Appendix A
Topology and Manifolds

In this appendix we provide some definitions that might be useful for readers interested in a more formal approach to General Relativity and black holes. For detailed treatments we refer to the books of Isham (2005), Frankel (2012), and Nash and Sen (2011).

A.1 Topology

Topology is the study of those properties of a geometric shape that are unchanged under continuous deformation. In more technical terms, topology deals with topological spaces. One of the main aspects of topology is that it allows to make qualitative predictions when quantitative ones are impossible or extremely difficult.

A.1.1 Topological Spaces

Let X be any set and $T = \{X_\alpha\}$ a collection, finite or infinite, of subsets of X. Then the ordered pair (X, T) forms a *topological space* iff:

1. $X \in T$.
2. $\emptyset \in T$.
3. Any finite or infinite sub-collection $\{X_1, X_2, \dots, X_n\}$ of the X_α is such that $\bigcup_1^n X_i \in T$.
4. Any *finite* sub-collection $\{X_1, X_2, \dots, X_n\}$ of the X_α is such that $\bigcap_1^n X_i \in T$.

The set X is called a topological space and the X_α are called *open sets*. The assignation of T to X is said to "give" a topology to X.

A function f mapping from the topological space X onto the topological space X^* is continuous if the inverse image of an open set in X^* is an open set in X.

If a set X has two topologies $T^1 = \{X_\alpha\}$ and $T^2 = \{X_\alpha^*\}$ such that $T^1 \supset T^2$, we say that T^1 is *stronger* than T^2.

G.E. Romero, G.S. Vila, *Introduction to Black Hole Astrophysics*,
Lecture Notes in Physics 876, DOI 10.1007/978-3-642-39596-3,

A.1.2 Neighborhoods

Given a topology T on X, then N is a *neighborhood* of a point $x \in X$ if $N \subset X$ and there is some $X_\alpha \subset N$ such that $x \in X_\alpha$. Notice that it is not necessary for N to be an open set. However, all open sets X_α which contain x are neighborhoods of x since they are contained in themselves. Thus, neighborhoods are more general than open sets.

A.1.3 Closed Sets

Let T be a topology on X. Then any $U \subset X$ is *closed* if the complement of U in X ($\bar{U} = X - U$) is an open set. Since $\bar{\bar{U}} = U$ then a set is open when its complement is closed. The sets X and $\emptyset$ are open and closed regardless the topology T.

A.1.4 Closure of a Set

Given a set U, there will be in general many closed sets that contain U. Let F_α be the family of closed sets that contain U. The *closure* of U is $\tilde{U} = \bigcap_\alpha F_\alpha$. The closure is the smallest closed set that contains U. Notice that $\tilde{\tilde{U}} = \tilde{U}$.

A.1.5 Boundary and Interior

The *interior* U^0 of a set U is the union of all open sets O_α of U: $U_0 = \bigcup_\alpha O_\alpha$. The interior of U is the largest open set of U.

The *boundary* $b(U)$ of a set U is the complement of the interior of U in the closure of U: $b(U) = \tilde{U} - U^0$. Closed sets always contain their boundaries:

$$U \cap b(U) = \emptyset \Longleftrightarrow U \text{ is open,}$$

$$b(U) \subset U \Longleftrightarrow U \text{ is closed.}$$

Notice that the sets (a,b), $[a,b)$, $(a,b]$, and $[a,b]$ all have the same boundary: $b = a, b$.

A.1.6 Compactness

Given a family of sets $\{F_\alpha\} = F$, F is a *cover* of U if $U \subset \bigcup_\alpha F_\alpha$. If $(\forall F_\alpha)_F$ (F_α is an open set) then the cover is called an *open cover*.

A set U is *compact* if for *every* open covering $\{F_\alpha\}$ with $U \subset \bigcup_\alpha F_\alpha$ there *always* exists a *finite* sub-covering $\{F_1, \ldots, F_n\}$ of U such that $\bigcup_1^n F_\alpha \subset U$.

As an illustration consider $\Re^n$. A subset X of $\Re^n$ is compact iff it is closed and bounded. This means that X must have finite area and volume in n-dimensions.

A.1.7 Connectedness

A set X is *connected* if it cannot be written as $X = X_1 \cup X_2$ where X_1 and X_2 are both open sets and $X_1 \cap X_2 = \emptyset$.

A.1.8 Homeomorphisms and Topological Invariants

Let T_1 and T_2 be two topological spaces. An *homeomorphism* is a map f from T_1 to T_2:

$$f : T_1 \to T_2$$

such that f is continuous and its inverse map f^{-1} is also continuous. If there is a third topological space T_3 such that T_1 is homeomorphic to T_2 and T_2 is homeomorphic to T_3, then T_1 is homeomorphic to T_3. An homeomorphism defines an equivalence class, that of all spaces that are homeomorphic to a given topological space. If the homeomorphism f and its inverse f^{-1} are infinitely differentiable (C^∞), then f is called a *diffeomorphism*. All diffeomorphisms are homeomorphisms, but the converse is not always the case.

A *topological invariant* is a construct that does not change under homeomorphisms. They are characteristics of the equivalence class of the homeomorphism. An example of an invariant is the dimension n of $\Re^n$.

Homeomorphisms generate equivalence classes whose members are topological spaces. Instead, *homotopies* generate classes whose members are continuous maps. More specifically, let f_1 and f_2 be two continuous maps between the topological spaces T_1 and T_2:

$$f_1 : T_1 \to T_2,$$
$$f_2 : T_1 \to T_2.$$

Then f_1 is said to be homotopic to f_2 if f_1 can be deformed into f_2. Formally:

$$F : T_1 \times [0, 1] \to T_2, \qquad F \text{ continuous}$$

and

$$F(x, 0) = T_1(x),$$

$$F(x, 1) = T_2(x).$$

This means that as the real variable t changes continuously from 0 to 1 in the interval [0, 1] the map f_1 is deformed continuously into the map f_2. Homotopy is an equivalence relation that divides the space of continuous maps from T_1 to T_2 into equivalent classes. These homotopy equivalent classes are topological invariants of the pair of spaces T_1 and T_2.

Homotopy can be used to classify topological spaces. If we identify one of the topological spaces with the n-dimensional sphere S^n, then the space of continuous maps from S^n to T, $C(S^n, T)$, can be divided into equivalence classes according to the topological space T. The equivalent classes of $C(S^n, T)$ have a group structure and form the homotopy group $\Pi_n(T)$.

It is rather straightforward to show that both compactness and connectedness are topological invariants (Nash and Sen 2011).

A.2 Manifolds

Whereas topology is the natural mathematical framework to study continuity, differential geometry is a natural framework to study differentiability. Since differentiability implies continuity, but not the other way around, differential geometry is more specific than topology. The concept of *manifold* is central to differential geometry.

A.2.1 Manifolds: Definition and Properties

A set M is a differentiable manifold if:

1. M is a topological space.
2. M is equipped with a family of pairs $\{(M_\alpha, \varphi_\alpha)\}$.
3. The M_α's are a family of open sets that cover M: $M = \bigcup_\alpha M_\alpha$. The φ_α's are homeomorphisms from M_α to open subsets O_α of $\Re^n$: $\varphi_\alpha : M_\alpha \to O_\alpha$.
4. Given M_α and M_β such that $M_\alpha \cap M_\beta \neq \emptyset$, the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ from the subset $\varphi_\alpha(M_\alpha \cap M_\beta)$ of $\Re^n$ to the subset $\varphi_\beta(M_\alpha \cap M_\beta)$ of $\Re^n$ is infinitely differentiable (C^∞).

The family $\{(M_\alpha, \varphi_\alpha)\}$ is called an *atlas*. The individual members of the altas are *charts*. In informal language we can say that M is a space that can be covered by patches M_α which are assigned coordinates in $\Re^n$ by φ_α. Within each of these patches M looks like a subset of the Euclidean space $\Re^n$. M is not necessarily globally Euclidean or pseudo-Euclidean. If two patches overlap, then in $M_\alpha \cap M_\beta$ there are two assignations of coordinates, which can be transformed smoothly into each other. The dimension of the manifold M is the dimension n of the space $\Re^n$.

A manifold M is said to be Hausdorff if for any two distinct elements $x \in M$ and $y \in M$, there exist $O_x \subset M$ and $O_y \subset M$ such that $O_x \cap O_y = \emptyset$.

A given topological space M is said to be *metric* if the open sent are provided by a binary function $d(x, y)$ such that:

1. $d(x, y) \geq 0$.
2. $d(x, y) = 0$ iff $x = y$.
3. If $z \in M$, then $d(x, y) + d(y, z) = d(x, z)$.

An important property of manifolds is their *orientability*. Given a manifold M whose atlas is $\{(M_\alpha, \varphi_\alpha)\}$, M is orientable if $\det(\varphi_\beta \circ \varphi_\alpha^{-1}) > 0$ for all M_α and M_β such that $M_\alpha \cap M_\beta \neq \emptyset$. The manifold is orientable if one can define a preferred direction unambiguously.

A.2.2 Fiber Bundles

A fiber bundle is a topological space that is *locally, but not necessarily globally* the product of two spaces. All spaces that are globally products are called *trivial bundles*. Fiber bundles can be defined upon many spaces of use in physics and, in particular, in gravitation, where, as we shall see, the tangent spaces to a manifold form a bundle.

Formally, a collection (E, Π, F, G, X) is called a fiber bundle iff:

1. E is a topological space, usually called the *total space*.
2. X is a topological space, usually called the *base space*.
3. F is a topological space, usually called the *fibre*.
4. Π is an application $\Pi : E \to X$ of E onto X, called the *projection*.
5. G is a group of homeomorphisms on the fiber F.
6. There is a set of open coordinate neighborhoods $\{U_\alpha\}$ covering X, which reflects the local triviality of E. Specifically, with each U_α there is a given homeomorphism such that:

$$\varphi_\alpha : \Pi^{-1}(U_\alpha) \to U_\alpha \times F, \tag{A.1}$$

and

$$\Pi \varphi_\alpha^{-1}(x, f) = x, \quad \text{with } x \in U_\alpha,\ f \in F. \tag{A.2}$$

Let us consider a transformation of a local coordinate set $\{\varphi_\alpha, U_\alpha\}$ to another set $\{\varphi_\beta, U_\beta\}$. Let us suppose that $U_\alpha \cap U_\beta \neq \emptyset$. Then $g_{\alpha\beta}(x) = \varphi_\alpha \circ \varphi_\beta^{-1}$ is a continuous invertible map of the form

$$g_{\alpha\beta}(x) : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F. \tag{A.3}$$

The function $g_{\alpha\beta}(x)$ is a homeomorphism of the fiber F, called the *transition function*. The set of all these homeomorphisms for all choices of $\{\varphi_\alpha, U_\alpha\}$ form the group G. This group is called the *structure group* of the fiber bundle E.

Perhaps the simplest illustration of a fiber bundle is a Möbius strip. Such a space is illustrated in Fig. A.1. Here, the total space E is the whole Möbius strip. The

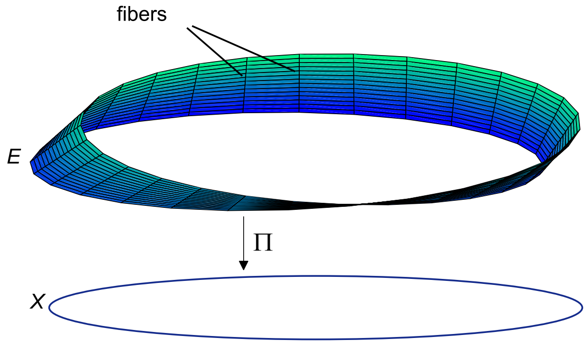

Fig. A.1 Simple example of a fiber bundle: the Möbius strip. See the text for explanation

base space X is the projection onto $\Re^2$. The fiber F is a line segment on the strip obtained from a point of $x \in X$ by $\Pi^{-1}(x)$. To an open set U_α of X corresponds a set of fibers on E. The functions φ_α transforms $\Pi^{-1}(U_\alpha)$ into the product $U_\alpha \times F$. We see, then, that whereas a manifold is locally like $\Re^n$ a fiber bundle is locally like a product of topological spaces.

Manifolds and fiber bundles are related since we can define a fiber bundle formed by all tangent spaces to a given manifold. Specifically, for each manifold M there is a fiber bundle $T(M)$ (called *tangent bundle*) given by:

$$T(M) = \bigcup_{p \in M} T_p(M). \tag{A.4}$$

The base space of the fiber bundle is M, the fiber at any point p of M is the tangent space $T_p(M)$, and the projection is defined by:

$$\Pi : T(M) \to M, \tag{A.5}$$

and

$$\mathbf{V} \in T_p(M) \to p. \tag{A.6}$$

The fiber $T_p(M)$ at p is a vector space of dimension n, equal to the dimension of the manifold. Let now $p \in U_\alpha \subset M$, then

$$\varphi_\alpha : \Pi^{-1}(U_\alpha) \to U_\alpha \times \Re^n, \tag{A.7}$$

and

$$\mathbf{V} \to \big(p, a^i(p)\big), \tag{A.8}$$

where $a^i(p)$ is a coordinate assignation to p. The group structure is given by the group of invertible $n \times n$ matrices.

The main role of the tangent bundle is to provide a domain and range for the derivative of a smooth function. Namely, if $f : M \to M'$ is a smooth function, with M and M' smooth manifolds, its derivative is also a smooth function.

References

T. Frankel, *The Geometry of Physics*, 3rd edn. (Cambridge University Press, Cambridge, 2012)
C. Isham, *Modern Differential Geometry for Physicists*, 2nd edn. (World Scientific, Singapore, 2005)
C. Nash, S. Sen, *Topology and Geometry for Physics* (Dover, Mineola, 2011)

Appendix B
Selected and Annotated Bibliography

The following bibliography intends to be merely orientative and by no way is complete. It reflects the authors' taste and contains some of those books we most frequently resort to in our library.

B.1 Books on General Relativity

- *Gravitation*, by C.W. Misner, K.S. Thorne and J.A. Wheeler, W.H. Freeman & Co., New York, 1973.
  Complete and detailed. Part VII is devoted to gravitational collapse and black holes.
- *General Relativity*, by R.M. Wald, The University of Chicago Press, Chicago, 1984.
  A modern introduction to General Relativity with emphasis on global techniques.
- *The Large Structure of Space-Time*, by S.W. Hawking and G.F.R. Ellis, Cambridge University Press, Cambridge, 1973.
  The classic reference on global techniques in General Relativity.
- *Introducing Einstein's Relativity*, by R. D'Inverno, Clarendon Press, Oxford, 1992.
  A complete and very useful reference on General Relativity and black holes.
- *Space-Time and Geometry*, by S. Carroll, Addison Wesley, San Francisco, 2004.
  Clear and readable introduction for undergraduate and graduate students.
- *Relativity, Second Edition*, by W. Rindler, Oxford University Press, Oxford, 2006.
  An excellent textbook.
- *General Relativity*, by M.P. Hobson, G. Efstathiou and A.N. Lasenby, Cambridge University Press, Cambridge, 2006.
  A very clear and complete introduction to mathematical aspects of General Relativity.
- *General Relativity*, by N. Straumann, Springer, Berlin, 2004.
  Complete, mathematically strong, with several astrophysical applications.

G.E. Romero, G.S. Vila, *Introduction to Black Hole Astrophysics*,
Lecture Notes in Physics 876, DOI 10.1007/978-3-642-39596-3,

- *Gravity*, by J.B. Hartle, Addison Wesley, San Francisco, 2003.
  Undergraduate textbook with strong emphasis on the physical interpretations.
- *General Relativity and the Einstein's Equations*, by Y. Choquet-Bruhat, Oxford University Press, Oxford, 2009.
  A mathematically sophisticated monograph on General Relativity.
- *Introduction to General Relativity*, by L. Ryder, Cambridge University Press, Cambridge, 2009.
  Student-friendly and well-illustrated basic textbook.
- *General Relativity*, by M. Ludvigsen, Cambridge University Press, Cambridge, 1999.
  A geometric and abstract presentation of General Relativity.
- *Global Aspects in Gravitation and Cosmology*, by P.S. Joshi, Clarendon Press, Oxford, 1993.
  A good complement to Hawking and Ellis' book.
- *Numerical Relativity*, by T.W. Baumgarte and S.L. Shapiro, Cambridge University Press, Cambridge, 2010.
  Essential to the working scientist.
- *A Relativist's Toolkit*, by E. Poisson, Cambridge University Press, Cambridge, 2004.
  Faithful to the title. It is a must for every relativist.
- *Relativity on Curved Manifolds*, by F. De Felice and C.J.S. Clarke, Cambridge University Press, Cambridge, 1990.
  An advanced monograph.
- *Advanced General Relativity*, by J. Stewart, Cambridge University Press, Cambridge, 1991.
  Good reference for advanced topics such as spinors and asymptopia.
- *Rotating Fields in General Relativity*, by J.N. Islam, Cambridge University Press, Cambridge, 1985.
  Monograph fully devoted to axially symmetric solutions of Einstein's equations.
- *An Introduction to the Relativistic Theory of Gravitation*, by P. Hajicek, Springer, Heidelberg, 2008.
  Good introductory course.
- *Lecture Notes on the General Theory of Relativity*, by Ø. Grøn, Springer, Heidelberg, 2009.
  Short overview of the topic.
- *An Introduction to Relativity*, by J.V. Narlikar, Cambridge University Press, Cambridge, 2010.
  Undergraduate level. It deals with some controversial topics.
- *A First Course on General Relativity, Second Edition*, by B.F. Schutz, Cambridge University Press, Cambridge, 2009.
  Standard textbook.
- *Tensor Relativity and Cosmology*, by M. Dalarsson and N. Dalarsson, Elsevier, Amsterdam, 2005.
  An introduction which includes many explicit calculations.
- *Gravitation*, by T. Padmanabhan, Cambridge University Press, Cambridge, 2010.
  Complete, modern, with advanced topics.

- *Relativity: the General Theory*, by J.L. Synge, North-Holland Publishing Company, Amsterdam, 1960.
  Excellent book by one of the greatest relativists. Still a unique source on several topics.
- *Gravitation and Spacetime, Second Edition*, by H.C. Ohaniam and R. Ruffini, W.W. Norton & Co., New York, 1994.
  The chapter on black holes is particularly good.
- *The Theory of Space, Time and Gravitation, Second Revised Edition*, by V.A. Fock, Pergamon Press, Oxford, New York, 1964.
  A deep exposition a là Landau.
- *The Theory of Relativity, Second Edition*, by R.K. Pathria, Pergamon Press, Oxford, 1974.
  A nice oldie.
- *Space Time Matter*, by H. Weyl, Dover, New York, 1952.
  The first and still one of the best books on General Relativity (first edition 1918).
- *Theory of Relativity*, by W. Pauli, Dover, New York, 1958.
  A classic (first edition 1921).
- *The Mathematical Theory of Relativity*, by A.S. Eddington, Cambridge University Press, Cambridge, 1923.
  Classic.
- *Relativity, Thermodynamics and Cosmology*, by R.C. Tolman, Dover, New York, 1987.
  A classic and one of the few books dealing with relativistic thermodynamics. Originally published in 1934.
- *Introduction to the Theory of Relativity*, by P.G. Bergmann, Dover, New York, 1976.
  A book strongly recommended by Einstein himself. One of the few books discussing the later field theories developed by Einstein. Originally published in 1942.
- *General Theory of Relativity*, by P.A.M. Dirac, Princeton University Press, Princeton, 1996.
  The lectures given by Dirac on the topic. Originally published in 1975.
- *Principles of Relativity Physics*, by J.L. Anderson, Academic Press, New York, 1967.
  An excellent book, full of physical insight.

B.2 Books on Black Holes

- *Black Holes*, by J.-P. Luminet, Cambridge University Press, Cambridge, 1992.
  A popular thought-provoking introduction.
- *Gravity's Fatal Attraction*, by M. Begelman and M. Rees, Scientific American, New York, 1998.
  Superbly illustrated, conceptually clear.
- *Exploring Black Holes*, by E.F. Taylor and J.A. Wheeler, Addison Wesley, San Francisco, 2000.

A didactic primer.

- *Stars and Relativity*, by Y.B. Zel'dovich and I.D. Novikov, Dover, New York, 1996.
  Originally published in 1971, it was one of the first books to discuss black holes by then named "frozen stars".
- *Black Holes: the Membrane Paradigm*, by K. Thorne, R.H. Price and D.A. McDonald, Yale University Press, New Haven, 1986.
  The main reference on a much debated analogy.
- *Black Holes*, by D. Raine, E. Thomas, Imperial College Press, London, 2005.
  Good, well-written and concise.
- *Black Holes and Relativistic Stars*, by R.M. Wald (ed.), Chicago University Press, Chicago, 1998.
  An outstanding collection of original papers dedicated to the memory of S. Chandrasekhar.
- *The Mathematical Theory of Black Holes*, by S. Chandrasekhar, Oxford University Press, Oxford, 1983.
  There is a lot of material that you will only find in this book, but not recommended for the beginner.
- *Introduction to Black Hole Physics*, by P.V. Frolov and A. Zelnikov, Oxford University Press, Oxford, 2011.
  Perhaps the best book available on black holes.
- *Black Hole Gravito-Hydromagnetics, Second Edition*, by B. Punsly, Springer, Berlin, 2008.
  Strongly focused on ergospheric effects and relativistic magnetohydrodynamics.
- *Black Holes, White Dwarfs and Neutron Stars*, by S.L. Shapiro and S.A. Teukolsky, John Wiley & Sons, New York, 1983.
  Classic but a bit outdated on some of the astrophysical aspects.
- *Black Hole Physics*, by V.P. Frolov and I.D. Novikov, Kluwer Academic Publishers, Dordretch, 1998.
  As complete as expensive, but if you can afford it very worthy.
- *Black Hole Uniqueness Theorems*, by M. Heusler, Cambridge University Press, Cambridge, 1996.
  Advanced and unique in its kind.
- *Physics and Astrophysics of Neutron Stars and Black Holes, Second Edition*, by R. Giacconi and R. Ruffini, Cambridge Scientific Publishers, Cambridge, 2009.
  A rich resource of material on both Physics and Astrophysics of black holes.

B.3 Books on Related Topics in Astrophysics

- *High-Energy Radiation from Black Holes*, by C.D. Dermer and G. Menon, Princeton University Press, Princeton, 2009.
  Mostly devoted to radiative processes.
- *Relativistic Astrophysics and Cosmology*, by P. Hoyng, Springer, Heidelberg, 2006.
  Concise and insightful.

- *Compact Objects in Astrophysics*, by M. Camenzind, Springer, Berlin, 2007.
  Very complete with many applications.
- *MHD Flows in Compact Astrophysical Objects*, by V.S. Beskin, Springer, Heidelberg, 2010.
  Key reference for jets and outflows.
- *Active Galactic Nuclei*, by J.H. Krolik, Princeton University Press, Princeton, 1999.
  Covers both observational and theoretical aspects of AGN.
- *The Physics of Extragalactic Radio Sources*, by D.S. De Young, The University of Chicago Press, Chicago, 2002.
  Includes a very good discussion of the hydrodynamics of jets.
- *Quasars and Active Galactic Nuclei*, by A.K. Kembhavy and J.V. Narlikar, Cambridge University Press, Cambridge, 1999.
  Highly-recommended as an introduction to the topic.
- *High-Energy Astrophysics, Third edition*, by M.S. Longair, Cambridge University Press, Cambridge, 2011.
  Outstanding textbook.
- *Very High-Energy Cosmic Gamma Radiation*, by F.A. Aharonian, World Scientific, New Jersey, 2004.
  Broad coverage of gamma-ray astronomy.
- *Beams and Jets in Astrophysics*, by P.A. Hughes (ed.), Cambridge University Press, Cambridge, 1991.
  Very useful source of information for the researcher.
- *Theory of Black Hole Accretion Disk*, by M.A. Abramowicz, G. Björnsson and J.E. Pringle (eds.), Cambridge University Press, Cambridge, 1998.
  A menagerie of valuable reviews. We specially recommend those on ADAFs and stability in black hole binaries.
- *Accretion Power in Astrophysics, Second Edition*, by J. Frank, A. King and D. Raine, Cambridge University Press, Cambridge, 1992.
  Perhaps the most widely used book on accretion by astrophysicists.
- *Accretion*, by A. Treves, L. Maraschi and M.A. Abramowicz, World Scientific, Singapur, 1989.
  The subtitle of this book is "A Collection of Influential Papers". It delivers what it promises.
- *X-ray Binaries*, by W.H.G. Lewin, J. van Paradijs and E.P.J. van den Heuvel (eds.), Cambridge University Press, Cambridge, 1997.
  A complete introduction.
- *High-Energy Astrophysics*, by F. Melia, Princeton University Press, Princeton, 2009.
  Discusses black holes in binaries, gamma-ray bursts and supermassive black holes.
- *Relativistic Astrophysics of the Transient Universe*, by M.H.P.M. van Putten and A. Levinson, Cambridge University Press, Cambridge, 2012.
  Updated and engaging treatment of many astrophysical manifestations of black holes.

Index

G.E. Romero, G.S. Vila, *Introduction to Black Hole Astrophysics*, Lecture Notes in Physics 876, DOI 10.1007/978-3-642-39596-3,

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