Appendix: Open Problems

In this appendix, we give a little guidance to some interesting open research projects and research directions in the vicinity of the PAM. These items have been mentioned already in this text, but they are scattered, and we felt it would be useful to collect them at one place. The list is certainly influenced by our personal taste. We do not repeat here areas that are too far away from the main body of this text, like random walk in random scenery (Sect. 7.4), self-attractive path measures (Sect. 7.5), drift and directed polymers (Sect. 7.10), and time-dependent potentials (Chap. 8). Certainly, if the PAM is put into a much broader connection, a lot of new exciting research areas open up, like the consideration of more general types of PDEs with random coefficients, or the addition of terms that introduce new physical effects, or other classes of time-depending potentials. However, recall that the scope of this book is defined by the characteristics that

- the solution admits an explicit solution,
- the analysis of its long-time behaviour can be based on large-deviation analysis and extreme-value statistics,
- the arising variational formulas are explicit and admit interpretations and deeper investigation,
- the solution shows a clear geometric picture,
- there is a clear connection with the spectrum of the Anderson operator.

For many interesting PDEs with random coefficients, already the solution theory is challenging, and it cannot be hoped for deriving a clear picture in near future; the research on them is entirely different from the study of the PAM as we reported on in this text. Let us give a list of open problems that we find interesting.

Other potential distributions. The PAM has not yet been analysed for some interesting potential distributions that are popular in the study of the spectrum of the Anderson Hamiltonian, like alloy-type potentials (see Example 1.19). Furthermore, random potentials with long correlations (see Sect. 7.2), in particular the Gaussian free field both on $\mathbb{Z}^d$ and on $\mathbb{R}^d$ and, more generally, log-correlated random fields,

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are currently much studied with respect to their local and global maxima. They show some interesting new structures like certain relations to branching processes and Poisson process descriptions with random intensity measures. Hence, it appears rather interesting to initiate a study of the PAM with such fields as the random input and to investigate how the correlation lengths may result into a new intermittency picture featuring new classes of intermittent islands (larger asymptotics of their radii, new variational formulas) or even a transition to a homogenised behaviour, possibly after inserting some extra changes, like some time-dependent pre-factors. See the last remark in Sect. 7.3 for a general ansatz for the study of the PAM with correlated fields, which seems to be particularly suitable for many Gaussian fields on $\mathbb{Z}^d$.

The PAM with a Gaussian white-noise potential in the spatially continuous setting, see Example 1.21, is currently very much in fashion. One reason is that the PAM presents one of the few important and prominent types of stochastic partial differential equations to which the florishing theory of regularity structures, rough paths and other related methods on the edge between analysis and probability theory is applied. The interest stems from the facts that (1) the continuous-space PAM with Gaussian white noise should arise as the rescaling limit of the discrete-space PAM in the spirit of Donsker's invariance principle, (2) the construction of a solution is not possible in all the dimensions and requires a subtle smoothing and rescaling, and (3) for future investigations with respect to long-time features like ageing and intermittency, the construction has to fulfill high requirements. Currently, the latest achievements are constructions of solutions to the PAM in selected dimensions in the full space, but only locally in time, and the mathematical foundation of the spectral theory of the Anderson Hamiltonian is in its infancy. The big enterprise will be to see how far the new techniques can be extended. Furthermore, the study of the local maxima of the Gaussian white noise will certainly lead to other results than the study of maxima of the local eigenvalues of the Anderson Hamiltonian (unlike in the case of a smooth Gaussian potential).

Eigenvalue order statistics, one-island concentration and time evolution. The most comprehensive and detailed description of the mass flow modeled by the PAM can be given in terms of a concentration property of the solution in just one of the intermittent islands (see Sect. 6.4) and an description of its location as a function of time (see Sect. 6.5). This in turn requires a full control on the behaviour of all the top eigenvalues and eigenfunctions of the Anderson Hamiltonian as given in terms of a Poisson process approximation; see Sect. 6.3. This programme has been carried out for practically all heavier-tailed potentials (in the sense of Example 1.14) and for the double-exponential distribution of Example 1.12. However, it is still open for such important distributions like the Bernoulli traps or, more generally, bounded potentials, also in the spatially continuous setting, i.e., Poisson traps, neither for any kind of Gaussian field. At the end of Sect. 6.3.3 we briefly pointed out that we expect that much of the techniques derived in [BisKön16] and [BisKönSan16] will be useful. But a substantial new input will be necessary to overcome problems related to the characteristic feature that small changes in the eigenvalue will not come from larger values of the potential, but larger sizes of the intermittent island. For general Gaussian potentials in the continuous setting, working out each of these points seems to be widely open and promising, even in the classical case where a high degree of regularity is assumed, like in [GärKönMol00]. For deriving such detailed information as an order statistics, a very precise and explicit control on the distribution of the potential seems necessary.

Anderson localisation via local maximisation. A further, quite ambitious, question is about the geometric interpretation of Anderson localisation deeper in the spectrum of the Anderson operator $\Delta^{\mathrm{d}} + \xi$. With the help of spatial versions of extreme-value statistics, one was able to characterise local areas of concentration of the leading eigenfunctions in large boxes and to derive a kind of Anderson localisation picture from that, as we explained in Sect. 6.3.4. However, one knows from much less explicit methods developed in the community of Anderson localisation that much more eigenfunctions are localised, not only the leading ones. We think it should be highly interesting to derive a geometric characterisation also of high local peaks in areas away from the highest potential peaks, which give rise to localisation of eigenfunctions away from the leading ones. The goal would be to use methods from extreme-value theory to describe such structures of local potential maxima and their influence on the spectrum of $\Delta^{\mathrm{d}} + \xi$. Since these methods would depend on an analysis in large boxes, one cannot strictly speak of Anderson localisation, hence, the next step must be to relate the findings in the large-box setting to the spectral properties in $\mathbb{Z}^d$.

Transition between concentration and homogenisation. As we reported on in Sect. 7.3, interesting critical regimes and phase transitions and variational formulas arose in the study of the PAM with an accelerated motion or, equivalently, a weakly interacting potential. A deep study was carried out for Brownian motion in a scaled Poisson trap field only (see Sect. 7.3.3), but in the general case on $\mathbb{Z}^d$, a variational formula (see (7.15)) was derived that might contain a phase transition in great generality if the parameter θ ranges from small to large values: the formula should have no solutions for small θ and should be compact for large ones. This has not yet been explored, and it should be done both on the level of the variational analysis and the behaviour of the PAM. Different behaviours in the respective dimensions should arise, and it will be interesting to find criteria on H (i.e., on the random potential) for them to hold. Intimately connected is the study of the (top of the) spectrum of $\Delta^{\mathrm{d}} + \varepsilon\xi$ in large, ε-depending boxes of various choices of the radii $\gg \varepsilon^{-2}$ in the limit $\varepsilon \downarrow 0$.

PAM in random environment. As we discussed in Sect. 7.9.1, another interesting enterprise is the PAM in random environment, i.e., when the simple random walk is replaced by a random walk in random environment. One natural choice is the random walk among random conductances (RWRC). Here, already some precise heuristics have been formulated (see Sect. 7.9.1), but yet only for the behaviour of the moments. Deeper insight and possibly the introduction of new methods will be necessary if one wants to study any of the almost sure settings, where one takes the average over the potential only, the environment only, or none of them. The RWRC is the easiest and most comfortable random environment to study, since it admits still the exploitation of a well-functioning ℓ^2-theory and an explicit variational analysis, because there is a symmetric generator. The study becomes much more challenging if instead a general random walk in random environment is considered. As a (already highly intriguing) pre-study, its long-time behaviour in boxes, possibly with slowly diverging radius, is interesting, i.e., an extension of the work reported on in Sect. 7.9.1 to non-symmetric random walks.

PAM with stable diffusivity. Replacing the driving motion (simple random walk and Brownian motion) by some random motion in the domain of attraction of stable processes, or replacing the Laplace operator by some fractional version of it, gives rise to new, interesting formulas and effects, both on the probabilistic and the analytic side. As we reported in Sect. 7.8, we are aware only of one work in that direction, and it is clear that the non-continuity of the paths, respectively the non-locality of the operator, give rise to additional effects that wait for exploration.

PAM with drift. In spite of a substantial interest in drifted random motions in random potential, most of the relevant works rely on subadditivity and do therefore not offer any explicit formulas, see Sect. 7.10. A notable exception is [Rue14] for the spatially continuous setting, i.e., Brownian motion in a quite general random potential. Here the logarithmic long-distance and the large-time asymptotics are expressed in terms of a variational formula involving the well-known energy term $\|\nabla\varphi\|_2^2$ and another one that describes the influence of the potential. The interpretation of these terms for the motion is not direct and requires some manipulations, but nevertheless it is there and presumably will give rise to some interesting discoveries, as soon as one undertakes efforts to make this relation more explicit. Also the understanding of the quenched setting will greatly benefit from a deeper analysis of the relation between the variational formula and suitable objects encoded in the Feynman-Kac formula.

More realistic biological population models. One of the main interpretations and applications of the PAM is in terms of spatial stochastic particle systems with branching and killing, as we remarked at some places, notably in Remark 2.1.1. However, in those cases of a static random potential that can produce large branching rates, the growth of the population is ridiculously large in the long-time limit, and the contrast between single sites with such a gigantic offspring production and the ample regions around with almost no growth is far beyond all reasonably observed real population histories. One obvious drawback of this model for the explanation of population histories is the absence of any kind of birth control, even at places where the local population is enormous. There is a lack of reasonable population models in random environments that include such effects, but can still be handled mathematically. One possibility is to combine spatial versions in random environment of the Moran model, which is well-known in biological stochastic modeling, where the number of individuals is kept fixed over the entire duration of the process, or of the Lenski experiment, where the population is randomly thinned out after certain time lags. However, it seems as if no substitute for the Feynman-Kac formula is in sight for such models.

Asymptotics for the (non-parabolic, time-dependent) Anderson Schrödinger equation. The Anderson Schrödinger equation in (1.4), i.e., the non-parabolic version, has been shown in [Wag14] to be amenable to a probabilistic analysis, see Remark 2.2. Indeed, the papers by Wagner seem to have opened the door to a comprehensive analysis of the Schrödinger equation with the help of the theory of spatial marked branching processes. This makes possible the application of a powerful probabilistic toolbox. Looking at the representation of the solution in (2.1), the biggest mathematical obstacle seems to be to find a useful Feynman-Kac formula (possibly on an enlarged state space, e.g., $\mathbb{Z}^d \times \{-1, 1\} \times \{+, -\}$) and to master the technical problems coming from the difference of the particle numbers, causing a lot of extinction.

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