A

Appendix

The purpose of this short appendix is twofold. In the first three sections, which concern point set topology, functional analysis, and measure theory, we simply repeat some standard notation and concepts and list a number of results that are used throughout the book. No proofs are given and we tend to be very brief because the reader will have a firm background in these areas. However, because we cannot expect the reader to be familiar with basic harmonic analysis, a different viewpoint is taken in the remaining three sections where we treat Abelian topological groups. Although we just cite the existence and uniqneness of Haar measure on a locally compact group, we prove a number of facts about convolution of functions since the group algebra $L^1(G)$ serves as a prominent example in the book and also the Hilbert space $L^2(G)$ is used substantially. In Section 5 we deduce the Pontryagin duality theorem from the Plancherel formula and point out the bijection between closed subgroups of G and closed subgroups of the dual group $\widehat{G}$. Finally, in Section 6 we describe the coset ring of an Abelian group and the closed sets in the coset ring of a topological Abelian group.

A.1 Topology

Let X be a topological space. Then $C(X)$ denotes the set of all continuous complex-valued functions on X and $C^b(X)$ the subspace of all bounded functions in $C(X)$. For $f \in C(X)$, the support of f, $\operatorname{supp} f$, is the closure of the set of all $x \in X$ at which $f(x) \neq 0$. The set of all functions in $C(X)$ with compact support is denoted $C_c(X)$. A function f on X is said to *vanish at infinity* if for each $\epsilon > 0$, there exists a compact subset K_ϵ of X such that $|f(x)| < \epsilon$ for all $x \in X \setminus K_\epsilon$. Then $C_0(X)$ stands for the set of all $f \in C(X)$ which vanish at infinity. Clearly, $C_c(X) \subseteq C_0(X) \subseteq C^b(X)$ and all these spaces coincide with $C(X)$ when X is compact. Also, all these spaces are algebras under pointwise operations. On $C^b(X)$ we can introduce the supremum norm defined by $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. This norm turns $C^b(X)$ and $C_0(X)$

into Banach spaces. If X is a locally compact Hausdorff space, then $C_c(X)$ is dense in $C_0(X)$. This is a consequence of the Stone–Weierstrass theorem (Theorem A.1.3 below).

A topological space X is called *normal* if it is Hausdorff and for each pair $\{A, B\}$ of disjoint closed subsets of X there exist open subsets U and V of X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Theorem A.1.1. *(Urysohn's lemma)*

(i) *Let X be a normal topological space, and let A and B be disjoint closed subsets of X. Then there exists a continuous function $f : X \to [0,1]$ such that $f|_A = 1$ and $f|_B = 0$.*
(ii) *Let X be a locally compact Hausdorff space, and let C be a compact subset of X and U an open set containing C. Then there exists $f \in C_c(X)$ with $f|_C = 1$, $0 \leq f(x) \leq 1$ for all $x \in X$ and $\operatorname{supp} f \subseteq U$.*

Theorem A.1.2. *(Tietze's extension theorem) A Hausdorff space X is normal if and only if every real valued function, which is defined and continuous on a closed subset of X, admits a continuous extension to all of X.*

A family F of complex-valued functions on a topological space X is said to *strongly separate the points* of X if for each $x \in X$, there exists $f \in F$ with $f(x) \neq 0$, and for each $x, y \in X$ with $x \neq y$, there exists $g \in F$ such that $g(x) \neq g(y)$. The family F is said to be self-adjoint if it contains with a function f the conjugate complex function $\overline{f}$.

Theorem A.1.3. *(Stone–Weierstrass theorem) Let X be a locally compact Hausdorff space, and let A be a self-adjoint subalgebra of $C_0(X)$. Suppose that A strongly separates the points of X. Then A is uniformly dense in $C_0(X)$.*

Theorem A.1.4. *(Arzela–Ascoli) Let X be a locally compact Hausdorff space and $F \subseteq C_0(X)$. Suppose that F satisfies the following two conditions.*

(i) *The set $F(x) = \{f(x) : f \in F\}$ is bounded for every $x \in X$.*
(ii) *F is equicontinuous; that is, for each $x \in X$ and $\epsilon > 0$, there exists a neighbourhood U of x such that $|f(y) - f(x)| < \epsilon$ for all $f \in F$ and $y \in U$.*

Then F is relatively compact in $(C_0(X), \|\cdot\|_\infty)$.

Theorem A.1.5. *(Baire's category theorem) Let X be either a locally compact Hausdorff space or a complete metric space.*

(i) *If X is the union of countably many closed subsets, then one of them contains a nonempty open set.*

(ii) *The intersection of a countable collection of dense open subsets of X is dense in X.*

Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of topological spaces. Let X be a nonempty set and for each $\lambda \in \Lambda$, let $f_\lambda : X \to X_\lambda$ be a mapping. Then there exists a weakest (or coarsest) topology on X with respect to which all the mappings f_λ are continuous. This topology can be characterized by the universal property that for any topological space Y and any mapping $f : Y \to X$, f is continuous if and only if $f_\lambda \circ f : Y \to X_\lambda$ is continuous for all $\lambda \in \Lambda$. In the special case where X is the Cartesian product of the sets X_λ and, for each λ, p_λ is the projection from X onto X_λ, this topology is called the *product topology* on X.

Theorem A.1.6. *(Tychonoff's theorem) Let X be the product of topological spaces X_λ, $\lambda \in \Lambda$. Then X is compact in the product topology if and only if all X_λ are compact.*

A compact space C is a compactification of a topological space X if there exists a continuous injective mapping from X onto a dense subset of C. Let X be a locally compact Hausdorff space. Then there exists a compact Hausdorff space $\widetilde{X}$ together with an embedding $j : X \to \widetilde{X}$ such that $\widetilde{X} \setminus j(X)$ is a singleton. $\widetilde{X}$ is uniquely determined up to homeomorphisms and is called the *one-point compactification* of X. The space $\widetilde{X}$ can be constructed as follows. Let $\widetilde{X} = X \cup \{\infty\}$ as a set and take the open sets in $\widetilde{X}$ to be the open sets in X together with the complements in $\widetilde{X}$ of the compact subsets of X.

Note that each $f \in C_0(X)$ extends to a continuous function on $\widetilde{X}$, also denoted f, by setting $f(\infty) = 0$.

Let X be a compact space and Y a Hausdorff space. If f is a continuous and injective mapping from X into Y, then f is a homeomorphism from X onto its range $f(X)$.

Proposition A.1.7. *If f is a continuous open map of a locally compact Hausdorff space X onto a Hausdorff space Y and if K is a compact subset of Y, then there exists a compact subset C of X such that $f(C) = K$.*

Proposition A.1.8. *Let X be a locally compact Hausdorff space. A subset Y of X is locally compact (in the induced topology) if and only if there exist a closed subset A of X and an open subset V of X such that $Y = A \cap V$. In particular, a dense subset of X is locally compact if and only if it is open in X.*

Occasionally, a topology is introduced by designating the closed subsets rather than the open subsets. The procedure is as follows. A *closure operation* on a set X is an assignment $A \to \overline{A}$ from $\mathcal{P}(X)$, the collection of all subsets of X, to itself such that $\overline{\emptyset} = \emptyset$, $A \subseteq \overline{A} = \overline{\overline{A}}$, and $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for all $A, B \subseteq X$. If such a closure operation is given, there exists a unique topology on X such that for each $A \subseteq X$, $\overline{A}$ equals the closure of A in X with respect to this topology.

A.2 Functional analysis

Let E and F be normed linear spaces (over the complex number field $\mathbb{C}$). Note that a linear transformation T from E into F is continuous if and only if it is bounded. The set $\mathcal{B}(E,F)$ of bounded linear transformations $T : E \to F$ is itself a normed linear space with the norm given by

$$\|T\| = \sup\{\|Tx\| : x \in E, \|x\| \leq 1\},$$

and $\mathcal{B}(E,F)$ is complete if F is a Banach space. It is common to write $\mathcal{B}(E)$ instead of $\mathcal{B}(E,E)$. Composition of bounded linear operators turns $\mathcal{B}(E)$ into a Banach algebra. For $T \in \mathcal{B}(E,F)$, the *adjoint* T^* of T is the linear map from F^* into E^* defined by $(T^*g)(x) = g(Tx)$ for all $g \in F^*$ and $x \in E$. Clearly, $T^* \in \mathcal{B}(F^*,E^*)$.

For a normed space E, let E^* denote the *dual space* of E; that is, $E^* = \mathcal{B}(E,\mathbb{C})$, the vector space of all continuous linear functionals on E. Thus E^* is a Banach space when equipped with the norm

$$\|f\| = \sup\{|f(x)| : x \in E, \|x\| \leq 1\},$$

$f \in E^*$. The space E embeds isometrically into the second dual space E^{**} as follows. For each $x \in E$, define $\widehat{x} : E^* \to \mathbb{C}$ by $\widehat{x}(f) = f(x)$ for $f \in E^*$. Then $\widehat{x} \in E^{**}$, and it is a consequence of the Hahn–Banach theorem (Theorem A.2.1 below) that $\|\widehat{x}\| = \|x\|$.

The *weak topology* $\sigma(E,E^*)$ on E is the coarsest topology with respect to which all the functionals $f \in E^*$ are continuous on E. Similarly, the *weak**-*topology* (or w^*-*topology*) $\sigma(E^*,E)$ is the coarsest topology on E^* with respect to which all the linear functionals $\widehat{x}$ on E^*, $x \in E$, are continuous. Thus a neighbourhood basis of $f_0 \in E^*$ in the w^*-topology is formed by the sets

$$U(f_0,F,\epsilon) = \{f \in E^* : |f(x) - f_0(x)| < \epsilon \text{ for all } x \in F\},$$

where $\epsilon > 0$ and F is any finite subset of E.

We now collect some fundamental results about dual spaces and bounded linear operators.

Theorem A.2.1. *(Hahn–Banach) Let E be a normed space and F a (not necessarily closed) linear subspace of E. If f is a bounded linear functional on F, then there exists $g \in E^*$ such that $g(x) = f(x)$ for all $x \in F$ and $\|g\| = \|f\|$.*

Corollary A.2.2. *If F is a linear subspace of E and x is an element of E which is not contained in the closure of F, then there exists $g \in E^*$ such that $g|_F = \{0\}$ and $g(x) \neq 0$.*

Theorem A.2.3. *(Banach–Alaoglu) Let E be a normed space. Then the unit ball $E_1^* = \{f \in E^* : \|f\| \leq 1\}$ of E^* is w^*-compact.*

However, E_1^* is compact in the norm topology only if E is finite-dimensional.

Corollary A.2.4. *If M is a w^*-closed linear subspace of E^* and $f \in E^* \setminus M$, then there exists $x \in E$ such that $f(x) \neq 0$ but $g(x) = 0$ for all $g \in M$.*

Theorem A.2.5. *(Closed graph theorem) Let E and F be Banach spaces, and let $T : E \to F$ be a linear map. Then the following conditions on T are equivalent.*

(i) *T is continuous.*
(ii) *The graph $G_T = \{(x, Tx) : x \in E\}$ of T is closed in $E \times F$.*
(iii) *If $x_n \to 0$ in E and $Tx_n \to y$ in F, then $y = 0$.*

Theorem A.2.6. *(Open mapping theorem) Let E and F be Banach spaces, and let $T : E \to F$ be a continuous linear mapping. If T is surjective, then T is open. In particular, if $T \in \mathcal{B}(E, F)$ is bijective, then $T^{-1} \in \mathcal{B}(F, E)$.*

Corollary A.2.7. *If a vector space E is a Banach space with respect to two norms, say $\|\cdot\|_1$ and $\|\cdot\|_2$, and if there is a constant c such that $\|x\|_2 \leq c\|x\|_1$ for all $x \in E$, then the two norms are equivalent, that is, there is a constant d such that $\|x\|_1 \leq d\|x\|_2$ for all $x \in E$.*

Theorem A.2.8. *(Uniform boundedness principle) Let E be a Banach space, F a normed space, and $\{T_\lambda : \lambda \in \Lambda\}$ a family of continuous linear maps from E into F. Suppose that $\{T_\lambda x : \lambda \in \Lambda\}$ is bounded in F for each $x \in E$. Then there exists a constant $C \geq 0$ such that $\|T_\lambda\| \leq C$ for all $\lambda \in \Lambda$.*

Theorem A.2.9. *(Krein–Milman) Let E be a locally convex topological vector space and let C be a nonempty convex subset of E. If C is compact, then C is the closed convex hull of the set of its extreme points.*

Let E and F be complex vector spaces. The algebraic tensor product $E \otimes F$ of E and F can be introduced in different ways. However, it is uniquely determined up to isomorphism by the following universal property. Given any complex vector space G and a bilinear map $T : E \times F \to \mathbb{C}$, there exists a unique linear map $S : E \otimes F \to G$ such that $S(x \otimes y) = T(x, y)$ for all $x \in E$ and $y \in F$.

Suppose that E and F are Banach spaces. A basic natural requirement for a norm γ on $E \otimes F$ is to satisfy $\gamma(x \otimes y) = \|x\| \cdot \|y\|$ for all $x \in E$ and $y \in F$. Such a norm is called a *cross-norm*. We now introduce the two cross-norms which play a role in this book.

Let $\mathcal{B}^2(E^* \times F^*, \mathbb{C})$ be the space of all bounded bilinear maps from $E^* \times F^*$ into $\mathbb{C}$, equipped with the norm given by

$$\|T\| = \sup\{|T(f, g)| : f \in E_1^*, g \in F_1^*\}.$$

Then $\mathcal{B}^2(E^* \times F^*, \mathbb{C})$ is complete. Given $x \in E$ and $y \in F$, let $B_{x,y}$ denote the element of $\mathcal{B}^2(E^* \times F^*, \mathbb{C})$ defined by

$$B_{x,y}(f, g) = f(x)g(y), \quad f \in E^*, \quad g \in F^*.$$

Then there is an injective linear map from $E \otimes F$ into $\mathcal{B}^2(E^* \times F^*, \mathbb{C})$ sending $x \otimes y$ to $B_{x,y}$. The norm ϵ on $E \otimes F$, inherited from $\mathcal{B}^2(E^* \times F^*, \mathbb{C})$, is called the *injective tensor norm*. So

$$\epsilon(u) = \sup\left\{\sum_{j=1}^{n} f(x_j)g(y_j) : f \in E_1^*, g \in F_1^*\right\},$$

where the supremun is taken over all representations $u = \sum_{j=1}^{n} x_j \otimes y_j$ of u. Obviously, ϵ is a cross-norm. The completion of $E \otimes F$ in $\mathcal{B}^2(E^* \times F^*, \mathbb{C})$ is called the *injective tensor product* of E and F and is denoted $E \widehat{\otimes}_\epsilon F$.

The *projective tensor norm* π on $E \otimes F$ is defined by

$$\pi(u) = \inf\left\{\sum_{j=1}^{n} \|x_j\| \cdot \|y_j\| : u = \sum_{j=1}^{n} x_j \otimes y_j\right\},$$

where the infimum is taken over all such representations of u. Then $\pi(x \otimes y) = \|x\| \cdot \|y\|$ for all $x \in E$ and $y \in F$. Actually, π is the largest cross-norm on $E \otimes F$. The completion of $E \otimes F$ with respect to π is called the *projective tensor product* of E and F and denoted $E \widehat{\otimes}_\pi F$.

Proposition A.2.10. *Let E and F be Banach spaces and let $u \in E \widehat{\otimes}_\pi F$ and $\epsilon > 0$. Then there exist bounded sequences $(x_n)_n$ in E and $(y_n)_n$ in F such that the series $\sum_{n=1}^{\infty} x_n \otimes y_n$ converges to u and $\sum_{n=1}^{\infty} \|x_n\| \cdot \|y_n\| < \pi(u) + \epsilon$. In particular, for any $u \in E \widehat{\otimes}_\pi F$,*

$$\pi(u) = \inf\left\{\sum_{n=1}^{\infty} \|x_n\| \cdot \|y_n\| : u = \sum_{n=1}^{\infty} x_n \otimes y_n, \sum_{n=1}^{\infty} \|x_n\| \cdot \|y_n\| < \infty\right\},$$

where the infimum is taken over all such representations of u.

It follows from Proposition A.2.10 that every element u of $E \widehat{\otimes}_\pi F$ has a representation of the form $u = \sum_{j=1}^{\infty} x_j \otimes y_j$, where $\sum_{j=1}^{\infty} \|x_j\| \cdot \|y_j\| < \infty$ and $u(f,g) = \sum_{j=1}^{\infty} f(x_j)g(y_j)$ for $f \in E^*$ and $g \in F^*$.

The following proposition provides a useful description of the projective tensor product.

Proposition A.2.11. *Let E and F be Banach spaces. There exists an isometric isomorphism from $\mathcal{B}(E, F^*)$ onto $(E \widehat{\otimes}_\pi F)^*$ with the property that*

$$\langle T, \sum_{j=1}^{n} x_j \otimes y_j \rangle = \sum_{j=1}^{n} \langle Tx_j, y_j \rangle$$

for any $T \in \mathcal{B}(E, F^*)$, $x_1, \ldots, x_n \in E$ and $y_1, \ldots, y_n \in F$.

Let $u = \sum_{n=1}^{\infty} x_n \otimes y_n \in E \widehat{\otimes}_\pi F$, $\sum_{n=1}^{\infty} \|x_n\| \cdot \|y_n\| < \infty$. Then the series $\sum_{n=1}^{\infty} B_{x_n, y_n}$ converges in $\mathcal{B}^2(E^* \times F^*, \mathbb{C})$. This element of $\mathcal{B}^2(E^* \times F^*, \mathbb{C})$ does not depend on the representation of u. Indeed, if $\sum_{n=1}^{\infty} x_n \otimes y_n = 0$ in $E \widehat{\otimes}_\pi F$, then $\sum_{n=1}^{\infty} \langle Sx_n, y_n \rangle = 0$ for all $S \in \mathcal{B}(E, F^*)$ by Proposition A.2.11, and hence, taking for S the operator defined by $Sx = f(x)g$ for $x \in E$,

$$\sum_{n=1}^{\infty} B_{x_n,y_n}(f,g) = g\left(\sum_{n=1}^{\infty} f(x_n)y_n\right) = \sum_{n=1}^{\infty} f(x_n)g(y_n) = 0$$

for all $f \in E^*$ and $g \in F^*$. Thus we have a natural continuous linear mapping $u \to B_u = \sum_{n=1}^{\infty} B_{x_n,y_n}$ from $E \widehat{\otimes}_\pi F$ into the injective tensor product $E \widehat{\otimes}_\epsilon F$.

This map is not injective in general, and the problem of when it is injective is closely related to the approximation property of a Banach space. A Banach space E has the *approximation property* if for each compact subset C of E and each $\epsilon > 0$, there exists a finite rank operator S on E with $\|Sx - x\| \leq \epsilon$ for all $x \in C$. The class of Banach spaces with the approximation property includes all spaces $C_0(X)$ for X a locally compact Hausdorff space, all Banach spaces with a Schauder basis, $c_0(I)$ and $l^p(I)$, $1 \leq p < \infty$, for any index set I, spaces $L^p(\mu)$, $1 \leq p < \infty$, for any measure μ and the disc algebra. Also, if E^* has the approximation property, then so does E. The first example of a Banach space which was shown to not have the approximation property, is $\mathcal{B}(l^2, l^2)$. A very good reference to the approximation property and tensor products of Banach spaces in general is [114].

Theorem A.2.12. *For a Banach space E, the following two conditions are equivalent.*

(i) *E has the approximation property.*
(ii) *The natural mapping from $E \widehat{\otimes}_\pi F$ into $E \widehat{\otimes}_\epsilon F$ is injective for every Banach space F.*

A.3 Measure and integration

In the following, if μ is a positive measure on a set X, $L^p(X,\mu)$ or $L^p(\mu)$, for short, denotes the set of (equivalence classes of) p-integrable complex-valued μ-measurable functions on X.

Theorem A.3.1. *(Hölder's inequality) Let μ be a positive measure and let $1 \leq p \leq \infty$, and $(1/p) + (1/q) = 1$. If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Theorem A.3.2. *(Minkowski's inequality) Let μ be a positive measure, $1 \leq p \leq \infty$, and $f, g \in L^p(\mu)$. Then $f + g \in L^p(\mu)$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

It follows that $L^p(\mu)$ with the norm $\|\cdot\|_p$ is a Banach space, and $L^2(\mu)$ is a Hilbert space for the scalar product $\langle f, g\rangle = \int_X f(x)\overline{g(x)}d\mu(x)$.

A consequence of Hölder's inequality is that every $g \in L^q(\mu)$ defines a bounded linear functional F_g of $L^p(\mu)$ by

$$\langle F_g, f\rangle = \int_X f(x)g(x)d\mu(x)$$

for all $f \in L^p(\mu)$.

Theorem A.3.3. *Let μ be a positive measure and suppose that either $1 < p < \infty$ or that $p = 1$ and μ is σ-finite. Let q be the conjugate index to p. Then, for each $F \in L^p(\mu)^*$, there exists a unique $g \in L^q(\mu)$ such that $F = F_g$. The map $g \to F_g$ is an isometric isomorphism from $L^q(\mu)$ onto $L^p(\mu)^*$.*

Theorem A.3.4. *Let μ be a positive regular Borel measure on a locally compact Hausdorff space X. Then, for $1 \le p < \infty$, $C_c(X)$ is dense in $L^p(X, \mu)$.*

Theorem A.3.5. *(Riesz representation theorem) Let X be a locally compact Hausdorff space. For each $\mu \in M(X)$, define $F_\mu \in C_0(X)^*$ by*

$$\langle F_\mu, f\rangle = \int_X f(x)d\mu(x), \quad f \in C_0(X).$$

Then the map $\mu \to F_\mu$ is an isometric isomorphism from $M(X)$ onto $C_0(X)^*$.

Let X and Y be locally compact Hausdorff spaces and let μ and ν be positive regular Borel measures on X and Y, respectively. Then the assignment

$$f \to \int_X \left(\int_Y f(x,y)d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f(x,y)d\mu(x)\right) d\nu(y)$$

defines a positive linear functional F on $C_c(X \times Y)$. Hence, by the Riesz representation theorem, there exists a unique positive regular Borel measure, denoted $\mu \times \nu$, on $X \times Y$ such that

$$\langle F, f\rangle = \int_{X\times Y} f(x,y)d(\mu \times \nu)(x,y)$$

for all $f \in C_c(X \times Y)$.

Theorem A.3.6. *(Fubini's theorem) Let X and Y be locally compact Hausdorff spaces and let μ and ν be positive regular Borel measures on X and Y, respectively. Let $f \in L^1(\mu \times \nu)$ and suppose that there exist σ-finite Borel sets A and B of X and Y, respectively, such that f vanishes on $(X \times Y)\setminus(A \times B)$. Then*

$$\int_X \left(\int_Y f(x,y)d\nu(y)\right) d\mu(x) = \int_{X\times Y} f(x,y)d(\mu \times \nu)(x,y)$$
$$= \int_Y \left(\int_X f(x,y)d\mu(x)\right) d\nu(y).$$

We now briefly discuss vector-valued integration. Let (X, μ) be a measure space and E a Banach space. A function $f : X \to E$ is said to be a μ-measurable simple function if it is of the form $f(x) = \sum_{j=1}^n 1_{M_j}(x)a_j$, where the a_j are elements of E, the M_j are disjoint μ-measurable subsets of X with $\mu(M_j) < \infty$ and 1_{M_j} denotes the characteristic function of M_j, $j = 1, \ldots, n$. The integral $\int_X f(x)d\mu(x)$ of such a μ-measurable simple function is defined to be the element $\sum_{j=1}^n \mu(M_j)a_j$ of E. An arbitrary function $f : X \to E$ is called μ-measurable if there exists a sequence of μ-measurable simple functions converging to f almost everywhere. Such an f can then be defined to be Bochner integrable if the scalar-valued function $x \to \|f(x)\|$ is integrable. In this case, the Bochner integral of f is

$$\int_X f(x)d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)d\mu(x),$$

where $(f_n)_n$ is any sequence of μ-measurable simple functions such that

$$\int_X \|f(x) - f_n(x)\|d\mu(x) \to 0.$$

The space of E-valued Bochner integrable functions is denoted $L^1(\mu, E)$ or $L^1(X, E)$, if the measure μ is understood.

A.4 Haar measure and convolution on locally compact groups

A locally compact group G is always understood to be a group which is also a locally compact Hausdorff space and for which the map $(x, y) \to xy^{-1}$ of the product space $G \times G$ to G is continuous.

For a function f on G and $x \in G$, the left and right translation $L_x f$ and $R_x f$ of f are defined by $L_x f(y) = f(x^{-1}y)$ and $R_x f(y) = f(yx)$ for $y \in G$, respectively. A nonzero positive regular Borel measure μ on G is called a *left Haar measure* if it satisfies $\mu(xE) = \mu(E)$ for all Borel sets E and all $x \in G$. This left invariance condition is equivalent to $\int_G L_x f(y)d\mu(y) = \int_G f(y)d\mu(y)$ for all $f \in L^1(G, \mu)$. Likewise, a right Haar measure is defined.

Theorem A.4.1. *On any locally compact group G there exists a left invariant (right invariant) Haar measure. If μ and λ are two left Haar measures on G, then there exists a constant $c > 0$ such that $\mu = c\lambda$.*

If the Haar measure is fixed, we most times denote the Haar measure of a set M by $|M|$.

Remark A.4.2. Let μ be a Haar measure on G.

(1) Then $\mu(U) > 0$ for every nonempty open set and $\int_G f(x)dx > 0$ for each $f \in C_c^+(G)$ which is not identically zero.

(2) The support of μ equals G, and $\mu(G) < \infty$ if and only if G is compact.

Example A.4.3. (1) On $\mathbb{R}$ (more generally $\mathbb{R}^n$) Lebesgue measure is a Haar measure.

(2) On any discrete group counting measure, that is, $\mu(f) = \sum_{x \in G} f(x)$ is a left and right invariant Haar measure.

(3) On the circle group $\mathbb{T}$ a Haar measure is given by

$$\mu(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})dt,$$

$f \in C(\mathbb{T})$, where dt is Lebesgue measure on $[0, 2\pi]$.

(4) If G_1 and G_2 are two locally compact groups with left Haar measures μ_1 and μ_2, respectively, then the product measure $\mu_1 \times \mu_2$ is a left Haar measure on the product group $G_1 \times G_2$.

Let dx be a left Haar measure on G. For every $x \in G$ the measure

$$f \to \int_G f(yx^{-1})dy, \quad f \in C_c(G),$$

is left invariant. So there is a unique positive number $\Delta(A)$ such that

$$\int_G f(yx^{-1})dy = \Delta(x) \int_G f(y)dy$$

for all $f \in C_c(G)$. The function $\Delta : x \to \Delta(x)$ is a continuous homomorphism from G into $\mathbb{R}_+^\times$, the multiplicative group of positive real numbers.

The function Δ is called the *modular function* of G and G is called *unimodular* if $\Delta(x) = 1$ for all $x \in G$.

Abelian groups are unimodular, and so are compact groups because $\{1\}$ is the only compact subgroup of $\mathbb{R}_+^\times$.

Proposition A.4.4. *Let $1 \leq p < \infty$ and let $f \in L^p(G)$. Given $\epsilon > 0$, there exists a neighbourhood U of e in G such that $\|L_x f - L_y f\|_p < \epsilon$ for all $x, y \in G$ such that $x^{-1}y \in U$.*

Proof. Because $C_c(G)$ is dense in $L^p(G)$, we find $g \in C_c(G)$ with $\|f - g\|_p < \epsilon/3$. Choose a compact neighbourhood V of e in G. Then $0 < |V \cdot \operatorname{supp} g| < \infty$ since $V \cdot \operatorname{supp} g$ is compact and has nonempty interior. Since g is uniformly continuous, there exists a symmetric neighbourhood U of e in G such that $U \subseteq V$ and

$$|g(x) - g(y)| < \frac{\epsilon}{3} \cdot |V \cdot \operatorname{supp} g|^{-1/p}$$

for all $x, y \in G$ with $y^{-1}x \in U$. For such x and y, we have

$$\begin{aligned}\|L_x g - L_y g\|_p^p &= \int_G |g(x^{-1}t) - g(y^{-1}t)|^p dt \\ &= \int_{V \cdot \operatorname{supp} g} |g(x^{-1}yt) - g(t)|^p dt \\ &\leq \left(\frac{\epsilon}{3}\right)^p |V \cdot \operatorname{supp} g|^{-1} |V \cdot \operatorname{supp} g| = \left(\frac{\epsilon}{3}\right)^p,\end{aligned}$$

and this implies

$$\begin{aligned}\|L_x f - L_y f\|_p &\leq \|L_x(f-g)\|_p + \|L_x g - L_y g\|_p + \|L_y(f-g)\|_p \\ &= 2\|f-g\|_p + \|L_x g - L_y g\|_p,\end{aligned}$$

which, by the above and the choice of g, is $< \epsilon$. □

Proposition A.4.5. *Let $1 < p < \infty$ and let q be the conjugate index to p; that is, $(1/p) + (1/q) = 1$. Suppose that $f \in L^p(G)$ and $g \in L^q(G)$. Then $f * \check{g} \in C_0(G)$ and $\|f * \check{g}\|_\infty \leq \|f\|_p \|g\|_q$.*

Proof. For each $x \in G$, we have

$$\int_G |f(xy)\check{g}(y^{-1})| dy = \int_G |L_{x^{-1}} f(y)| \cdot |g(y)| dy \leq \|L_{x^{-1}} f\|_p \|g\|_q$$

by Hölder's inequality. So $f * \check{g}$ is defined everywhere and bounded on G by $\|f\|_p \|g\|_q$. For x and y in G, Hölder's inequality gives

$$|f * \check{g}(x) - f * \check{g}(y)| \leq \|L_{x^{-1}} f - L_{y^{-1}} f\|_p \|g\|_q.$$

The map $t \to L_t f$ from G into $L^p(G)$ is continuous (Proposition A.4.4), and therefore we obtain that $f * \check{g}$ is continuous.

To prove that $f * \check{g}$ vanishes at infinity, note first that $f * \check{g} \in C_c(G)$ whenever $f, g \in C_c(G)$. If $f \in L^p(G)$ and $g \in L^q(G)$ then, since $C_c(G)$ is dense in $L^r(G)$ for each $1 \leq r < \infty$, there exist sequences $(f_n)_n$ and $(g_n)_n$ in $C_c(G)$ such that $\|f - f_n\|_p \to 0$ and $\|g - g_n\|_q \to 0$. Then, for all $x \in G$,

$$\begin{aligned}|f * \check{g}(x) - f_n * \check{g_n}(x)| &\leq |(f - f_n) * \check{g}(x)| + |f_n * (\check{g} - \check{g_n})(x)| \\ &\leq \|f - f_n\|_p \|g\|_q + \|f_n\|_p \|g - g_n\|_q,\end{aligned}$$

which tends to 0 as $n \to \infty$. It follows that $f * \check{g} \in C_0(G)$. □

Proposition A.4.6. *Let G be a locally compact group. For every relatively compact symmetric open neighbourhood V of e in G, let $u_V \in L^1(G)$ be such that $u_V \geq 0$, $\|u_V\|_1 = 1$ and $u_V = 0$ almost everywhere on $G \setminus V$. Then, given $f \in L^p(G)$, $1 \leq p < \infty$, and $\epsilon > 0$,*

$$\|u_V * f - f\|_p < \epsilon$$

for all sufficiently small V.

Proof. Since $C_c(G)$ is dense in $L^p(G)$, we can choose $g \in C_c(G)$ such that $\|f - g\|_p < \epsilon/3$. For g it follows that

$$\begin{aligned}
\|u_V * g - g\|_p^p &= \int_G \left| \int_G u_V(xy)g(y^{-1})dy - g(x) \right|^p dx \\
&= \int_G \left| \int_G u_V(y)L_y g(x)dy - g(x) \right|^p dx \\
&= \int_G \left| \int_G u_V(y)[L_y g(x) - g(x)]dy \right|^p dx \\
&\leq \int_G \left(\int_G u_V(y)|L_y g(x) - g(x)|dy \right)^p dx \\
&\leq |V \cdot \operatorname{supp} g| \cdot \sup\{\|L_y g - g\|_\infty^p : y \in V\}.
\end{aligned}$$

Now, since the map $y \to L_y g$ from G into $L^p(G)$ is continuous, we find a neighbourhood W of e in G such that, for all $y \in W$,

$$\|L_y g - g\|_\infty \leq \frac{\epsilon}{3|V \cdot \operatorname{supp} g|^{1/p}}.$$

Together with the above estimate we get for all $V \subseteq W$,

$$\begin{aligned}
\|u_V * f - f\|_p &\leq \|u_V * (f - g)\|_p + \|u_V * g - g\|_p + \|g - f\|_p \\
&\leq (\|u_V\|_1 + 1)\|f - g\|_p + \frac{\epsilon}{3},
\end{aligned}$$

which is $< \epsilon$ since $\|u\|_1 = 1$. □

In Proposition A.4.6, u_V can, for instance, be taken to be $|V|^{-1}1_V$.

Proposition A.4.7. *Suppose that* $1 \leq p \leq \infty$, $g \in L^p(G)$, *and* $f \in L^1(G)$. *Then* $f * g(x)$ *is defined for almost all* $x \in G$, *and we have* $f * g \in L^p(G)$ *and* $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. *If* $p = \infty$, *then* $f * g(x)$ *is defined for all* $x \in G$ *and* $f * g$ *is continuous.*

Proof. Assume first that $p = \infty$. By Hölder's inequality, the integral $f * g(x) = \int_G f(xy)g(y^{-1})dy$ converges for every $x \in G$ and satisfies $|f * g(x)| \leq \|f\|_1 \|g\|_\infty$. Moreover, for all $x, y \in G$,

$$\begin{aligned}
|f * g(x^{-1}) - f * g(y^{-1})| &\leq \int_G |L_x f(t) - L_y f(t)| \cdot |g(t^{-1})|\, dt \\
&\leq \|g\|_\infty \|L_x f - L_y f\|_1.
\end{aligned}$$

Proposition A.4.4 shows that $f * g$ is continuous.

Now let $1 \leq p < \infty$ and let q be the conjugate exponent to p. Since the map $y \to L_y g$ from G into $L_p(G)$ is continuous (Proposition A.4.4) and bounded by $\|g\|_p$, the L^p-valued function $y \to f(y)L_y g$ is Bochner integrable and satisfies, for each $h \in L^q(G)$,

$$\begin{aligned}\int_G \left(\int_G f(y)L_y g\, dy\right)(x)h(x)dx &= \left\langle \int_G f(y)L_y g\, dy, h\right\rangle \\ &= \int_G f(y)\langle L_y g, h\rangle dy \\ &= \int_G \int_G f(y)g(y^{-1}x)h(x)dxdy.\end{aligned}$$

It follows from Fubini's theorem and Hölder's inequality that the order of integration can be reversed. Since $h \in L^q(G)$ is arbitrary, we conclude that

$$\left(\int_G f(y)L_y g\right)(x) = \int_G f(y)g(y^{-1}x)dy = f * g(x)$$

for almost all $x \in G$. Finally, using this, we get

$$\begin{aligned}\|f * g\|_p &= \left(\int_G \left|\left(\int_G f(y)L_y g\right)(x)\right|^p dx\right)^{1/p} \\ &= \left\|\int_G f(y)L_y g\right\|_p \leq \int_G \|f(y)L_y g\|_p\, dy \\ &= \int_G |f(y)| \cdot \|L_y g\|_p\, dy \\ &= \|g\|_p \|f\|_1.\end{aligned}$$

This finishes the proof of the proposition. □

Let H be a closed normal subgroup of a locally compact group G. For $f \in C_c(G)$, define the function $T_H f$ on G/H by $T_H f(xH) = \int_H f(xh)dh$, $x \in G$. Then $T_H f \in C_c(G/H)$ and T_H maps $C_c(G)$ onto $C_c(G/H)$. Given a left Haar integral on G/H, the assignment $f \to \int_{G/H} T_H f(xH)d(xH)$ defines a left Haar integral on G. Hence there exists a unique left Haar measure dx on G such that

$$\int_{G/H} \left(\int_H f(xh)dh\right) d(xH) = \int_G f(x)dx.$$

This formula is called *Weil's formula*. If two of the left Haar integrals on G, H and G/H are given, the third can always be normalized so that Weil's formula holds. It follows from Weil's formula that $T_H(f * g) = T_H(f) * T_H(g)$ and $\|T_H f\|_1 \leq \|f\|_1$ for all $f, g \in C_c(G)$. Hence T_H extends to a continuous homomorphism from $L^1(G)$ into $L^1(G/H)$, also denoted T_H. Since $\|T_H f\|_1$ equals the quotient norm, T_H is actually surjective.

Theorem A.4.8. *Let H be a closed normal subgroup of G and let $f \in L^1(G)$.*

(i) *There exists a set* S *of measure zero in* G/H *such that the function* $h \to f(xh)$ *is in* $L^1(H)$ *for each* $x \in G$ *with* $xH \notin S$.
(ii) *The function* $xH \to \int_H f(xh)dh$, *which is defined on* $G/H \setminus S$, *is integrable.*
(iii) *If left Haar measures on* H, G/H, *and* G *are chosen so that Weil's formula holds, then*

$$\int_{G/H} \left(\int_H f(xh)dh \right) d(xH) = \int_G f(x)dx.$$

The formula in (iii) is often called the *extended Weil formula.* A function $f \in L^1(G)$ is in the kernel of the homomorphism $T_H : L^1(G) \to L^1(G/H)$ if and only if $\int_H f(xh)dh = 0$ for almost all $x \in G$.

A.5 The Pontryagin duality theorem

The famous Pontryagin duality theorem asserts that there is a canonical topological isomorphism between a locally compact Abelian group and its second dual group. In this section we deduce this duality theorem from the results in Sections 4.4 and 2.7. In the sequel, G will always denote a locally compact Abelian group and $\widehat{G}$ the dual group of G.

Proposition A.5.1. *For* $x \in G$, $\epsilon > 0$, *and a compact subset* Γ *of* $\widehat{G}$, *let*

$$V(x, \Gamma, \epsilon) = \{y \in G : |\gamma(y) - \gamma(x)| < \epsilon \textit{ for all } \gamma \in \Gamma\}.$$

Then $V(x, \Gamma, \epsilon)$ *is open in* G, *and the sets* $V(x, \Gamma, \epsilon)$ *form a neighbourhood basis of* x *in* G.

Proof. Let $(y_\alpha)_\alpha$ be a net in $G \setminus V(x, \Gamma, \epsilon)$ converging to some $y \in G$. For each α, there exists $\gamma_\alpha \in \Gamma$ with $|\gamma_\alpha(y_\alpha) - \gamma_\alpha(x)| \geq \epsilon$. Since Γ is compact, passing to a subnet if necessary, we can assume that $\gamma_\alpha \to \gamma$ for some $\gamma \in \Gamma$. Since the function $(t, \lambda) \to \lambda(t)$ on $G \times \widehat{G}$ is continuous (Lemma 2.7.4), it follows that $|\gamma(y) - \gamma(x)| \geq \epsilon$. This shows that $y \notin V(x, \Gamma, \epsilon)$. So $V(x, \Gamma, \epsilon)$ is open in G.

Since $V(x, \Gamma, \epsilon) = xV(e, \Gamma, \epsilon)$, it remains to show that if U is an open neighbourhood of e in G, then there exist a compact subset Γ of $\widehat{G}$ and $\epsilon > 0$ such that $V(e, \Gamma, \epsilon) \subseteq U$. To that end, choose symmetric open neighbourhoods V and W of e in G such that $W \subseteq V$, $V^2 \subseteq U$, $\overline{W^2} \subseteq V$ and V is relatively compact. Let

$$f = \|1_W * 1_W\|_2^{-1}(1_W * 1_W).$$

Then $f \in C_c(G)$, $f(x) \geq 0$ for all $x \in G$ and $\operatorname{supp} f \subseteq V$. If $x \in G \setminus U$, then $\operatorname{supp} f$ and $\operatorname{supp}(L_x f)$ are disjoint and therefore

$$\|f - L_x f\|_2^2 = \|f\|_2^2 + \|L_x f\|_2^2.$$

In particular, if $x \in G$ is such that $\|f - L_x f\|_2 \leq 1$, then $x \in U$. Now, choose $u \in L^1(G)$ such that $u \geq 0$, $\|u\|_1 = 1$ and $\|u * f - f\|_2 < 1/3$ (Proposition A.4.6). Since the $\|\cdot\|_2$-norm is translation invariant, it follows that, for all $x \in G$,

$$\|u * L_x f - L_x f\|_2 = \|L_x(u * f - f)\|_2 = \|u * f - f\|_2 < frac13.$$

Thus, if $x \in G$ satisfies $\|u * f - f\|_2 \leq 1/3$, then

$$\|f - L_x f\|_2 \leq \|f - u * f\|_2 + \|u * f - u * L_x f\|_2 + \|u * L_x f - f\|_2 < 1,$$

and hence $x \in U$. Applying the regular representation (Section 4.4), we have

$$\begin{aligned} \|u * f - u * L_x f\|_2 &= \|\lambda_u f - \lambda_{L_x u} f\|_2 \\ &\leq \|\lambda_u - \lambda_{L_x u}\| \cdot \|f\|_2 \\ &= \|\widehat{u} - \widehat{L_x u}\|_\infty . \end{aligned}$$

So, if $x \in G$ is such that $\|\widehat{u} - \widehat{L_x u}\|_\infty \leq 1/3$, then $x \in U$ by the above.

Since $\widehat{u} \in C_0(\widehat{G})$, there exists a compact subset Γ of $\widehat{G}$ such that $|\widehat{u}(\gamma)| < 1/6$ for all $\gamma \in \widehat{G} \setminus \Gamma$. Let $x \in V(e, \Gamma, 1/3$. Then, for $\gamma \in \Gamma$,

$$\left|\widehat{u}(\gamma) - \widehat{L_x u}(\gamma)\right| = |\widehat{u}(\gamma)| \cdot |1 - \overline{\gamma(x)}| \leq \|u\|_1 \cdot |1 - \overline{\gamma(x)}| < \frac{1}{3}$$

since $\|u\|_1 = 1$, and if $\gamma \in \widehat{G} \setminus \Gamma$, then

$$\left|\widehat{u}(\gamma) - \widehat{L_x u}(\gamma)\right| = |\widehat{u}(\gamma)| \cdot |1 - \overline{\gamma(x)}| < \frac{1}{3}$$

as $|\widehat{u}(\gamma)| < \frac{1}{6}$. Therefore, if $x \in V(e, \Gamma, \frac{1}{3})$, then

$$\left|\widehat{u}(\gamma) - \widehat{L_x u}(\gamma)\right| < \frac{1}{3}$$

for all $\gamma \in \Gamma$, and this implies $x \in U$ by what we have seen above. $\square$

Let $\widehat{\widehat{G}}$ denote the second dual of G, that is, the dual group of $\widehat{G}$. Each $x \in G$ defines an element $\widehat{x}$ of $\widehat{\widehat{G}}$ by setting $\widehat{x}(\alpha) = \alpha(x)$ for $\alpha \in \widehat{G}$.

Theorem A.5.2. *(Pontryagin duality theorem) Let G be a locally compact Abelian group. The map $\wedge : x \to \widehat{x}$ is a topological isomorphism from G onto the second dual group $\widehat{\widehat{G}}$.*

Proof. Clearly, $\wedge$ is a homomorphism. If x and y are elements of G such that $\widehat{x} = \widehat{y}$, then for all $f \in L^1(G)$ and $\alpha \in \widehat{G}$,

$$\widehat{L_x f}(\alpha) = \overline{\alpha(x)}\widehat{f}(\alpha) = \overline{\alpha(y)}\widehat{f}(\alpha) = \widehat{L_y f}(\alpha).$$

Since the Gelfand homomorphism of $L^1(G)$ is injective, it follows that $L_x f = L_y f$ for all $f \in L^1(G)$ and this implies $x = y$. Proposition A.5.1 shows that $\wedge$ is a homeomorphism of G onto its range $\wedge(G) \subseteq \widehat{\widehat{G}}$. In particular, $\wedge(G)$ is a locally compact group. Since a locally compact subset of a Hausdorff space is the intersection of an open set and a closed set, it follows that $\wedge(G)$ is open in its closure $\overline{\wedge(G)}$. Now, $\overline{\wedge(G)}$ is a topological group and an open subgroup of a topological group is automatically closed. So $\wedge(G)$ is closed in $\widehat{\widehat{G}}$.

Towards a contradiction, assume that $\wedge(G) \neq \widehat{\widehat{G}}$. Since $L^1(\widehat{G})$ is a regular commutative Banach algebra, there exists $g \in L^1(\widehat{G})$, $g \neq 0$, such that $\widehat{g}(\widehat{x}) = 0$ for all $x \in G$. As $g \neq 0$, we find $h \in C_0(\widehat{G})$ with $\int_{\widehat{G}} g(\alpha)h(\alpha)d\alpha \neq 0$. Now, the Gelfand homomorphism maps $C^*(G)$ onto $C_0(\widehat{G})$ (Section 4.4). Thus there exists $T \in C^*(G)$ with $\widehat{T} = h$. Since $L^1(G)$ is dense in $C^*(G)$ and $C_c(G)$ is dense in $L^1(G)$ in the $\| \cdot \|_1$-norm and hence in the C^*-norm, there exists $f \in C_c(G)$ with

$$\int_{\widehat{G}} g(\alpha)\widehat{f}(\alpha)d\alpha = \int_{\widehat{G}} g(\alpha)\widehat{\lambda_f}(\alpha)d\alpha \neq 0.$$

Fubini's theorem yields

$$\begin{aligned}
\int_{\widehat{G}} g(\alpha)\widehat{f}(\alpha)d\alpha &= \int_{\widehat{G}} g(\alpha)\left(\int_G f(x)\overline{\alpha(x)}dx \right) d\alpha \\
&= \int_G f(x)\left(\int_{\widehat{G}} g(\alpha)\overline{\alpha(x)}d\alpha \right) dx \\
&= \int_G f(x)\left(\int_{\widehat{G}} g(\alpha)\overline{\widehat{x}(\alpha)}d\alpha \right) dx \\
&= \int_G f(x)\widehat{g}(\widehat{x})dx,
\end{aligned}$$

which is zero since $\widehat{g}$ vanishes on $\wedge(G)$. This contradiction shows that $\wedge$ is surjective and finishes the proof of the duality theorem. □

For any subset Γ of $\widehat{G}$, let $A(G, \Gamma)$ denote the *annihilator* of Γ in G, that is,

$$A(G, \Gamma) = \{x \in G : \gamma(x) = 1 \text{ for all } \gamma \in \Gamma\}.$$

Similarly, the annihilator of a subset M of G is defined to be

$$A(\widehat{G}, M) = \{\alpha \in \widehat{G} : \alpha(x) = 1 \text{ for all } x \in M\}.$$

Clearly, $A(G, \Gamma)$ is a closed subgroup of G and $A(\widehat{G}, M)$ is a closed subgroup of $\widehat{G}$.

Corollary A.5.3. *Suppose that Γ is a closed subgroup of $\widehat{G}$. Then*

$$\Gamma = \{\alpha \in \widehat{G} : \alpha(A(G,\Gamma)) = \{1\}\}.$$

Proof. The subgroup Γ of $\widehat{G}$ identifies canonically with a closed subgroup of $(G/A(G,\Gamma))^\wedge$ which separates the points of $G/A(G,\Gamma)$. Therefore, we can assume that $A(G,\Gamma) = \{e\}$.

Since $\Delta(L^1(\widehat{G}/\Gamma)) = \widehat{\widehat{G}/\Gamma}$, it suffices to show that if α is a character of $\widehat{G}$ with $\alpha|_\Gamma = 1$, then $\alpha = 1_{\widehat{G}}$. By the duality theorem, there exists $x \in G$ such that $\alpha = \widehat{x}$. Then $\gamma(x) = \alpha(x) = 1$ for all $\gamma \in \Gamma$ and hence $x = e$ by hypothesis. So $\alpha = 1$ on all of G. □

It follows from the preceding corollary that the map $\Gamma \to A(G,\Gamma)$ is a bijection between the closed subgroups of G and the closed subgroups of $\widehat{G}$.

Corollary A.5.4. *Let G be a locally compact Abelian group. If $\mu \in M(G)$ and*

$$\widehat{\mu}(\alpha) = \int_G \overline{\alpha(x)}\,d\mu(x) = 0$$

for all $\alpha \in \widehat{G}$, then $\mu = 0$.

Proof. If $f \in L^1(\widehat{G})$, then $\widehat{f} \in C_0(\widehat{\widehat{G}})$ and hence, since $x \to \widehat{x}$ is a homeomorphism from G to $\widehat{\widehat{G}}$, the function $x \to \widehat{f}(\widehat{x})$ belongs to $C_0(G)$. By Fubini's theorem,

$$\begin{aligned}\int_G \widehat{f}(\widehat{x})\,d\mu(x) &= \int_G \left(\int_{\widehat{G}} f(\alpha)\overline{\widehat{x}(\alpha)}\,d\alpha\right) d\mu(x)\\ &= \int_{\widehat{G}} f(\alpha)\left(\int_G \overline{\alpha(x)}\,d\mu(x)\right) d\alpha\\ &= \int_{\widehat{G}} f(\alpha)\widehat{\mu}(\alpha)\,d\alpha,\end{aligned}$$

whence $\int_G \widehat{f}(\widehat{x})\,d\mu(x) = 0$ for all $f \in L^1(\widehat{G})$. Thus, denoting by ν the image of μ under the homeomorphism $x \to \widehat{x}$,

$$\int_{\widehat{\widehat{G}}} \widehat{f}(\chi)\,d\nu(\chi) = 0$$

for all $f \in L^1(\widehat{G})$.

However, the image of $L^1(\widehat{G})$ under the Gelfand homomorphism is norm-dense in $C_0(\widehat{\widehat{G}})$ (Lemma 2.7.3(iii)). It follows that $\int_{\widehat{\widehat{G}}} g(\chi)\,d\nu(\chi) = 0$ for all $g \in C_0(\widehat{\widehat{G}})$ and this implies $\nu = 0$ and hence $\mu = 0$. □

In passing we recall the notion of a compactly generated topological group. For a subset M of G and $n \in \mathbb{N}$, let M^n denote the set of all n-fold products $x_1 x_2 \cdot \ldots \cdot x_n$ of elements x_j of M. Suppose that G is a topological group. Then G is said to be *compactly generated* if there exists a compact subset C of G such that $G = \bigcup_{n=1}^{\infty} C^n$. If C is any compact symmetric neighbourhood of the identity of G, then $\bigcup_{n=1}^{\infty} C^n$ is an open compactly generated subgroup of G. Conversely, every compactly generated open subgroup of G arises in this manner.

Clearly, $\mathbb{R}$ and $\mathbb{Z}$ are compactly generated, and so is the direct product of two compactly generated topological groups. In particular, groups of the form $\mathbb{R}^n \times \mathbb{Z}^m \times K$, where $n, m \in \mathbb{N}_0$ and K is a compact group, are compactly generated. The following structure theorem, for the proof of which we refer to the literature, says that within the class of locally compact Abelian groups these groups are the only compactly generated ones.

Theorem A.5.5. *Let G be a compactly generated locally compact Abelian group. Then G is topologically isomorphic to a direct product $\mathbb{R}^n \times \mathbb{Z}^m \times K$, where $n, m \in \mathbb{N}_0$ and K is a compact Abelian group.*

A.6 The coset ring of an Abelian group

Let G be a locally compact Abelian group. In Section 5.6 we have described explicitly the closed ideals in $L^1(G)$ with bounded approximate identities. As an essential tool we have used a characterisation of the closed sets in the coset ring of an Abelian topological group. This characterisation, Theorem A.6.9 below, was established by Gilbert [43] and, independently and with a much simpler proof, by Schreiber [117]. Accordingly, our presentation follows very closely the one of [117]. Schreiber's approach, in turn, is based on a result due to Cohen [22] (Proposition A.6.5). We start with the relevant definitions.

The *coset ring* of an Abelian group G, denoted $\mathcal{R}(G)$, is the smallest Boolean algebra of subsets of G containing the cosets of all subgroups of G. That is, $\mathcal{R}(G)$ is the smallest family of subsets of G which contains all the cosets of subgroups of G and which is closed under the processes of forming finite unions, finite intersections and complements.

Suppose now that G is a topological Abelian group. Then the *closed coset ring* of G, $\mathcal{R}_c(G)$, is defined to be

$$\mathcal{R}_c(G) = \{E \in \mathcal{R}(G) : E \text{ is closed in } G\}.$$

We start with a description of the sets in $\mathcal{R}(G)$.

Proposition A.6.1. *Let G be an Abelian group. A subset E of G belongs to $\mathcal{R}(G)$ if and only if E is of the form*

$$E = \bigcup_{i=1}^{n} \left(C_i \setminus \bigcup_{j=1}^{n_i} C_{ij} \right), \quad n, n_i \in \mathbb{N},$$

where C_i and C_{ij} are (possibly void) cosets of subgroups of G.

Proof. Let $\mathcal{E}$ denote the collection of all such sets E. By definition of $\mathcal{R}(G)$ and since $\mathcal{E}$ is closed under forming finite unions, it suffices to show that if $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$ and $E \setminus F \in \mathcal{E}$. Let E be as above and let

$$F = \bigcup_{k=1}^{m} \left(D_k \setminus \bigcup_{l=1}^{m_k} D_{kl} \right), \quad m, m_k \in \mathbb{N},$$

where D_k and D_{kl} are cosets of subgroups of G (or empty). Since

$$E \cap F = \bigcup_{i=1}^{n} \bigcup_{k=1}^{m} \bigcap_{j=1}^{n_i} \bigcap_{l=1}^{m_k} ((C_i \setminus C_{ij}) \cap (D_k \setminus D_{kl})),$$

it will follow that $E \cap F \in \mathcal{E}$ once we have shown that if C, C', D, D' are cosets in G, then $(C \setminus C') \cap (D \setminus D') \in \mathcal{E}$. However, for that we only have to observe that

$$(C \setminus C') \cap (D \setminus D') = (C \cap D) \setminus (C' \cup D')$$

and that $C \cap D$ is either empty or a coset. Turning to complements, note that

$$(C \setminus C') \setminus (D \setminus D') = (C \setminus (C' \cup D)) \cup ((C \cap D') \setminus C')$$

belongs to $\mathcal{E}$. Finally, with the above notation, we have

$$E \setminus F = \bigcup_{i=1}^{n} \bigcap_{k=1}^{m} \left(\bigcap_{j=1}^{n_i} \bigcup_{l=1}^{m_k} ((C_i \setminus C_{ij}) \setminus (D_k \setminus D_{kl})) \right).$$

Because $\mathcal{E}$ is closed under forming finite unions and intersections, we conclude that $E \setminus F \in \mathcal{E}$. □

Remark A.6.2. Let H and K be subgroups of G and $a, b \in G$ such that $aH \cap bK \neq \emptyset$. Then there exists $h \in H$ such that

$$aH \setminus bK = ah(H \setminus (H \cap K)),$$

and $H \cap K$ has infinite index in H whenever $aH \setminus bK$ is infinite. Thus Proposition A.6.1 can be reformulated as follows. A subset E of G belongs to $\mathcal{R}(G)$ if and only if E can be written as

$$E = F \cup \bigcup_{i=1}^{m} \left(a_i H_i \setminus \bigcup_{j=1}^{m_i} b_{ij} K_{ij} \right),$$

where F is finite, H_i is a subgroup of G and K_{ij} is a subgroup of infinite index in H_i, $1 \leq i \leq m, 1 \leq j \leq m_i$.

The following lemma is used to show that homomorphisms map coset rings into coset rings (Theorem A.6.6).

Lemma A.6.3. *Let G be an Abelian group, H a subgroup of G, and $K_1, \ldots, K_n$ cosets in G. Then the set*

$$E = \{x \in G : xH \subseteq K_1 \cup \ldots \cup K_n\}$$

belongs to $\mathcal{R}(G)$.

Proof. We prove this by induction on $n \in \mathbb{N}$. If $n = 1$, then either $E = \emptyset$ or some coset of H is contained in K_1. In the latter case, since K_1 is a coset, E is a coset and hence $E \in \mathcal{R}(G)$.

Assume the statement is true for n and let

$$E = \{x \in G : xH \subseteq K_1 \cup \ldots \cup K_{n+1}\}.$$

If $E \neq \emptyset$ then, replacing E by a suitable translate, we can assume that $H \subseteq K_1 \cup \ldots \cup K_{n+1}$. Set $H_i = H \cap K_i$ and let K_i be a coset of the subgroup G_i of G $(i = 1, \ldots, n+1)$. Since $xH = \bigcup_{i=1}^{n+1} xH_i$ for all $x \in G$, we have

$$\begin{aligned} E &= \bigcap_{i=1}^{n+1} \{x \in G : xH_i \subseteq K_1 \cup \ldots \cup K_{n+1}\} \\ &= \bigcap_{i=1}^{n+1} \left(\{x \in G : xH_i \subseteq K_i\} \bigcup \left\{ x \in G : xH_i \subseteq \bigcup_{j \neq i} K_j \right\} \right) \\ &= \bigcap_{i=1}^{n+1} \left(G_i \cup \left\{ x \in G : xH_i \subseteq \bigcup_{j \neq i} K_j \right\} \right), \end{aligned}$$

which belongs to $\mathcal{R}(G)$ by the induction hypothesis. □

Let $\mathcal{F}(G)$ be the space of all finite linear combinations of characteristic functions 1_A, where A is a coset of a subgroup of G. Observe the following simple facts.

(1) The intersection of two cosets is a coset.
(2) $1_{A \cap B} = 1_A 1_B$.
(3) $1_{A \cup B} = 1_A + 1_B - 1_A 1_B$.
(4) $1_{G \setminus A} = 1_G - 1_A$.

It follows from (1), (2) and (3) that $\mathcal{F}(G)$ is an algebra of functions on G.

The next proposition is shown in [22]. For sake of brevity, we refrain from giving the proof. As pointed out by Schreiber, the following corollary is actually equivalent to Proposition A.6.4.

Proposition A.6.4. *Let $f \in \mathcal{F}(G)$, and let $B_1, \ldots, B_r$ be the finite family of sets in G on which f takes on its different values. Then the Boolean algebra generated by $B_1, \ldots, B_r$ and all of their translates contains a finite collection $\{K_1, \ldots, K_s\}$ of cosets in G such that the Boolean algebra generated by $\{K_1, \ldots, K_s\}$ contains every B_k.*

Corollary A.6.5. *Let $E \in \mathcal{R}(G)$. Then the Boolean algebra generated by E and all of its translates contains a finite collection $\{K_1, \ldots, K_r\}$ of cosets in G such that the Boolean algebra generated by $\{K_1, \ldots, K_r\}$ contains E.*

Proof. Recall that E can be written as a finite union of finite intersections of sets of the form $K \setminus L$, where K and L are cosets and L may be empty (Proposition A.6.1). The characteristic function of $K \setminus L$ is equal to $1_K - 1_{K \cap L}$. Since these functions are in $\mathcal{F}(G)$ and $\mathcal{F}(G)$ is an algebra, we have $1_E \in \mathcal{F}(G)$. Now, the sets of constancy of 1_E are just E and $G \setminus E$, so that the statement follows. □

Even though we need the following theorem only in the special case where G^* is a quotient group of G and ϕ the quotient homomorphism, we present it in slightly more generality.

Theorem A.6.6. *Let G and G^* be Abelian groups and let $\phi : G \to G^*$ be a homomorphism. If $E \in \mathcal{R}(G)$, then $\phi(E) \in \mathcal{R}(G^*)$.*

Proof. Because ϕ preserves unions and translations we only need to consider sets of the form $E = H \setminus \bigcup_{i=1}^{n} K_i$, where H is a subgroup of G and $K_1, \ldots, K_n$ are cosets in H of subgroups of H.

Let $N = \ker \phi$ and let $q : H \to H/(H \cap N)$ be the quotient homomorphism. Then there is an injective homomorphism $j : H/(H \cap N) \to G^*$ such that $\phi|_H = j \circ q$. We show that $q(E) \in \mathcal{R}(H/(H \cap N))$. Since j is an injective homomorphism, it then follows that $\phi(E) = j(q(E)) \in \mathcal{R}(G^*)$.

Therefore it suffices to show that if G is an Abelian group, H is a subgroup of G and $K_1, \ldots, K_n$ are cosets in G then, with $q : G \to G/H$ the quotient homomorphism,

$$q\left(G \setminus \bigcup_{i=1}^{n} K_i\right) \in \mathcal{R}(G/H).$$

This is equivalent to showing that the complement

$$F = \{\xi \in G/H : q^{-1}(\xi) \subseteq K_1 \cup \ldots \cup K_n\}$$

belongs to $\mathcal{R}(G/H)$. Let

$$E = \{x \in G : q(x) \in K_1 \cup \ldots \cup K_n\}.$$

Then $q(E) = F$ and $E \in \mathcal{R}(G)$ by Lemma A.6.3. We show that this implies that $q(E) \in \mathcal{R}(G/H)$.

If $E \neq \emptyset$, then E is a union of cosets of H, and the same is true of every member of the Boolean algebra $\mathcal{A}$ generated by E and all its translates. By Corollary A.6.5, $\mathcal{A}$ contains a finite collection $\mathcal{C}$ of cosets such that the Boolean algebra $\mathcal{B}$ generated by $\mathcal{C}$ contains E. The quotient homomorphism q induces a Boolean algebra homomorphism on $\mathcal{A}$, and hence on $\mathcal{B}$. It follows that $q(E) \in \mathcal{R}(G/H)$. This finishes the proof. □

Lemma A.6.7. *Let G be an Abelian topological group and G_0 a dense subgroup of G. Suppose that $K_1, \ldots, K_n$ are cosets in G_0 and let $E = G_0 \setminus \bigcup_{i=1}^n K_i$. Then there exists an open subgroup H of G such that $\overline{E}$ is a union of cosets of H.*

Proof. Let K_i be a coset of the subgroup $G_i, i = 1, \ldots, n$. Let $\mathcal{S}$ denote the smallest (and necessarily finite) collection of subgroups of G which contains all G_i, $i = 0, 1, \ldots, n$, and is closed under forming intersections. Since G_0 is dense in G and $G_0 \in \mathcal{S}$, there exists $K \in \mathcal{S}$ which is minimal with respect to the property that $\overline{K}$ is open G. Then there is a (possibly void) subset I of $\{1, \ldots, n\}$ such that

(1) $K = G_0 \cap \left(\bigcap_{i \in I} G_i\right)$.
(2) $i \in F$ and $G_i = G_j$ implies $j \in I$.

Set $H = \overline{K}$, and let C be any coset of H. We have to show that either $C \cap \overline{E} = \emptyset$ or $C \subseteq \overline{E}$. To that end, suppose that $x \in C \cap E$ (equivalently, $C \cap \overline{E} \neq \emptyset$ since C is open in G). Then xK is a dense subset of $C = xH$. Let $L_i = K_i \cap xK, i = 1, \ldots, n$. We claim that even

$$xK \setminus \bigcup_{i=1}^n K_i = xK \setminus \bigcup_{i=1}^n L_i$$

is dense in C. Since xK is dense in C, it suffices to verify that $\bigcup_{i=1}^n L_i$ is nowhere dense in G. If $i \in I$ then K is a subgroup of G_i. Thus $L_i = \emptyset$ since $xK \not\subseteq K_i$. If $i \notin I$ then L_i is either void or a coset of $K \cap G_i$. By the choice of K and I, $\overline{K \cap G_i}$ is not open, so $K \cap G_i$ is nowhere dense and hence so is L_i. Since $x \in G_0$ and $K \subseteq G_0$, it follows that

$$C = \overline{\left(xK \setminus \bigcup_{i=1}^n K_i\right)} \subseteq \overline{\left(G_0 \setminus \bigcup_{i=1}^n K_i\right)} = \overline{E},$$

as was to be shown. □

The preceding lemmas now lead to the characterisation of closed sets in the coset ring at which we were aiming.

Theorem A.6.8. *Let G be an Abelian topological group and $E \in \mathcal{R}(G)$. Then $\overline{E} \in \mathcal{R}(G)$ and E is closed if and only if E can be written*

$$E = \bigcup_{j=1}^m \left(C_j \setminus \bigcup_{l=1}^{m_j} C_{jl}\right),$$

where C_j and C_{jl} are (possibly void) closed cosets in G and C_{jl} is contained in C_j and open in C_j.

Proof. Clearly, a set E of the above form is closed and an element of $\mathcal{R}(G)$. Conversely, let $E \in \mathcal{R}(G)$ and suppose first that E is of the form $E = G_0 \setminus \bigcup_{l=1}^n K_l$, where G_0 is a subgroup of G and the K_l are cosets contained in G_0. By Lemma 5.6.13 there exists an open subgroup H of $\overline{G}_0$ such that $\overline{E}$ is a union of cosets of H. If $q : \overline{G}_0 \to \overline{G}_0/H$ is the quotient homomorphism then, by Lemma 5.6.12, $q(E) \in \mathcal{R}(\overline{G}_0/H)$, say

$$q(E) = \bigcup_{j=1}^{m} \left(D_j \setminus \bigcup_{l=1}^{m_j} D_{jl} \right),$$

where the D_j and D_{jl} are cosets in $\overline{G}_0/H$ (Lemma 5.6.9). Moreover, $q(E) = q(\overline{E})$ since q is continuous and $\overline{G}_0/H$ is discrete. Thus

$$\overline{E} = q^{-1}(q(\overline{E})) = q^{-1}(q(E)) = \bigcup_{j=1}^{m} \left(q^{-1}(D_j) \setminus \bigcup_{l=1}^{m_j} q^{-1}(D_{jl}) \right),$$

and each $q^{-1}(D_{jl})$ and $q^{-1}(D_j)$ is open in $\overline{G}_0$. This proves that $\overline{E} \in \mathcal{R}(G)$.

Now let E be an arbitrary set in $\mathcal{R}(G)$. Then $E = E_1 \cup \ldots \cup E_m$, where each E_i is a translate of a set of the type considered above. It follows that $\overline{E} = \overline{E}_1 \cup \ldots \cup \overline{E}_m \in \mathcal{R}(G)$ and $\overline{E}$ has the desired form. □

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