sq^{s+1}) U + dy_2^1 = 0 \]

We next show that \( y_1^2 \) exists. In \( A \otimes A^* \) one may easily calculate \( d\lambda^{0,0} = Sq^1 \lambda^0 \).
Hence, by the arguments in the proof of 2.5,

\[ d\lambda^0,0v_s^2 = \theta(d\lambda^0,0v_s^2) = \theta(Sq^1 \lambda^0v_s^2) \]
\[ = Sq^1 \circ v_s \lambda^s \text{ mod } F_{2s-1} \]
\[ = \sum_{i>0} Sq^i \circ v_{s+1-i} \lambda^s \text{ mod } F_{2s-1} = y_1^i \text{ mod } F_{2s-1} \]

Thus \( u = d\lambda^0,0v_s^2 + y_1^i \in F_{2s-1} \) and \( du = 0 \). Therefore there is a \( z \in F_{2s-1}(D_2) \) such that \( dz = u \). Let \( y_1^2 = \lambda^0,0v_s^2 + z \). The existence of \( y_2^2, y_3^2, \) and \( y_3^3 \) are proven in an analogous fashion.

We now define \( r: A(BO) \otimes V_l \to A(BO) \otimes \tilde{V}_l \). For \( l \neq 2 \) and \( l \neq 3 \), \( s \) odd, \( \tilde{V}_1 = V_1 \) and \( r \) is the identity; \( \tilde{V}_l \subset V_l \) and \( r \circ A(BO) \otimes \tilde{V}_l \) is the identity. \( \tilde{V}_2 \) is formed from \( V_2 \) by omitting the basis elements \( \lambda^0,0w_s^2, \lambda^0,-1w_s^2 \) and \( \lambda^{-1,-2}w_s^2 \). By 3.4, \( v_i \) involves \( w_i = s_{(1,1,\ldots,1)} \) when \( v_i \) is expressed in the \( u_w \) basis. Let

\[ r(\lambda^0,0w_s^2) = y_1^2 - \lambda^0,0w_s^2 \]
\[ r(\lambda^0,-1w_{s+1}) = y_2^2 - \lambda^0,-1w_{s+1} \]
\[ r(\lambda^{-1,-2}w_{s+2}) = y_3^2 - \lambda^{-1,-2}w_{s+2} \]

We define \( r \) on \( A(BO) \otimes V_3 \) analogously. Then \( r(y_1^2) = r(y_2^3) = 0 \).

We conclude this section with an algebraic lemma about the \( y_i \)'s. Let \( L_l \subset A(BO) \otimes V_l \) be defined as follows: \( L_l = 0 \) for \( l = 0 \), \( l = 3 \) and \( s \) even, and \( l > 3 \).

\[ L_1 = A(BO)\{y_1^1\} + S_1 \]
where \( S_1 = \{v_3SQ^2\lambda^2\} \) when \( s = 2 \) and \( S_1 = 0 \) for \( s \neq 2 \).

\[ L_2 = A(BO)\{y_1^2\} + S \]
where \( S_2 = \{v_3\lambda^{1,2}\} \) when \( s = 2 \) and \( S_2 = 0 \), \( s \neq 2 \).

\[ L_3 = A(BO)\{y_3^2\} \]
\[ (d(v_3\lambda^{1,2}) = v_3SQ^2\lambda^2). \]

**Lemma 3.6.** \( d(L_l) \subset L_{l-1}, r(L_1) = 0 \) for \( l > 1 \) and the sequence

\[ \longrightarrow L_l \xrightarrow{d} L_{l-1} \longrightarrow \cdots \longrightarrow L_0 \]

is exact at \( L_l^0 \) for all \( l \) and \( q \leq 2s + 2 \).
Proof. The first part of 3.6 is clear from the definition of $L_t$. One easily checks that if $x \in A(BO)$, $|x| \leq 1$ and $d(xy^2) = 0$, then $x = 0$ and therefore $d : L_2 \to L_3^{q+1}$ is an injection for $q \leq 2s + 2$. $d : L_2 \to L_1$ is clearly onto. To check exactness at $L_2^q$, $q \leq 2s + 2$ one must verify that if $y = x_1y_1^1 + x_2y_2^1 + x_3y_3^1 + x_4v_3Sq^2\lambda^2 = 0$, $x_i \in A(BO)$ and $|y| \leq 2s + 3$, then $x_1 = x_3 = x_4 = 0$ and $x_2 = 0$ or $s$ is odd and $x_2 = Sq^1 + w_1$. This is a tedious but straightforward calculation, made somewhat simpler by the following observation. Let

$$F : A(BO) \otimes \{\lambda^n\} \to H^*(MO \wedge K(Z_2, N))$$

be given by

$$F(a \circ u^\lambda) = a(u^\lambda(Sq^{n+1})U \otimes t_N)$$

Then

$$F(y_1^1) = v_{s+1}U \otimes t_N + U \otimes Sq^{n+1}t_N$$
$$F(y_2^1) = U \otimes Sq^{n+2}t_N$$
$$F(v_3Sq^2\lambda^2) = v_3^2U \otimes Sq^2t_N$$

We leave the details to the reader.

§4. Proofs of 2.4 and 2.8

Let $\{A \otimes \Lambda^n_i, d\}$ be the chain complex described in Proposition 3.2.

PROPOSITION 4.1. For each integer $k$, there are $\Omega$-spectra $Y_t = Y_t(k)$ and maps $\rho_t = \rho_t(k) : Y_{t-1} \to K(\Lambda^k_i)$ of degree one, $l = 0, 1, 2, \ldots$ such that

(i) $Y_0 = K(\Lambda^0_k)$. $Y_t$ is a fibration over $Y_{t-1}$ induced by $\rho_t$ from the contractible fibration over $K(\Lambda^k_i)$.

(ii) If $i : K(\Lambda^k_{i-1}) \to Y_{t-1}$ is the inclusion of the fibre,

$$(\rho i)^* = d : A \otimes \Lambda^k_i \to A \otimes \Lambda^k_{i-1}$$

where $d$ is as in 3.2.

(iii) If $M$ is a smooth, compact $n$-manifold and $\nu$ is its normal bundle, then

$$[T(\nu), Y_t]_p \to [T(\nu), Y_{t-1}]_p$$

is an epimorphism for $p < 2k + 2$. 
Suppose $k = 0$. Let $I(l, 0) = (0, \ldots, 0)$ have length $l$.

$$
\rho_l^{*} \lambda^{I(l,0)} = \delta_l
$$

where $e \in H^0(Y_{l-1}; Z_2)$, $e$ reduced modulo two is the generator $e \in H^0(Y_{l-1}) \approx Z_2$ and $\delta_l$ is the Bockstein associated to $Z_2 \rightarrow Z_{2l+1} \rightarrow Z_2$.

Proof. For $k \geq 0$, 4.1(i), (ii), and (iii) were proved in [5]. For $k < 0$, $\{A \otimes \Lambda^k_l, d\}$ is a free acyclic resolution of the zero $A$ module so that the existence of $Y_l$ and $\rho_l$ easily follow by induction on $l$. If $M$ is as in (iii), $v : T(v) \rightarrow Y_{l-1}$ has degree $p$, $p < 2k + 2$ and $k < 0$, then $|\rho_l v^*(\lambda^l)| > n$ and (iii) follows.

Finally we prove (iv). The formula for $d$ in 3.1 shows that $d\lambda^{I(l,0)} = Sq^1 \lambda^{I(l-1,0)}$.

The complex,

$$
\cdots \rightarrow A \otimes \{\lambda^{I(l,0)}\} \rightarrow A \otimes \{\lambda^{I(l-1,0)}\} \rightarrow \cdots \rightarrow A \otimes \{\lambda^{I(0,0)}\}
$$

is realized by the tower

$$
\rightarrow K(Z_2) \rightarrow K(Z_{2l}) \rightarrow \cdots \rightarrow K(Z_2)
$$

with $k$-invariants, $\delta_l : K(Z_{2l}) \rightarrow K(Z_2)$. Except for $\lambda^{I(l,0)}$, the generators of $\Lambda^0_l$ have dimension $> 0$ and hence kill classes of dimension $> 1$. Thus $Y_l = K(Z_{2l+1})$ in dimensions $\leq 1$. Therefore (iv) holds.

Proof of 2.4: We wish to realize the complex $\{A \otimes U_\sigma, d\}$ by a tower of spectra, $X_\sigma$. Let $Y_l(k)$ and $\rho_l(k)$ be as in 4.1. For a spectrum $Z$, let $SZ$ denote the shift suspension, i.e., $(SZ)_q = Z_{q+1}$. Define $X_\sigma$ and $\alpha_\sigma : X_{\sigma-1} \rightarrow K(Y_l)$ by

$$
X_\sigma = \prod_{u_\sigma \in U_\sigma} S^{h_{u_\sigma}} Y_l([(n - |u_\sigma|)/2])
$$

$$
\alpha_\sigma = \prod S^{h_{u_\sigma}} \rho_l([(n - |u_\sigma|)/2])
$$

The map $\alpha_\sigma$ takes $X_{\sigma-1}$ into $K(U_l)$ since

$$
\prod S^k K(\Lambda^k) = K(U_l)
$$

where $k$ ranges over $[(n - |u_\sigma|)/2], |u_\sigma| \in U_\sigma$. Proposition 2.4 now follows directly from 4.1.
\textbf{Proof of 2.8:} Using induction on \( l \), one easily proves that if \( I \) is admissible and \( l = l(I) \),

\[
|\lambda^I| \geq 2t(I) \left( 1 - \frac{1}{2^l} \right)
\]

Suppose \( l \geq n \) and \( \lambda^I u_\omega \in U_l \). Then \( 2(t(I) + 1) > n - |u_\omega| \). Therefore

\[
|\lambda^I u_\omega| \geq 2t(I) \left( 1 - \frac{1}{2^l} \right) + |u_\omega| \geq n - 1 - \frac{n - |u_\omega| - 1}{2^l} > n - 2
\]

Also if \( |u_\omega| > n - 1 \), \( |\lambda^I u_\omega| > n - 1 \). If \( |u_\omega| < n - 1 \), \( t(I) \geq 1 \) and hence \( |\lambda^I| \geq l \geq n \).

Therefore \( U_l^q = 0 \) for \( q < n - 1 \) and \( U_l^{n-1} = \{ \lambda^I u_\omega \mid u_\omega \in U_0^{n-1} \} \) since \( \lambda^I \) is the only \( \lambda^I \) with \( t(I) \geq 0 \) and \( |\lambda^I| = 0 \). If \( r > l \) and \( \omega, \not\in \{ \}, |u_\omega| \geq |u_\omega^2| \geq 2^r > n \). Hence \( V_l^r = U_l^q \) for \( q \equiv n - 1 \).

By the definition of \( \beta_I : B_{l-1} \to K(V_l) \),

\[
\Phi(\beta_I^* (\lambda^I u_\omega)) = f_1^* \alpha_{l-1}^* (\lambda^I u_\omega)
\]

By 4.1(iv) \( \alpha_{l-1}^* (\lambda^I u_\omega) = \delta_{l-1}^* \) where \( \delta_{l-1}^* \in H^*(X_{l-1}; \mathbb{Z}_2) \) comes from the factor of \( X_{l-1}, Y([n - |u_\omega|/2]) \). Since the diagram

\[
\begin{array}{ccc}
T_{l-1} & \xrightarrow{f_{l-1}} & X_{l-1} \\
| & \quad \downarrow p_1 \quad | \\
T_0 & \xrightarrow{f_0} & X_0
\end{array}
\]

commutes, \( \bar{u} = f_{l-1}^* \bar{c} \) reduced modulo two is \( p_1^* f_0^* u_\omega = p_1^* u_\omega U_0 = u_\omega U_{l-1} \), where \( U_I \) is the Thom class of \( T_I \) and the proof of 2.8 is complete.

\section{Proof of 2.7}

If \( G_1 \) and \( G_2 \) are graded groups and \( h : G_1 \to G_2 \) is a homomorphism of degree \( i \), we will say that \( h \) is \( k \) connected if \( h : G_1^q \to G_2^{q+i} \) is an epimorphism for \( q < k \) and a monomorphism if \( q \leq k \). We will say that a sequence of graded groups and homomorphisms,

\[
\cdots \to G_1 \to G_{l-1} \to \cdots
\]
is $k$-exact if
\[ G^{q-i}_{l+1} \rightarrow G^q_l \rightarrow G^{q+i}_{l-1} \]
is exact for all $l$ and $q \leq k$.

In §3 we constructed isomorphisms $\theta : A \otimes U_l \rightarrow A(BO) \otimes V_l$ and a subcomplex $\{L_0, d\} \subset \{A(BO) \otimes V_l, d\}$ such that
\[ A(BO) \otimes V_l \rightarrow L_1 \rightarrow \cdots \rightarrow L_0 = 0 \]
is $2s + 2$ exact, $s = [n/2]$. In §4 we constructed a tower of fibrations $\rightarrow X_i \rightarrow X_{i-1} \rightarrow \cdots$ with $k$-invariants $\alpha_i : X_{i-1} \rightarrow K(U_i)$ associated to the complex $\{A \otimes U_i, d\}$. Let
\[ \tilde{H}^*(K(U_i)) = H^*(K(U_i))/\theta^{-1}(L_4) \]
\[ \tilde{H}^*(X_i) = H^*(X_i)/\alpha_i^* \theta^{-1}(L_{i-1}) \]

**LEMMA 5.1:** The maps
\[ K(U_i) \xrightarrow{i} X_i \xrightarrow{p} X_{i-1} \xrightarrow{\alpha_i} K(U_i) \]
induce a $2s + 2$-exact sequence
\[ \rightarrow \tilde{H}^*(K(U)) \rightarrow \tilde{H}^*(X_{i-1}) \rightarrow \tilde{H}^*(X_i) \rightarrow \]

**Proof:** Let $E_i$ be the kernel of
\[ H^*(X_i) \rightarrow \lim_{k \rightarrow \infty} H^*(X_k) \]
Then $H^*(X_i) \approx H^*(MO)/\Phi(I_n) \oplus E_i$ and $E_i$ and $A \otimes U_i$ are related by the diagram
\[ \begin{array}{ccc}
\rightarrow & A \otimes U_1 & \rightarrow A \otimes U_{i-1} & \rightarrow A \otimes U_{i-2} \\
\uparrow & \alpha_i & \alpha_i & \alpha_i \\
E_{i-1} & 0 & E_{i-2} & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array} \]
where the $\tilde{\alpha}_i$ and $\tilde{\epsilon}_i$ are defined by $\alpha_i^*$ and $\epsilon_i^*$ and each pair of composable arrows is exact. Dividing $A \otimes U_i$ and $E_{i-1}$ by $\theta^{-1}(L_i)$ and $\tilde{\alpha}_i\theta^{-1}(L_{i-1})$, respectively, produces the same type of diagram with exactness replaced by $2s+2$-exactness. The desired result then follows.

In §2 we defined maps

$$g_i^*: K(U_i) \to T_{i-1}/T_i$$

In §6 we prove:

**Lemma 5.2.** The map $g_i^*$ induces a $2s+2$-connected map

$$F_i : \tilde{H}^*(K(U_i)) \to H^*(T_{i-1}/T_i)$$

for $l \geq 1$.

**Proof of 2.7:** We first prove 2.7(ii). Suppose $M$ is a smooth $n$-manifold, $h : M \to B_0 = BO$ classifies $\nu$, the normal bundle of $M$ and $\tilde{h} : M \to B_{i-1}$ is a lifting of $h$. Let $T(\tilde{h}) : T(\nu) \to T_{i-1}$ denote the associated Thom space map. Then $f_{i-1}T(\tilde{h}) : T(\nu) \to X_{i-1}$ is a lifting of $f_0T(h) : T(\nu) \to X_0$ and hence by 2.4(iv), $f_{i-1}T(\tilde{h})$ lifts to $X_i$ and therefore $\alpha_{f_{i-1}}T(h) = 0$. Thus for $v \in \tilde{V}_i$

$$\Phi h^*\beta_i^*(v_1) = T(\tilde{h})^*\Phi(\beta_i^*(v_1)) = T(\tilde{h})^*f_{i-1}^*\alpha_i^*(v) = 0$$

Thus $\beta_i\tilde{h} = 0$ and $\tilde{h}$ lifts to $h' : M \to B'_i$.

If $u \in U_{i+1}$, $\tilde{u} = \{u\} \in W_{i+1} = U_{i+1}/\ker \Delta$ and $\nu \theta(u) = \sum x_iu_i$,

$$\Phi((h')^*\gamma_i^*(\tilde{u}_i)) = T(h')^*\Phi(\gamma_i^*\tilde{u}_i) = T(h')\Delta(u).$$

Recall,

$$\Delta(u) = (f_{i+1}^*)\alpha_{i+1}^*u - \sum x_i(f_{i+1}^*)^*\alpha^*u_i$$

But $T(h')^*$ is $A(BO)$ linear and $\alpha_{i+1}f'T(h') = 0$ as above. Thus $T(h')^*\Delta(u) = 0$ and hence $\gamma h' = 0$. Therefore $h'$ lifts to $B_i$ and the proof of 2.7(ii) is complete. We note for further reference:

**Lemma 5.3:** $T(h')^*\Delta(u) = 0$ for $u \in U_{i+1}$.

**Lemma 5.4.** If $\delta^* : H^*(T_i') \to H^*(T_{i-1}/T_i')$, $\delta^*\Delta(u) = 0$ for $u \in U_{i+1}$.
Proof. Consider the commutative diagram:

$$
H^*(X_i) \xrightarrow{i^*} H^*(K(U_i)) \\
\downarrow f_i \downarrow \xi \o i \o d \\
H^*(T'_i) \xrightarrow{\alpha^*} H^*(T_{i-1}/T'_i)
$$

Recall, $g_0'$ realizes $\Psi \theta$, $i^* \alpha_{i+1}^* = d$ and $\Psi$, $r$, and $d : A(BO) \otimes V_{i-1} \to A(BO) \otimes V$ are $A(BO)$ linear. Hence,

$$
\delta^* \Delta(u) = \delta^*((f_i')^* \alpha_{i+1}^* u + \sum x_i (f'_i)^* \alpha_{i+1}^* u_i) \\
= (g_i')^* i^* \alpha_{i+1}^* u + \sum x_i (g'_i)^* i^* \alpha_{i+1}^* u_i \\
= \Psi r \theta du + \sum x_i \Psi r \theta du_i = \Psi \theta du + \sum \Psi r dx_i \theta (u_i)
$$

where $r(\theta(u)) = \sum x_i u_i$, $x_i \in A(BO)$ and $u_i \in V_{i+1}$. But for $v \in V_{i+1}$, $\theta(v) = v$. Thus

$$
\sum x_i \theta (u_i) = \sum x_i u_i = r \theta u = \theta u + z
$$

where $z \in L_{i+1}$. Furthermore $dz \in L_i$. Hence $\delta^* \Delta(u) = \Psi rdz = \Psi \theta^{-1} dz = (g')^* \theta^{-1} dz$.

But by 5.2, $\theta^{-1}(L_i)$ is the kernel of $(g'_i)^*$.

We now prove that $f_i$ induces a $2s + 2$-connected map $f_i^* : H^*(X_i) \to H^*(T'_i)$ by induction on $l \geq 0$. We first show that $f_i^*$ is well defined.

$$
H^*(X_i) = H^*(X_i)/\alpha_{i+1}^*(\theta^{-1}(L_{i+1}))
$$

From the commutative diagram:

$$
\begin{array}{ccc}
T_i & \xrightarrow{i} & T_i/T_{i+1} \\
\downarrow f_i & & \downarrow \xi_{i+1} \\
X_i & \xrightarrow{\alpha_{i+1}} & K(U_{i+1})
\end{array}
$$

we see that

$$
f^* \alpha_{i+1}^*(\theta^{-1}(L_{i+1})) = j^* (g'_{i+1})^*(\theta^{-1}(L_{i+1}))
$$

By 5.2, $\theta^{-1}(L_{i+1})$ is in the kernel of $(g'_{i+1})^*$. 
Since \( f_0^* \) is an isomorphism, \( \tilde{f}_0 = f_0^* \) and \( \tilde{f}_0 \) is an isomorphism.

Suppose \( \tilde{f}_{i-1} \) is \( 2s+2 \) connected. If \( u \in U_{i+1} \), \( \Delta(u) \in H^q(T_i) \) pulls back to \( H^q(T_{i-1}) \) since, by 5.4, \( \delta^* \Delta(u) = 0 \) and it pulls back to \( H^q(X_{i-1}) \) if \( q < 2s + 2 \), that is, if \( |u| < 2s + 1 \), \( \Delta(u) = (f_i)^* p^* x \) where \( p: X_i \to X_{i-1} \). But since the \( X_i \)'s are constructed from an acyclic complex, image \( p^* \) = image \( (H^*(X_0) \to H^*(X_i)) \). Therefore image \( (f_i)^* p^* = \text{image} \ (H^*(T_0) \to H^*(T_i)) = H^*(MO) / \Phi(I_n) \). But by 5.3, \( \Delta(u) \) is zero on all \( n \)-manifolds. Hence \( \Delta(u) = 0 \) and we have shown that \( W_{i+1} = (U_{i+1} / \ker \Delta)^a = 0 \) for \( q < 2s + 2 \). Therefore \( H^a(B_i) \to H^a(B_i) \) is an isomorphism for \( q \leq 2s + 2 \) since \( B_i \) is a fibration over \( B_{i-1} \) induced by \( \gamma_i : B_i \to K(W_{i+1}) \). Then \( H^a(T_i/T_i) = H^a(B_i, B_i) = 0 \) for \( q < 2s + 2 \) and hence

\[
H^*(T_{i-1}/T_i) \to H^*(T_{i-1}/T_i)
\]
is \( (2s+2) \)-connected. Let \( g_i \) be the composition

\[
T_{i-1}/T_i \to T_{i-1}/T_i \xrightarrow{\pi} K(U_i)
\]
and let \( \tilde{g}_i : \tilde{H}^*(K(U_i)) \to H^*(T_{i-1}/T_i) \) be induced by \( g_i \). Then \( \tilde{g}_i \) is \( (2s+2) \)-connected by 5.2. Consider the commutative diagram:

\[
\begin{array}{cccccc}
\longrightarrow & \tilde{H}^*(K(U_i)) & \longrightarrow & \tilde{H}^*(X_{i-1}) & \longrightarrow & \tilde{H}^*(X_i) & \longrightarrow & \tilde{H}^*(K(U)) \\
\downarrow{\tilde{g}} & \downarrow{\tilde{f}_{i-1}} & \downarrow{\tilde{f}} & \downarrow & \downarrow & \downarrow & \downarrow \\
\longrightarrow & H^*(T_{i-1}/T_i) & \longrightarrow & H^*(T_{i-1}) & \longrightarrow & H^*(T_i) & \longrightarrow & H^*(T_{i-1}/T_i)
\end{array}
\]

A five lemma argument and the fact that \( \tilde{f}_{i-1} \) and \( \tilde{g}_i \) are \( (2s+2) \)-connected shows that \( \tilde{f}_i \) is \( 2s+2 \)-connected.

Since \( L_4 = 0 \) for \( l > 3 \), \( \tilde{H}^*(X_i) = H^*(X_i) \) for \( l \geq 3 \) and therefore \( f_i^* : H^q(X_i) \to H^q(T_i) \) is an isomorphism for \( q \leq n < 2s + 2 \). This completes the proof of 2.7.

§6. Proof of 5.2

LEMMA 6.1.

\[
H^q(B_{i-1}) \to H^q(B_i)
\]
is an isomorphism for \( l > 1 \) and \( q \leq s + 1 \). For \( l = 1 \) it is an epimorphism for \( q \leq s + 1 \) and \( v_{s+1}, w_1 v_{s+1}, S q^1 v_{s+1} \) and \( v_{s+2} \) generate the kernel for \( q \leq s + 2 \).
Proof. As we saw in the proof of 2.8, if $\lambda^l u_0 \in V_l$, $|\lambda^l u_0| \geq (n-1) - (n - |u_0| - 1)/2$. Hence the lowest dimensional element in $V_l$ is of the form $\lambda^l$ with $t(I) = s$. For such an $I$, $|\lambda^l| \geq s + 2$ except for $l = 1$ or $l = 2$ and $s = 1$ and 2. The space $B_l'$ is a fibration over $B_{l-1}$ induced by $\beta_l : B_{l-1} \to K(V_l)$ and for $l > 1$, $K(V_l)$ is $s + 2$ connected except when $l = 2$ and $s = 1$ or 2. For $s = 1$ or 2, the lowest dimensional elements in $V_2$ are $\lambda^{1,1}$ and $\lambda^{1,2}$ respectively; $d\lambda^{1,1} \neq 0$ and $d\lambda^{1,2} \neq 0$ so these elements kill nonzero classes in $B_l$. Thus for $l > 1$, $H^q(B_{l-1}) \approx H^q(B_l')$ for $q \leq s + 1$.

Suppose $l = 1$. From 3.1 one sees that $d\lambda^l = \chi(Sq^{l+1})U = \Phi(v_{l+1})$ where $U$ is the Thom class and $v_{l+1}$ is the Wu class. Hence $\beta_1 : B_0 \to K(V_1)$ takes $\lambda^l$ into $v_{l+1}$. One easily checks that $V_0^q = 0$ for $q < s$, $V_1^q = \{\lambda^s\}$ and $V_2^q = \{\lambda^{s+1}\}$. The remainder of 6.1 now follows by a simple Serre spectral sequence argument.

Let $K_l = K(V_l)$. Viewing $\beta_l : B_{l-1} \to K_l$ as a fibre map with fibre $B_l'$, consider the pair of fibrations $p_1$ and $p_2$:

$$(B_{l-1}, B_l') \xrightarrow{c} (B_{l-1} \times K_l, B_{l-1} \times \{\ast\})$$

where $p_1$ is defined by $\beta_l$, $p_2$ is projection on the second factor and $c = id \times p$. Note $c$ is a fibre preserving map so we may use it to compare the Serre spectral sequences of $p_1$ and $p_2$.

LEMMA 6.2. For $l > 1$, $c^* : H^q(B_{l-1} \times K_l, B_{l-1} \times \{\ast\}) \to H^q(B_{l-1}, B_l')$ is an isomorphism for $q \leq 2s + 3$. For $l = 1$, $c^*$ is an epimorphism for $q \leq 2s + 2$ and for $q \leq 2s + 3$ the kernel is generated by

- $v_{l+1} \otimes \lambda^s + 1 \otimes (\lambda^s)^2$
- $v_{l+1} \otimes Sq^1 \lambda^s + 1 \otimes \lambda^s Sq^1 \lambda^s$
- $v_{l+1} \otimes \lambda^{s+1} + 1 \otimes \lambda^s \lambda^{s+1}$
- $w_1 v_{l+1} \otimes \lambda^s + w_1 \otimes (\lambda^s)^2$
- $Sq^1 v_{l+1} \otimes \lambda^s + 1 \otimes \lambda^s Sq^1 \lambda^s$
- $v_{l+2} \otimes \lambda^s + 1 \otimes \lambda^s \lambda^{s+1}$

Proof. Let $E_r^{p,q}$ and $\tilde{E}_r^{p,q}$ denote the Serre spectral sequences for $p_1$ and $p_2$ respectively.

- $E_2^{p,q} = H^p(K_l, \ast) \otimes H^q(B_{l-1})$
- $\tilde{E}_2^{p,q} = H^p(K_l, \ast) \otimes H^q(B_l')$
As we saw above, for \( l > 1 \), \( K_l \) is \( s+2 \) connected and \( H^q(B_{l-1}) \approx H^q(B_l) \) for \( q \leq s + 1 \). Therefore \( c \) induces an isomorphism at the \( E_2 \) level for \( p + q \leq 2s + 3 \) and the differentials are trivial for \( p_2 \) because it is a product fibration. This proves 6.2 for \( l > 1 \).

For \( l = 1 \), 6.2 is true at the \( E_2 \) level with the first summands in the above list of elements as a basis for the kernel; the second summands are of lower filtration. The same is true at the \( E_\infty \) level, so to complete the proof, we must show that these elements are in the kernel of \( c^* \).

Under the map \( H^*(B_0, B'_1) \to H^*(B_0) \), \( c^*(1 \otimes \lambda^1_i) \) goes to \( v_{s+1} \). Hence

\[
c^*(v_{s+1} \otimes \lambda^1_i + 1 \otimes (\lambda^1_i)^2) = v_{s+1}c^*(1 \otimes \lambda^1_i) + c^*(1 \otimes \lambda^1_i)^2 = 0
\]

(If \( j : X \subset (X, A) \) and \( x \in H^*(X, A) \), \( x^2 = (j^*x)x \).) The same argument applies to the other five elements.

Let

\[
\phi : (A(BO) \otimes \tilde{V}_l)^{\alpha} \to H^{q+1}(T_{l-1} \wedge K_l)
\]

be defined by

\[
\phi((a \otimes w)u) = a(wU \otimes u_1)
\]

where \( U \) is the Thom class, \( a \in A, w \in H^*(BO) \) and \( u \in \tilde{V}_l \).

**Lemma 6.3.** For \( q \leq 2s + 1 \), \( \phi \) is an epimorphism. For \( q \leq 2s + 2 \) the kernel of \( \phi \) is zero for \( l > 1 \) and \( (l, s) \neq (2, 2) \), is \( \{v_3 \lambda^{1,2} \} \) for \( (l, s) = (2, 2) \) and is \( \{(\sum S^l q \circ v_{s+2-l})\lambda^s \} \) for \( l = 1 \).

**Proof.** Let \( \mu, \mu' : A(BO) \to A(BO) \) be defined by

\[
\mu(a \circ w) = \sum a_i \circ ws(a_i)
\]

\[
\mu'(a \circ w) = \sum a_i \circ w\chi(a_i)
\]

(Recall, \( w\alpha \) is defined by \( (w\alpha)/U - \chi(a)(wU). \))

Where \( a \to \sum a_i' \otimes a''_i \) in the diagonal in \( A \). Then \( \mu\mu' = \mu'\mu = \text{identity} \) and thus \( \mu \) is a \( Z_2 \)-isomorphism. Let \( \phi' = \phi(\mu \otimes id) \). Then

\[
\phi'(a \circ w)u = \sum a_i'(\chi(a_i)(wU) \otimes u_1) = wU \otimes au_1
\]

Let \( \lambda^l \) be the lowest dimensional element in \( \tilde{V}_l; |\lambda^l| > s \) for \( l = 1 \). The lowest
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dimensional element in \( H^*(T_{l-1} \wedge K_1) \) not in the image of \( \phi' \) is \( U \otimes (\lambda_1^s \cup Sq^1 \lambda_1^s) \), an element of dimension \( \geq 2s + 3 \). Hence \( \phi \) is an epimorphism for \( q < 2s + 2 \). The lowest dimensional elements in the kernel of \( \phi' \) are \( 1 \circ v_{s+1} \lambda_1^s \) or \( (Sq^m \circ 1) \lambda_1^s \) where \( m = |\lambda_1^s| + 1 \). For \( l > 2 \), \( (l, s) \neq (2, 2) \), \( \lambda_1^s > s + 1 \) and hence these elements occur in dimensions \( > 2s + 3 \). For \( (l, s) = (2, 2) \), \( \phi(v_3 \lambda^{1,2}) = \phi'(v_3 \lambda^{1,2}) = 0 \). For \( l = 1 \)

\[
0 = \phi'((Sq^{s+2} \circ 1) \lambda^s) = \phi\left(\left(\sum Sq^i \circ v_{s+2-i}\right) \lambda^s\right)
\]

This proves the last part of 6.3.

**Proof of 5.2:** We must show that

\[
(g'_f)^* = \Psi T \theta : (A \otimes U_l)^q \to H^{q+1}(T_{l-1}/T_l'
\]

is an epimorphism for \( q \leq 2s + 2 \) and \( (L_q)^q \) is the kernel for \( q \leq 2s + 2 \). By 2.5, \( \theta \) is an isomorphism. Let \( \phi \) be the map in 6.3 and \( c \) the map in 6.2. Lifting \( c \) to the Thom space level we obtain a map

\[
T(c) : T_{l-1}/T_l' \to T_{l-1} \wedge K_l
\]

Furthermore \( \Psi = T(c)^* \). Thus by 6.2 and 6.3, \( \Psi \) is an epimorphism for \( q \leq 2s + 1 \) and since \( r \) is an epimorphism, \( (g'_f)^* \) is an epimorphism for \( q \leq 2s + 1 \). For \( l > 1 \) and \( (l, s) \neq (2, 2) \), \( T(c)^* \) and \( \phi \) are monomorphisms for \( q \leq 2s + 2 \) and \( L_q^s \) is the kernel of \( r \). When \( (l, s) = (2, 2) \) \( r(L_q^s) = \{v_3 \lambda^{1,2}\} \). This completes the proof of 5.2 for \( l > 1 \).

Suppose \( l = 1 \). Then \( r = \text{identity} \). We wish to show that \( L_1 = \phi^{-1}(\ker T(c)^*) \). In 6.2 a basis for \( \ker c^* \) was given for \( q \leq 2s + 2 \). Since image \( \phi = \text{image} \phi' \) cannot involve cup products (except squares) in \( H^*(K_l) \), the above basis shows that the following is a basis for image \( \phi \cap \ker T(c)^* \):

\[
\begin{align*}
&v_{s+1} U \otimes \lambda_1^s + U \otimes Sq^{s+1} \lambda_1^s \\
&w_1 v_{s+1} U \otimes \lambda_1^s + w_1 U \otimes Sq^{s+1} \lambda_1^s \\
&v_{s+1} U \otimes Sq^1 \lambda_1^s + (Sq^1 v_{s+1}) U \otimes \lambda_1^s \\
&v_{s+1} U \otimes \lambda_1^{s+2} + v_{s+2} U \otimes \lambda_1^s
\end{align*}
\]

Thus a basis for \( \phi^{-1}(\ker c^*) \) is \( \phi^{-1} \) of these elements and \( (\sum Sq^i \circ v_{s+2-i}) \lambda^s \) from the kernel of \( \phi \). A simple calculation shows that these elements form a basis for \( L_1^s, q \leq 2s + 2 \), completing the proof of 5.2.
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Received June 30, 1978