A Class of Solvable Potentials (*)

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Summary. — The problem of the construction of solvable one-variable Schrödinger potentials is formulated. A class of simple potentials for which the Schrödinger equation can be solved in terms of special functions of physics is constructed.

1. — Introduction.

A study of solvable potentials is of considerable interest (1,2). In ref. (2) several known solvable potentials were constructed and the analytic structure of some of the corresponding scattering amplitudes studied. This paper presents a more systematic method of construction of solvable potentials. A short summary of this work has already been given (2a).

In Sec. 2 the problem of the construction of solvable potentials is formulated in a natural way. In Sec. 3 a class of functions is introduced to transform the hypergeometric and the confluent hypergeometric equations to the form of a Schrödinger equation and thus a class of one-variable solvable Schrödinger potentials—in fact, the known solvable ones—is obtained in a unified manner. Section 4 discusses the application of the method to other linear equations of mathematical physics.

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2. – Formulation of the problem.

Consider a one-variable Schrödinger equation

$$\frac{d^2 \psi}{dx^2} + (k^2 - U(x)) \psi = 0.$$  

It is convenient to call $k^2$ an energy and $U$ a potential.

The formulation of the problem involves the transformation of a linear differential equation of second order

$$\frac{d^2 u}{dz^2} + p(z) \frac{du}{dz} + q(z) u = 0$$

to the form of eq. (1). Now, eq. (2) can be transformed by the substitution

$$u = v \exp \left[-\frac{1}{2} \int p(z) dz \right]$$

to the normal form

$$\frac{d^2 v}{dz^2} + I(z) v = 0,$$

where the invariant $I(z)$ is given by

$$I(z) = q(z) - \frac{1}{2} \frac{dp(z)}{dz} - \frac{1}{4} (p(z))^2.$$

Consequently we proceed to transform the normal form (3) to the Schrödinger normal form (1). Changing the independent variable in eq. (3) from $z$ to $x$ and then reducing, as before, the resulting equation, one easily obtains the transformed normal form

$$\frac{d^2 f}{dx^2} + I_t(x) f = 0,$$

where

$$v = (z')^2 f \quad \left( z' = \frac{dx}{dz} \right),$$

$$I_t = z'^2 I(z) + \frac{1}{2} \{z, x\},$$

with

$$\{z, x\} = \frac{d^2 \log z'}{dx^2} - \frac{1}{2} \left( \frac{d \log z'}{dx} \right)^2 = \frac{z^w}{z^t} - \frac{3}{2} \left( \frac{z'}{z} \right)^2.$$
The quantity (7) is known as the Schwarzian derivative of x with respect to z. We now define a Schrödinger invariant (S.I. hereafter) $I_s$ by

\begin{equation}
I_s(x) = k^2 - U(x),
\end{equation}

where $k^2$ is a purely parametric term. Clearly eq. (4) is of the form of eq. (1) if the transformed invariant $I_t$ is a S.I.

Thus the problem of the construction of potentials for which eq. (1) is solvable in terms of functions corresponding to a given $I(z)$ (cf. eq. (3)) is the problem of determining transformations $z(x)$ such that the relation

\begin{equation}
\frac{d^2}{dz^2} I(z) + \frac{1}{2} \{z, x\} = I_s(x)
\end{equation}

holds. A S.I. obtained in this manner is thus characterized by two elements, namely, $I(z)$ and $z(x)$. The purely parametric term of a S.I. is to be identified as the energy and the variable term with the sign changed as the solvable potential. As is evident from the method of construction, a potential so constructed is not necessarily an acceptable potential.

We conclude this Section by listing (3) the following properties of the Schwarzian derivative. This quantity is the differential invariant of the linear group, i.e.,

\begin{equation}
\begin{bmatrix} Az + B \\ Cz + D \end{bmatrix} = \{z, x\},
\end{equation}

where $A, B, C, D$ are constants such that

\begin{equation}
AD - BC \neq 0.
\end{equation}

The quotient $s$ of two linearly independent solutions of eq. (2) satisfies the nonlinear Schwarzian equation

\begin{equation}
\{s, z\} = 2 I.
\end{equation}

Hence all equations of the form (2) with the same invariant lead to the same Schwarzian equation. It is noteworthy that eq. (9) can be obtained directly from eq. (11) by changing the independent variable in the Schwarzian derivative from $z$ to $x$.


3. – A class of solvable potentials.

In this Section we construct a class of Schrödinger potentials for which eq. (1) is solvable in terms of special functions of physics. It is well known that these functions are related to the hypergeometric and the confluent hypergeometric equations.

3'1. Invariants of Riemann and of Whittaker. – The hypergeometric equation has singularities at 0, 1 and ∞, each of them being regular. The corresponding normal form is a Riemann equation (\(\xi\)) with the invariant

\[
I_\xi = \frac{1}{4} \left[ \frac{1 - \lambda^2}{z^2} + \frac{1 - \nu^2}{(1 - z)^2} + \frac{1 + \mu^2 - \lambda^2 - \nu^2}{z(1 - z)} \right].
\]

The parameters \(\lambda, \mu, \nu\) are related to the exponent differences. The complete set of solutions of the normal form is given by

\[
v = P \begin{pmatrix} 0 & 1 & \infty \\ \frac{1}{2}(1 - \lambda) & \frac{1}{2}(1 - \nu) & -\frac{1}{2}(1 + \mu) \\ \frac{1}{2}(1 + \lambda) & \frac{1}{2}(1 + \nu) & -\frac{1}{2}(1 - \mu) \end{pmatrix},
\]

where \(P\) denotes the Riemann \(P\)-function.

The confluent hypergeometric equation (\(\xi\)) is obtained by the confluence of the two singularities of the Riemann equation. The singularities of this equation are at 0 and ∞, the former being regular and the latter irregular. The normal form is the Whittaker equation with the invariant

\[
I_\omega = \left[ -\frac{1}{4} + \frac{\lambda}{z} + \frac{1 - \mu^2}{z^2} \right].
\]

When \(2\mu \neq \text{integer}\), two integrals of the Whittaker equation which are regular near 0 are given by

\[
v = z^{1+\mu} \exp\left[-\frac{1}{2}z\right], \, F_1(\frac{1}{2} \pm \mu - \lambda; 1 \pm 2\mu; z).
\]

Here, \(F_1(\cdot; \cdot; z)\) denotes a confluent hypergeometric function. We note that if \(\lambda = 0\) the Whittaker equation can be reduced to the Bessel equation; in

fact, one has
\[ J_\mu(z) = \frac{1}{I(\mu + 1)} \left( \frac{z}{2} \right)^\mu \exp \left[ -iz \right] \frac{1}{\Gamma(\mu)} \frac{1}{1 + 2\mu} \frac{1}{2i\mu} \frac{1}{2i\mu} \frac{1}{2i\mu} . \]

3.2. A class of transformations. - We have now to determine functions that can transform \( I \) and \( J_\mu \) to \( I_s \). The forms of the invariants at once suggest the class of functions defined by

\[ (16) \quad z'z = \alpha z^2 + \beta z + \gamma . \]

Here \( \alpha, \beta, \gamma \) are arbitrary constants. From eq. (16) one obtains

\[ (17) \quad \{z, x\} = -\frac{x^2}{2} - \frac{3\beta^2 - 4\gamma x}{8(\alpha z^2 + \beta z + \gamma)} . \]

It is easy to see that the functions (16) form a class of possible transformations.

The integrals of eq. (16) are five types of elementary functions. By choosing the constants \( \alpha, \beta, \gamma \) and the constants of integration suitably, we write them in the following simple forms

\[ (18) \quad \begin{align*}
  \text{i)} & \quad z_1 = g \exp[ \pm 2\alpha x] \quad (\beta = \gamma = 0, \; \alpha = 4a^2) , \\
  \text{ii)} & \quad z_2 = -\sinh^2 \alpha x \quad (x + \beta = \gamma = 0, \; x = 4a^2) , \\
  \text{iii)} & \quad z_3 = \sin^2 \beta x \quad (x + \beta = \gamma = 0, \; x = 4b^2) , \\
  \text{iv)} & \quad z_4 = e \beta x \quad (x = \gamma = 0, \; \beta = 4\epsilon) , \\
  \text{v)} & \quad z_5 = d x \quad (x = \beta = 0, \; \gamma = d^2) ,
\end{align*} \]

where \( g = \) a constant.

The property (10) of the Schwarzian derivative suggests the application of linear transformations on the functions (18) to obtain other possible transformations. Consider

\[ (19) \quad z_5 = \frac{Az + B}{Cz + D} , \]

where \( z \) satisfies eq. (16). Differentiating eq. (19) with respect to \( x \) and eliminating \( z \) we obtain

\[ (20) \quad \left( \frac{dz_5}{dx} \right)^2 = \frac{(Cz_5 - A)^2}{(BC - AD)^2} [x(B - Dz_5)^2 + \beta(B - Dz_5)(Cz_5 - A) + \gamma(Cz_5 - A)^2] . \]

It is easy to see that the solutions of eq. (20), which is a generalization of
eq. (16), are possible transformations. We observe that if the expression on
the right-hand side of eq. (20) has more than two roots, then one of them
must be a multiple root.

Thus we have found a class of functions for transforming $I_\omega$ and $I_\omega$ to $I_\omega$. It is noteworthy that this class of functions can be characterized differently. Generalizing eq. (16) let us write

$$z'^2 = R(z),$$

where $R$ denotes a rational function. The integrals of this equation, which is
of the Briot and Bouquet (6) type, have no movable branch points only if
$R(z)$ is a polynomial of degree 4 at most. Furthermore, if $R(z)$ is a polynomial
of degree 3 or 4 with distinct roots, the integrals are elliptic functions. Thus,
if one requires the transformations to be elementary functions one is at once
led to eq. (20).

3.3. Construction of simple Schrödinger invariants. — We now proceed to
construct simple Schrödinger invariants corresponding to the Riemann and
the Whittaker invariants with the aid of the transformations (20).

I) Riemann case. We begin with the simpler class of transforma-
tions (16). Inserting the expressions (12), (16) and (17) in eq. (9) we obtain

$$I_\omega = -\frac{1}{4} x\mu^2 + \frac{\gamma (1 - \lambda^2)}{4z^2} + \frac{(\beta + \gamma)(1 - \lambda^2) + \gamma (\mu^2 - \nu^2)}{4z} +$$

$$+ \frac{(x + \beta + \gamma)(1 - \nu^2)}{4(1 - z)^2} + \frac{(x + \beta + \gamma)(\mu^2 - \lambda^2) + (\gamma - \alpha)(1 - \nu^2)}{4(1 - z)} -$$

$$- \frac{3}{16} \frac{\beta^2 - 4xy}{(xz^2 + \beta z + \gamma)}.$$

Clearly $-\frac{1}{4} x\mu^2$ is the energy term. Note that then $\alpha \neq 0$. The form of
the potential is determined by the form $z(x)$ i.e., by $\alpha(\neq 0), \beta, \gamma$. We wish
to construct simple potentials. Clearly each of the choices

$$I_\omega = \gamma = 0, \quad x + \beta = \gamma = 0, \quad x + \beta + \gamma = 0, \quad x = \gamma,$$

yields a two-term potential. However, the choice iii) is essentially equivalent
to the choice i). Corresponding to the choices i)–ii) the transformations are

(6) E. L. INCE: Ordinary Differential Equations (New York, 1956), Sect. 13.8;
L. Bieberbach: Theorie der gewöhnlichen Differentialgleichungen (Berlin, 1953),
Sect. 4-5.
given by ((18), i–iii)). Inserting them in eq. (21) we obtain the following three Schrödinger invariants:

\[
\begin{aligned}
\text{i)} & \quad I_s(r1) = a^2 \left[ -\mu^2 - 1 + \lambda^2 - \mu^2 - v^2 \right] \\
& \quad = a^2 \left[ -\lambda^2 - (\lambda^2 - \mu^2) \frac{g \exp[2ax]}{1 - g \exp[2ax]} - (v^2 - 1) \frac{g \exp[2ax]}{1 - g \exp[2ax]^2} \right], \\
\text{ii)} & \quad I_s(r2) = a^2 \left[ -\mu^2 - \frac{1}{\sinh^2 ax} - \frac{1}{\cosh^2 ax} \right], \\
\text{iii)} & \quad I_s(r3) = b^2 \left[ \mu^2 - \frac{\lambda^2 - \frac{1}{4}}{\sin^2 bx} - \frac{v^2 - \frac{1}{4}}{\cos^2 bx} \right].
\end{aligned}
\]

(23)

We have introduced the symbol \((rn)\) to characterize the above invariants, \(r\) referring to \(I_r\) and \(n\) to \(z_n\). The corresponding solutions are obtained by inserting the transformations (16) and the function (13) in eq. (5).

We recognize the potential in the expression ((23), i)) as an Eckart potential \((7)\). Further, for \(v = 1\) it can be reduced to the Hulthén \((8)\) or the Woods-Saxon \((9)\) potential. The potentials corresponding to eqs. ((23), ii–iii)) were introduced by Pöschl and Teller \((10)\). The invariant of the reduced Schrödinger equation for the symmetric top \((11)\), which is given by

\[ I = \frac{1}{1 + \frac{2A}{\hbar^2} E - K^2} \left( \frac{1}{C} - 1 \right) - \frac{1}{4} \left[ \frac{(K - M)^2 - \frac{1}{4}}{\sin^2 (\beta/2)} + \frac{(K + M)^2 - \frac{1}{4}}{\cos^2 (\beta/2)} \right], \]

is of the type (23), iii).

We now turn to the transformations of the type (20). From the form of \(I_r\) one sees that the simplest choice is

\[
(24) \quad z' = ax^2(1 - z)^2.
\]

The function

\[
(25) \quad z_6 = \frac{\exp[2x]}{1 + \exp[2x]} = \frac{1}{2} (1 + \tgh x)
\]

[cf. eq. (18-i)] satisfies eq. (24) with \(z = 4\).

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Inserting the expressions (12) and (25) in eq. (9) and using the property (10) and the definition (7) we obtain

\[(26) \quad I_s(r) = -\frac{1}{2}(\nu^2 + \lambda^2) - \frac{1}{2}(\nu^2 - \lambda^2) \tgh x - \frac{1}{4}(1 - \mu^2) \sech^2 x.\]

The potential in eq. (26), which is an Eckart potential, was introduced by Rosen and Morse \((13)\).

We wish to make here a remark. We know that a linear transformation changes a Riemann equation into another Riemann equation \((12)\). One may utilize this property to write the above Schrödinger invariants and hence the potentials in different forms. In this connection we note that for special values of the parameters Riemann equations can undergo quadratic and higher transformations \((14)\).

II) Whittaker case. Inserting the expressions (14), (16) and (17) in eq. (9) we obtain

\[(27) \quad I_s = -\frac{\alpha x^2}{4} + \left(\alpha \lambda - \frac{\beta^2}{4}\right) z + \left(\beta \lambda - \alpha \mu^2 - \frac{\gamma}{4}\right) + \frac{\gamma \lambda + \beta(\frac{1}{2} - \mu^2)}{z} + \frac{\gamma(\frac{1}{4} - \mu^2)}{z^2} - \frac{3}{16} \frac{\beta^2 - 4a\gamma}{(\alpha z^2 + \beta z + \gamma)}.\]

Clearly each of the choices

\[i) \quad \beta = \gamma = 0, \quad ii) \quad \alpha = \gamma = 0, \quad iii) \quad \alpha = \beta = 0,\]

yields a two-term potential. The corresponding transformations are given by eq. (18), i), iv), v).

Proceeding as before we obtain

\[
\begin{aligned}
&i) \quad I_s(w1) = 4a^2 \left[-\mu^2 + g\lambda \exp[-2ax] - \frac{g^2}{4} \exp[-4ax]\right], \\
&ii) \quad I_s(w4) = 4c\lambda - c^2 x^2 - \frac{(16\mu^2 - 1)}{4x^2}, \\
&iii) \quad I_s(w5) = -\frac{d^2}{4} + \frac{d\lambda}{x} - \frac{(\mu^2 - \frac{1}{4})}{x^2}.
\end{aligned}
\]


\((13)\) Reference \((3)\), Sect. 21.

\((14)\) Reference \((3)\), Sect. 25.
We observe that in each of the above cases one of the terms is independent of the parameters $\lambda$, $\mu$.

The corresponding solutions are obtained by inserting the function (15) and the transformations (16) in eq. (5).

The potential in eq. (28, i)) is a Morse potential. For $\lambda = 0$ it reduces to an exponential potential, the solutions reducing to Bessel functions. The invariants (28, ii)–iii)) correspond to the well-known Coulomb and oscillator potentials. Clearly, only in these two cases one may include the centrifugal term. For $\lambda = 0$ the potential in (28, iii)) reduces to $x^{-2}$, the solutions again reducing to Bessel functions.

We note that in the Whittaker case the application of more general transformations does not lead to a simple S.I.

In this Section we have thus constructed the known solvable potentials and the corresponding general solutions in a unified manner. We know how by making use of the proper boundary conditions one then obtains the bound states (15) and the $S$-matrix (16,13).

4. – Discussion.

The general problem of the construction of solvable potentials is to find potentials for which eq. (1) can be solved in terms of functions that satisfy linear differential equations of mathematical physics (17–18). Some of these equations are of the Fuchsian type while others are their confluent forms. The simplest nontrivial equation of the Fuchsian type is the Riemann equation. In Sect. 3 a class of simple solvable potentials corresponding to this equation and its confluent form was constructed. It would be of interest to construct solvable potentials corresponding to the generalized Lamé equation. This is the most general linear equation of mathematical physics. We may mention that the functions (20) are suitable for transforming $I_{\ell}$ to $I_{\ell}$.

In the previous Section we have shown how the introduction of the invariant systematizes the treatment of solvable potentials. Here we wish to show that the Schwarzian eq. (11) yields at once the equation for the phase of a wave.

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(17) Reference (5), Sect. 10.6.

function. Writing an integral of eq. (1) as

\[ \Psi(x) = A(x) \exp \left( \frac{i}{\hbar} S(x) \right), \]

where \( A \) and \( S \) are real functions, we find

\[ s = \frac{\psi}{\psi^0} = \exp \left( \frac{2i}{\hbar} S(x) \right). \]

Hence by eq. (11)

\[ \left\{ \exp \left[ \frac{2i}{\hbar} S \right], x \right\} = 2(k^2 - U) = \frac{4m}{\hbar^2} (E - V(x)). \]

Recalling the definition (7) one obtains

\[ S'^2 = 2m(E - V) - \frac{\hbar^2}{2} \{S, z\}. \]

We recognize this to be the starting equation for the WBK method (19).

This work presents a method which systematizes the treatment of known solvable potentials and shows how the method can be applied to construct new solvable potentials.

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RIASSUNTO (*)

Si formula il problema della costruzione di potenziali di Schrödinger ad una variabile risolubili. Si costruisce una classe di potenziali semplici per cui l'equazione di Schrödinger può essere risolta in termini di speciali funzioni della fisica.

(*) Traduzione a cura della Redazione.