Appendix

A.1 Functional spaces and notation

In this section we introduce some definitions and notations which have been often used in the preceding chapters. For more detailed presentations and descriptions of the functional spaces useful in electromagnetism, see, e.g., Nečas [184], Adams [2], Adams and Fournier [3], Girault and Raviart [111], Dautray and Lions [94], Cessenat [76], Bossavit [59], Monk [179].

Let us consider an open, connected and bounded set Ω contained in $\mathbb{R}^3$, with a Lipschitz continuous boundary $\partial\Omega$, and let Σ be a Lipschitz continuous surface contained in $\partial\Omega$. The unit outward normal vector on $\partial\Omega$ is indicated by $\mathbf{n}$.

We denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions having compact support in Ω, i.e., vanishing outside an open set $\Omega' \subset \Omega$ which has a positive distance from the boundary $\partial\Omega$ of Ω.

The space of functions that are bounded in Ω (with the possible exception of a subset of measure equal to 0) is denoted by $L^\infty(\Omega)$, with norm $\| \cdot \|_{L^\infty(\Omega)}$.

For a function defined in Ω, for any $s \in \mathbb{R}$ the Sobolev space of order s is denoted by $H^s(\Omega)$. The norm in this space is indicated by $\| \cdot \|_{s,\Omega}$. For functions defined on the surface Σ, for any $t \in [-1, 1]$ the Sobolev space of order t is denoted by $H^t(\Sigma)$, with norm $\| \cdot \|_{t,\Sigma}$. As usual, the space $H^0(\Omega)$ (respectively, $H^0(\Sigma)$) is always denoted by $L^2(\Omega)$ (respectively, $L^2(\Sigma)$). We also recall that the space $H^{1/2}(\Sigma)$ is the space of the values on Σ (or, equivalently, the traces on Σ) of functions belonging to $H^1(\Omega)$, and that $H^{-t}(\Sigma)$ is the dual space of $H^t(\Sigma)$, $t \in [0, 1]$.

The space $H^1_{0,\Sigma}(\Omega)$ consists of those $H^1(\Omega)$-functions that have a vanishing value on Σ. When $\Sigma = \partial\Omega$, we simply write $H^1_0(\Omega)$ instead of $H^1_{0,\partial\Omega}(\Omega)$.

For a real number s with $0 \le s \le 1$ and a domain $D = \Omega$ or $D = \Sigma$, Σ a closed surface, we are also interested in the space $H^s(D)/\mathbb{C}$, whose elements are identified

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if they differ by a (complex) constant. This space is endowed with the following norm

$$\|v\|_{H^s(D)/\mathbb{C}} := \begin{cases} \left(\int_D |v - v_D|^2 \right)^{1/2} & \text{for } s = 0 \\ \left(\int_D |v - v_D|^2 + |v|^2_{s,D} \right)^{1/2} & \text{for } 0 < s \leq 1 \,, \end{cases}$$

where $v_D := (\text{meas}\, D)^{-1} \int_D v$ is the mean value of v and $|v|_{s,D}$ denotes the semi-norm of v in $H^s(D)$. In particular, if the function $v \in H^s(D)/\mathbb{C}$ is chosen with $v_D = 0$, we have $\|v\|_{H^s(D)/\mathbb{C}} = \|v\|_{s,D}$. Moreover, due to the Poincaré inequality (see, e.g., Dautray and Lions [94], Chap. IV, Sect. 7, Prop. 2), we have that in $H^1(\Omega)/\mathbb{C}$ the semi-norm $\|\operatorname{grad} v\|_{0,\Omega}$ is indeed an equivalent norm.

When considering vector-valued functions $\mathbf{v} : \Omega \to \mathbb{R}^3$, the space

$$H(\text{div}; \Omega) := \left\{ \mathbf{v} \in (L^2(\Omega))^3 \mid \text{div}\, \mathbf{v} \in L^2(\Omega) \right\}$$

is often used. It is endowed with the graph norm, i.e.,

$$\|\mathbf{v}\|_{H(\text{div};\Omega)} := (\|\mathbf{v}\|^2_{0,\Omega} + \|\,\text{div}\, \mathbf{v}\|^2_{0,\Omega})^{1/2} \,.$$

Similarly, we employ the space

$$H(\text{curl}; \Omega) := \left\{ \mathbf{v} \in (L^2(\Omega))^3 \mid \text{curl}\, \mathbf{v} \in (L^2(\Omega))^3 \right\} ,$$

with the norm

$$\|\mathbf{v}\|_{H(\text{curl};\Omega)} := (\|\mathbf{v}\|^2_{0,\Omega} + \|\,\text{curl}\, \mathbf{v}\|^2_{0,\Omega})^{1/2} \,.$$

Moreover, we set

$$H_{0,\Sigma}(\text{div}; \Omega) := \{ \mathbf{v} \in H(\text{div}; \Omega) \,|\, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Sigma \}$$

$$H_{0,\Sigma}(\text{curl}; \Omega) := \{ \mathbf{v} \in H(\text{curl}; \Omega) \,|\, \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Sigma \}$$
$$H^0(\text{div}; \Omega) := \{ \mathbf{v} \in H(\text{div}; \Omega) \mid \text{div}\, \mathbf{v} = 0 \text{ in } \Omega \}$$
$$H^0(\text{curl}; \Omega) := \{ \mathbf{v} \in H(\text{curl}; \Omega) \mid \text{curl}\, \mathbf{v} = \mathbf{0} \text{ in } \Omega \}$$
$$H^0_{0,\Sigma}(\text{div}; \Omega) := H_{0,\Sigma}(\text{div}; \Omega) \cap H^0(\text{div}; \Omega)$$
$$H^0_{0,\Sigma}(\text{curl}; \Omega) := H_{0,\Sigma}(\text{curl}; \Omega) \cap H^0(\text{curl}; \Omega) \,.$$

When $\Sigma = \partial\Omega$, we simply write $H_0(\text{div}; \Omega)$ instead of $H_{0,\partial\Omega}(\text{div}; \Omega)$, and similarly for the other cases.

For a symmetric matrix $\boldsymbol{\eta} = \boldsymbol{\eta}(\mathbf{x})$, uniformly positive definite in Ω and with entries belonging to $L^\infty(\Omega)$, we also set

$$H(\boldsymbol{\eta}, \text{div}; \Omega) := \{ \mathbf{v} \in (L^2(\Omega))^3 \mid \text{div}(\boldsymbol{\eta}\mathbf{v}) \in L^2(\Omega) \}$$

$$H_{0,\Sigma}(\boldsymbol{\eta}, \text{div}; \Omega) := \{ \mathbf{v} \in H(\boldsymbol{\eta}, \text{div}; \Omega) \,|\, \boldsymbol{\eta}\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Sigma \} \,.$$

To characterize the tangential boundary value of a vector belonging to $H(\text{curl}; \Omega)$ we need some preliminaries concerning tangential differential operators. The standard definition of the tangential gradient and the tangential curl on the flat surface $\{x_3 = 0\}$, having chosen the unit outward normal vector $\mathbf{n} = (0, 0, 1)$, is

$$\mathrm{grad}_\tau \phi = (\partial_1 \phi, \partial_2 \phi, 0) \ , \ \ \mathrm{Curl}_\tau\, \phi = \mathrm{grad}_\tau \phi \times \mathbf{n} = (\partial_2 \phi, -\partial_1 \phi, 0)\,.$$

Starting from this, if $\Sigma \subset \partial\Omega$ is a closed surface (namely, a surface without boundary), using local coordinates (see, e.g., Nečas [184]) it is possible to define the operators grad_τ and Curl_τ for functions belonging to $H^1(\Sigma)$, and one obtains $\mathrm{grad}_\tau \phi \in \mathcal{L}^2_t(\Sigma)$ and $\mathrm{Curl}_\tau \phi \in \mathcal{L}^2_t(\Sigma)$, where

$$\mathcal{L}^2_t(\Sigma) := \{\mathbf{v} \in (L^2(\Sigma))^3 \,|\, \mathbf{v} \cdot \mathbf{n} = 0\}\,.$$

By a duality argument, the adjoint operators

$$\mathrm{div}_\tau : \mathcal{L}^2_t(\Sigma) \to H^{-1}(\Sigma)$$

and

$$\mathrm{curl}_\tau : \mathcal{L}^2_t(\Sigma) \to H^{-1}(\Sigma)$$

are also introduced, and the Laplace–Beltrami operator

$$\Delta_\tau : H^1(\Sigma) \to H^{-1}(\Sigma)$$

is defined as $\Delta_\tau := \mathrm{div}_\tau\, \mathrm{grad}_\tau = -\, \mathrm{curl}_\tau\, \mathrm{Curl}_\tau$.

These operators can be restricted to other spaces: in particular, one can verify that the following relation holds

$$\mathrm{grad}_\tau \phi = (\mathbf{n} \times \mathrm{grad}\, \widetilde{\phi} \times \mathbf{n})_{|\Sigma} \ , \ \ \phi \in H^{3/2}(\Sigma)\,,$$

where we have set $H^{3/2}(\Sigma) := \{\varphi_{|\Sigma} \,|\, \varphi \in H^2(\Omega)\}$ and $\widetilde{\phi}$ is any extension of ϕ to $H^2(\Omega)$. Similarly, it holds

$$\mathrm{Curl}_\tau \phi = \mathrm{grad}_\tau \phi \times \mathbf{n} \ , \ \ \phi \in H^{3/2}(\Sigma)\,.$$

Clearly, in this case we have $\mathrm{grad}_\tau \phi \in H^{1/2}_T(\Sigma)$, where

$$H^{1/2}_T(\Sigma) := \{(\mathbf{n} \times \mathbf{v} \times \mathbf{n})_{|\Sigma} \,|\, \mathbf{v} \in (H^1(\Omega))^3\}\,, \tag{A.1}$$

and $\mathrm{Curl}_\tau \phi \in H^{1/2}_\times(\Sigma)$, where

$$H^{1/2}_\times(\Sigma) := \{(\mathbf{v} \times \mathbf{n})_{|\Sigma} \,|\, \mathbf{v} \in (H^1(\Omega))^3\}\,, \tag{A.2}$$

and moreover for each $\boldsymbol{\lambda} \in H^{1/2}_T(\Sigma)$ we have $(\boldsymbol{\lambda} \times \mathbf{n}) \in H^{1/2}_\times(\Sigma)$, and viceversa for each $\boldsymbol{\lambda} \in H^{1/2}_\times(\Sigma)$ we have $(\boldsymbol{\lambda} \times \mathbf{n}) \in H^{1/2}_T(\Sigma)$. Let us also note that the spaces $H^{1/2}_T(\Sigma)$ and $H^{1/2}_\times(\Sigma)$ are both equal to the space

$$H^{1/2}_t(\Sigma) := \{\boldsymbol{\lambda} \in H^{1/2}(\Sigma) \,|\, \boldsymbol{\lambda} \cdot \mathbf{n} = 0\}$$

if Σ is a smooth surface, while a characterization of them for a polyhedral domain is given in Buffa and Ciarlet [70].

The dual operators div_τ and curl_τ now read (all the integrals should be intended as duality pairings)

$$\int_\Sigma (\mathrm{div}_\tau \boldsymbol{\lambda})\, \phi := -\int_\Sigma \boldsymbol{\lambda} \cdot \mathrm{grad}_\tau \phi \ , \ \ \boldsymbol{\lambda} \in (H^{1/2}_T(\Sigma))' \ , \ \phi \in H^{3/2}(\Sigma)$$

$$\int_\Sigma (\mathrm{curl}_\tau \boldsymbol{\lambda})\, \phi := \int_\Sigma \boldsymbol{\lambda} \cdot \mathrm{Curl}_\tau \phi \ , \ \ \boldsymbol{\lambda} \in (H_\times^{1/2}(\Sigma))' \ , \ \phi \in H^{3/2}(\Sigma) \, ,$$

and clearly one has that $\mathrm{div}_\tau \boldsymbol{\lambda} \in (H^{3/2}(\Sigma))'$ and $\mathrm{curl}_\tau \boldsymbol{\lambda} \in (H^{3/2}(\Sigma))'$. It can also be checked that $\boldsymbol{\lambda} \in (H_\times^{1/2}(\Sigma))'$ if and only if $(\boldsymbol{\lambda} \times \mathbf{n}) \in (H_T^{1/2}(\Sigma))'$, and moreover that $\mathrm{div}_\tau(\boldsymbol{\lambda} \times \mathbf{n}) = \mathrm{curl}_\tau \boldsymbol{\lambda}$ for each $\boldsymbol{\lambda} \in (H_\times^{1/2}(\Sigma))'$.

We can now state the result concerning the characterization of the space of tangential traces on Σ or of tangential components on Σ of functions belonging to $H(\mathrm{curl};\Omega)$. In Buffa and Ciarlet [69], Buffa et al. [71] it has been proved that the space of tangential traces $(\mathbf{v} \times \mathbf{n})_{|\Sigma}$ on Σ for $\mathbf{v} \in H(\mathrm{curl};\Omega)$ is given by

$$H^{-1/2}(\mathrm{div}_\tau;\Sigma) := \{\boldsymbol{\lambda} \in (H_T^{1/2}(\Sigma))' \mid \mathrm{div}_\tau \boldsymbol{\lambda} \in H^{-1/2}(\Sigma)\} \, , \qquad \text{(A.3)}$$

with the graph norm

$$\|\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{div}_\tau;\Sigma)} := (\|\boldsymbol{\lambda}\|^2_{(H_T^{1/2}(\Sigma))'} + \|\,\mathrm{div}_\tau \boldsymbol{\lambda}\|^2_{-1/2,\Sigma})^{1/2} \, ,$$

while the space of tangential components $(\mathbf{n} \times \mathbf{v} \times \mathbf{n})_{|\Sigma}$ on Σ for $\mathbf{v} \in H(\mathrm{curl};\Omega)$ is given by

$$H^{-1/2}(\mathrm{curl}_\tau;\Sigma) := \{\boldsymbol{\lambda} \in (H_\times^{1/2}(\Sigma))' \mid \mathrm{curl}_\tau \boldsymbol{\lambda} \in H^{-1/2}(\Sigma)\} \, , \qquad \text{(A.4)}$$

with the graph norm

$$\|\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{curl}_\tau;\Sigma)} := (\|\boldsymbol{\lambda}\|^2_{(H_\times^{1/2}(\Sigma))'} + \|\,\mathrm{curl}_\tau \boldsymbol{\lambda}\|^2_{-1/2,\Sigma})^{1/2} \, .$$

It can be also shown that two these spaces are in duality, and that one has $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{curl}_\tau;\Sigma)$ if and only if $(\boldsymbol{\lambda} \times \mathbf{n}) \in H^{-1/2}(\mathrm{div}_\tau;\Sigma)$. Moreover, it holds $\mathrm{div}_\tau(\boldsymbol{\lambda} \times \mathbf{n}) = \mathrm{curl}_\tau \boldsymbol{\lambda}$ for each $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{curl}_\tau;\Sigma)$. Let us finally note that, when Σ is a smooth surface, these trace spaces can be described as

$$H^{-1/2}(\mathrm{div}_\tau;\Sigma) := \{\boldsymbol{\lambda} \in H^{-1/2}(\Sigma) \mid \boldsymbol{\lambda} \cdot \mathbf{n} = 0, \mathrm{div}_\tau \boldsymbol{\lambda} \in H^{-1/2}(\Sigma)\}$$
$$H^{-1/2}(\mathrm{curl}_\tau;\Sigma) := \{\boldsymbol{\lambda} \in H^{-1/2}(\Sigma) \mid \boldsymbol{\lambda} \cdot \mathbf{n} = 0, \mathrm{curl}_\tau \boldsymbol{\lambda} \in H^{-1/2}(\Sigma)\}$$

(see Paquet [190], Alonso and Valli [6], Cessenat [76]).

In this functional framework it is thus possible to extend the operators grad_τ and Curl_τ on $H^{1/2}(\Sigma)$ as

$$\int_\Sigma \mathrm{grad}_\tau \phi \cdot \boldsymbol{\lambda} = - \int_\Sigma (\mathrm{div}_\tau \boldsymbol{\lambda})\, \phi \ , \ \ \boldsymbol{\lambda} \in H^{-1/2}(\mathrm{div}_\tau;\Sigma),$$

and

$$\int_\Sigma \mathrm{Curl}_\tau \phi \cdot \boldsymbol{\lambda} = \int_\Sigma (\mathrm{curl}_\tau \boldsymbol{\lambda})\, \phi \ , \ \ \boldsymbol{\lambda} \in H^{-1/2}(\mathrm{curl}_\tau;\Sigma) \, ,$$

obtaining by duality $\mathrm{grad}_\tau \phi \in H^{-1/2}(\mathrm{curl}_\tau;\Sigma)$ and $\mathrm{Curl}_\tau \phi \in H^{-1/2}(\mathrm{div}_\tau;\Sigma)$. Again, $\mathrm{Curl}_\tau \phi = \mathrm{grad}_\tau \phi \times \mathbf{n}$ for each $\phi \in H^{1/2}(\Sigma)$. In particular, we have also obtained

$$\int_\Sigma \mathrm{grad}_\tau(\varphi_{|\Sigma}) \cdot \mathbf{u} \times \mathbf{n} = - \int_\Sigma \mathrm{div}_\tau(\mathbf{u} \times \mathbf{n})\, \varphi_{|\Sigma} \qquad \text{(A.5)}$$

for each $\mathbf{u} \in H(\mathrm{curl};\Omega)$ and $\varphi \in H^1(\Omega)$.

For each $\mathbf{u} \in H(\text{curl}; \Omega)$, $\mathbf{v} \in H_{0,\partial\Omega\setminus\Sigma}(\text{curl}; \Omega)$ it can be proved that the following formula of integration by parts holds true

$$\int_\Omega \text{curl}\, \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{u} \cdot \text{curl}\, \mathbf{v} - \int_\Sigma (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{v} \tag{A.6}$$

(where the last integral is indeed the duality pairing between $(\mathbf{u}\times\mathbf{n}) \in H^{-1/2}(\text{div}_\tau; \Sigma)$ and $(\mathbf{n} \times \mathbf{v} \times \mathbf{n}) \in H^{-1/2}(\text{curl}_\tau; \Sigma)$).

Taking $\mathbf{v} = \text{grad}\, \varphi$, with $\varphi \in H^1_{0,\partial\Omega\setminus\Sigma}(\Omega)$, from (A.6), (A.5) and the Gauss divergence theorem it follows

$$\begin{aligned} \textstyle\int_\Sigma \text{curl}\, \mathbf{u} \cdot \mathbf{n}\, \varphi_{|\Sigma} &= -\textstyle\int_\Omega (\text{div}\,\text{curl}\, \mathbf{u})\, \varphi + \int_\Sigma \text{curl}\, \mathbf{u} \cdot \mathbf{n}\, \varphi_{|\Sigma} \\ &= \textstyle\int_\Omega \text{curl}\, \mathbf{u} \cdot \text{grad}\, \varphi = \int_\Omega \mathbf{u} \cdot \text{curl}\,\text{grad}\, \varphi - \int_\Sigma (\mathbf{u} \times \mathbf{n}) \cdot \text{grad}\, \varphi \\ &= -\textstyle\int_\Sigma (\mathbf{u} \times \mathbf{n}) \cdot \text{grad}_\tau(\varphi_{|\Sigma}) = \int_\Sigma \text{div}_\tau(\mathbf{u} \times \mathbf{n})\, \varphi_{|\Sigma}\,, \end{aligned}$$

hence

$$\text{curl}\, \mathbf{u} \cdot \mathbf{n} = \text{div}_\tau(\mathbf{u} \times \mathbf{n}) \ \text{ on } \Sigma \tag{A.7}$$

for each $\mathbf{u} \in H(\text{curl}; \Omega)$.

We finally recall that the following trace inequalities hold true (in the second, third and fourth inequality we are assuming that Σ is a closed surface)

$$\|\phi_{|\Sigma}\|_{1/2,\Sigma} \le \kappa\, \|\phi\|_{1,\Omega} \quad \forall\, \phi \in H^1(\Omega) \tag{A.8}$$

$$\|(\mathbf{v} \cdot \mathbf{n})_{|\Sigma}\|_{-1/2,\Sigma} \le \kappa\, \|\mathbf{v}\|_{H(\text{div};\Omega)} \quad \forall\, \mathbf{v} \in H(\text{div}; \Omega) \tag{A.9}$$

$$\|(\mathbf{v} \times \mathbf{n})_{|\Sigma}\|_{H^{-1/2}(\text{div}_\tau;\Sigma)} \le \kappa\, \|\mathbf{v}\|_{H(\text{curl};\Omega)} \quad \forall\, \mathbf{v} \in H(\text{curl}; \Omega) \tag{A.10}$$

$$\|(\mathbf{n} \times \mathbf{v} \times \mathbf{n})_{|\Sigma}\|_{H^{-1/2}(\text{curl}_\tau;\Sigma)} \le \kappa\, \|\mathbf{v}\|_{H(\text{curl};\Omega)} \quad \forall\, \mathbf{v} \in H(\text{curl}; \Omega)\,, \tag{A.11}$$

where $\kappa > 0$ is a suitable constant only depending on Ω and Σ. Moreover, in all these cases there exist linear continuous extension operators form the trace space to the corresponding space of functions defined in Ω.

A.2 Nodal and edge finite elements

We present in this section a brief description of the finite element spaces used for the approximation of the spaces $H^1(\Omega)$ and $H(\text{curl}; \Omega)$. A more comprehensive presentation can be found, e.g., in Ciarlet [83], Quarteroni and Valli [199], Monk [179].

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz polyhedral domain and let us consider a finite decomposition of Ω given by

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K\,,$$

where, denoting by $\text{int}(K)$ and $\text{diam}(K)$ the internal part and the diameter of K, respectively,

- each K is a (closed) polyhedron with positive volume;
- if K_1 and K_2 are distinct elements in $\mathcal{T}_h$ then $\text{int}(K_1) \cap \text{int}(K_2) = \emptyset$;
- if K_1 and K_2 are distinct elements in $\mathcal{T}_h$ and $F = K_1 \cap K_2 \neq \emptyset$, then F is a common face, side, or vertex of K_1 and K_2;
- $\text{diam}(K) \leq h$ for each $K \in \mathcal{T}_h$.

Under these conditions, $\mathcal{T}_h$ is called a triangulation of Ω. In the sequel we will consider triangulations where each element K can be obtained as an affine trasformation of a reference element $\hat{K}$, i.e., $K = T_K(\hat{K})$, where T_K is an invertible affine map $T_K(\hat{\mathbf{x}}) = B_K\hat{\mathbf{x}} + \mathbf{b}_K$, B_K being a non-singular matrix. The reference element can be the tetrahedron $\hat{K}$ of vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ or the cube $\hat{K} = [0,1]^3$.

Finite element spaces are based on piecewise-polynomial functions, hence some notations for polynomial spaces are necessary. Let us denote by $\mathbb{P}_k$, $k \geq 0$, the space of polynomials of degree less than or equal to k in the three variables x_1, x_2, x_3, and by $\widetilde{\mathbb{P}}_k$ the space of homogeneous polynomials of degree k. Let $\mathbb{Q}_{l,m,n}$ be the polynomial space given by polynomials of maximum degree l in x_1, m in x_2, and n in x_3. In particular $\mathbb{Q}_k$ denotes the space of polynomials that are of degree less than or equal to k with respect to each variable.

For the definition of finite elements in $H(\text{curl};\Omega)$ the following space of vector polynomials is used

$$R_k := (\mathbb{P}_{k-1})^3 \oplus S_k \,,$$

where $k \geq 1$ and

$$S_k := \{\mathbf{q} \in (\widetilde{\mathbb{P}}_k)^3 \mid \mathbf{q}(\mathbf{x}) \cdot \mathbf{x} = 0\} \,.$$

Another space of incomplete vector polynomials is (for $k \geq 1$)

$$D_k := (\mathbb{P}_{k-1})^3 \oplus \widetilde{\mathbb{P}}_k \, \mathbf{x} \,.$$

We also use polynomial spaces defined on planes and lines. If e is a segment we denote by $\mathbb{P}_k(e)$ the space of polynomials of maximum degree k with respect to the arc length on e. If f is a plane subdomain in $\mathbb{R}^3$, $\mathbb{P}_k(f)$ denotes the space of polynomials of maximum degree k in two variables using an orthogonal coordinate system in the plane.

A.2.1 Grad-conforming finite elements

Let us first recall that a function $\phi : \Omega \to \mathbb{R}$ belongs to $H^1(\Omega)$ if and only if $\phi_{|K} \in H^1(K)$ for each $K \in \mathcal{T}_h$ and for each common face $f = K_1 \cap K_2$, $K_1, K_2 \in \mathcal{T}_h$, the value of $\phi_{|K_1}$ and $\phi_{|K_2}$ on f is the same.

Therefore, for any $k \geq 1$ the space

$$L_h^k := \{\phi_h \in C^0(\Omega) \mid \phi_{h|K} \in \mathbb{P}_k \quad \forall\, K \in \mathcal{T}_h\}$$

is a subspace of $H^1(\Omega)$.

In order to identify a basis of this finite dimensional subspace it is necessary to choose a set of degrees of freedom that are unisolvent on $\mathbb{P}_k$, namely, such that their values uniquely determine a polynomial belonging to $\mathbb{P}_k$.

Let us first assume that $\mathcal{T}_h$ is a triangulation of Ω composed by tetrahedra. Following Monk [179], for a regular enough function ϕ we consider the following set of degrees of freedom on a generic tetrahedron K:

- vertex degrees of freedom

$$m_v(\phi) := \{\phi(\mathbf{a}) \text{ for all vertices } \mathbf{a} \text{ of } K\} ;$$

- edge degrees of freedom (for $k \geq 2$)

$$m_e(\phi) := \left\{ \frac{1}{\text{length}(e)} \int_e \phi\, q\, ds\ \forall\, q \in \mathbb{P}_{k-2}(e) \text{ for all edges } e \text{ of } K \right\} ;$$

- face degrees of freedom (for $k \geq 3$)

$$m_f(\phi) := \left\{ \frac{1}{\text{area}(f)} \int_f \phi\, q\, dS\ \forall\, q \in \mathbb{P}_{k-3}(f) \text{ for all faces } f \text{ of } K \right\} ;$$

- volume degrees of freedom (for $k \geq 4$)

$$m_K(\phi) := \left\{ \frac{1}{\text{volume}(K)} \int_K \phi\, q\, dV\ \forall\, q \in \mathbb{P}_{k-4} \right\} .$$

It is easy to check that the total number of degrees of freedom in a tetrahedron concides with the dimension of $\mathbb{P}_k$; moreover it can be verified that a polynomial $\phi \in \mathbb{P}_k$ is vanishing in K provided that all its degrees of freedom are equal to 0. Hence these degrees of freedom are unisolvent on $\mathbb{P}_k$.

It can be also proved that, if all vertex, edge and face degrees of freedom of $\phi \in \mathbb{P}_k$ vanish for a particular face f of a tetrahedron, then $\phi = 0$ on that face. This means that, using these degrees of freedom for identifying a piecewise-polynomial functions that locally belongs to $\mathbb{P}_k$, we define a continuous function, hence an element of L_h^k. A basis of L_h^k is thus given by the collection of those functions that are locally in $\mathbb{P}_k$ and that have one degree of freedom equal to 1 and all the others equal to 0.

Remark A.1. A different and more often used set of degrees of freedom, consisting of the values of the function on different points of the tetrahedron, can be employed in order to describe these finite element spaces. Being expressed in terms of point values, this kind of finite dimensional spaces are often called *nodal* finite elements (see, e.g., Ciarlet [83], Quarteroni and Valli [199]). For instance, if $k = 2$ the values of the function ϕ at the vertices $\mathbf{a}_i$, $1 \leq i \leq 4$, and in the middle point of each edge constitutes another set of grad-conforming and unisolvent set of degrees of freedom; if $k = 3$ an analogous set of conditions is given by the values of ϕ at the 20 different points of the form $\frac{1}{3}\mathbf{a}_i + \frac{1}{3}\mathbf{a}_j + \frac{1}{3}\mathbf{a}_k$, with $1 \leq i, j, k \leq 4$.

Here we have preferred to adopt the vertex, edge, face and volume degrees of freedom for the sake of similarity with the curl-conforming finite elements introduced in Section A.2.2. □

For any $\phi \in H^{3/2+\delta}(K)$, $\delta > 0$, we can now define an interpolation operator π_K by requiring that

$$m_v(\phi - \pi_K\phi) = m_e(\phi - \pi_K\phi) = m_f(\phi - \pi_K\phi) = m_K(\phi - \pi_K\phi) = 0$$

for all the vertices, edges and faces of K; the corresponding global interpolation operator

$$\pi_h : H^{3/2+\delta}(\Omega) \to L_h^k$$

is defined by $(\pi_h\phi)_{|K} := \pi_K\phi_{|K}$ for each $K \in \mathcal{T}_h$. Note that the assumption on the regularity of ϕ ensures that ϕ is continuous in K, hence vertex values are well-defined.

We recall that a family of triangulations $\mathcal{T}_h$ is called regular if there exists a constant $\sigma > 0$ such that

$$\max_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K} \leq \sigma \qquad \forall\, h > 0\,,$$

where

$$\rho_K := \sup\{\mathrm{diam}(R) \mid R \text{ is a ball contained in } K\}\,.$$

The following interpolation error estimate holds:

Theorem A.2. *Let $\mathcal{T}_h$ be a regular family of triangulations of Ω. Then if $\phi \in H^{s+1}(\Omega)$, $1/2 + \delta \leq s \leq k$, there exists a constant $C > 0$, independent of h, such that*

$$\|\phi - \pi_h\phi\|_{0,\Omega} + h\|\phi - \pi_h\phi\|_{1,\Omega} \leq Ch^{s+1}\|\phi\|_{s+1,\Omega}\,.$$

It is also possible to construct a finite element space analogous to L_h^k when considering a triangulation of Ω consisting of parallelepipeds. In this case one works with piecewise-polynomial functions ϕ_h such that $\phi_{h|K} \circ T_K \in \mathbb{Q}_k$. The space of nodal finite elements for a mesh composed by parallelepipeds and for $k \geq 1$ is

$$\widetilde{L}_h^k := \{\phi_h \in C^0(\Omega) \mid \phi_{h|K} \circ T_K \in \mathbb{Q}_k \quad \forall\, K \in \mathcal{T}_h\}\,.$$

On the reference element $\hat{K}$ the degrees of freedom, unisolvent on $\mathbb{Q}_k$, are:

- vertex degrees of freedom

$$m_v(\hat{\phi}) := \left\{\hat{\phi}(\hat{\mathbf{a}}) \text{ for all vertices } \hat{\mathbf{a}} \text{ of } \hat{K}\right\}\,;$$

- edge degrees of freedom (for $k \geq 2$)

$$m_e(\hat{\phi}) := \left\{\int_{\hat{e}} \hat{\phi}\,\hat{q}\,ds\;\forall\,\hat{q} \in \mathbb{P}_{k-2}(\hat{e}) \text{ for all edges } \hat{e} \text{ of } \hat{K}\right\}\,;$$

- face degrees of freedom (for $k \geq 2$)

$$m_f(\hat{\phi}) := \left\{\int_{\hat{f}} \hat{\phi}\,\hat{q}\,dS\;\forall\,\hat{q} \in \mathbb{Q}_{k-2}(\hat{f}) \text{ for all faces } \hat{f} \text{ of } \hat{K}\right\}\,;$$

- volume degrees of freedom (for $k \geq 2$)

$$m_K(\hat{\phi}) := \left\{ \int_{\hat{K}} \hat{\phi}\, \hat{q}\, dV \;\forall\, \hat{q} \in \mathbb{Q}_{k-2} \right\} .$$

The degrees of freedom on a general element K can be obtained from those on $\hat{K}$ using the transformation $\phi \circ T_K = \hat{\phi}$.

For these finite element spaces the interpolation error estimate described in Theorem A.2 still holds.

Remark A.3. Let us assume $k \geq 0$. A finite element subspace of $L^2(\Omega)$ is easily defined by

$$C_h^k := \{q_h \in L^2(\Omega) \mid q_{h|K} \in \mathbb{P}_k \;\forall\, K \in \mathcal{T}_h\}$$

when the elements $K \in \mathcal{T}_h$ are tetrahedra, and by

$$\widetilde{C}_h^k := \{q_h \in L^2(\Omega) \mid q_{h|K} \circ T_K \in \mathbb{Q}_k \;\forall\, K \in \mathcal{T}_h\}$$

when the elements $K \in \mathcal{T}_h$ are parallelepipeds.

If $P_{0,h} : L^2(\Omega) \to C_h^k$ denotes the $L^2(\Omega)$-projection, then one has

$$\|\phi - P_{0,h}\phi\|_{0,\Omega} \leq C h^{s+1} \|\phi\|_{s+1,\Omega} \, ,$$

for all $\phi \in H^{s+1}(\Omega)$, $0 \leq s \leq k$. The same holds true for the $L^2(\Omega)$-projection $\widetilde{P}_{0,h} : L^2(\Omega) \to \widetilde{C}_h^k$. □

A.2.2 Curl-conforming finite elements

Here we introduce the finite element spaces used for the approximation of the space $H(\text{curl}; \Omega)$. We present the two families of elements proposed by Nédélec in [185] and [186], which are also called *edge* elements.

We start by considering a triangulation of Ω composed by tetrahedra. For $k \geq 1$, the first family is defined as

$$N_h^k := \{\mathbf{z}_h \in H(\text{curl}; \Omega) \mid \mathbf{z}_{h|K} \in R_k \quad \forall\, K \in \mathcal{T}_h\} \, .$$

We have the following set of degrees of freedom:

- edge degrees of freedom

$$m_e(\mathbf{z}) := \left\{ \int_e \mathbf{z} \cdot \boldsymbol{\tau}\, q\, ds \;\forall\, q \in \mathbb{P}_{k-1}(e) \text{ for all edges } e \text{ of } K \right\} ;$$

- face degrees of freedom (for $k \geq 2$)

$$m_f(\mathbf{z}) := \left\{ \int_f \mathbf{z} \times \boldsymbol{\nu} \cdot \mathbf{q}\, dS \;\forall\, \mathbf{q} \in (\mathbb{P}_{k-2}(f))^2 \text{ for all faces } f \text{ of } K \right\} ;$$

- volume degrees of freedom (for $k \geq 3$)

$$m_K(\mathbf{z}) := \left\{ \int_K \mathbf{z} \cdot \mathbf{q} \, dV \ \forall \mathbf{q} \in (\mathbb{P}_{k-3})^3 \right\} .$$

Here $\boldsymbol{\tau}$ denotes a unit vector with the direction of e, while $\boldsymbol{\nu}$ is the unit normal vector on f.

The total number of degrees of freedom on a tetrahedron K is equal to the dimension of R_k, and it can be shown that, if all the degrees of freedom are 0, then a polynomial $\mathbf{z} \in R_k$ is identically vanishing in K. Hence this set of degrees of freedom is unisolvent on R_k.

We recall that, if K_1, K_2 are two different elements of $\mathcal{T}_h$ with a common face $f = K_1 \cap K_2$, defining $\mathbf{z} \in L^2(K_1 \cup K_2)$ by

$$\mathbf{z} = \begin{cases} \mathbf{z}_1 & \text{in } K_1 \\ \mathbf{z}_2 & \text{in } K_2 \, , \end{cases}$$

where $\mathbf{z}_1 \in H(\text{curl}; K_1)$ and $\mathbf{z}_2 \in H(\text{curl}; K_2)$, it follows $\mathbf{z} \in H(\text{curl}; K_1 \cup K_2)$ provided that $\mathbf{z}_1 \times \boldsymbol{\nu} = \mathbf{z}_2 \times \boldsymbol{\nu}$ on f.

It can also be proved that, if a vector function $\mathbf{z} \in R_k$ has all its degrees of freedom vanishing on a face f of K and on the three edges contained in f, then the tangential component of $\mathbf{z}$ vanishes on f. This means that, using these degrees of freedom for identifying a piecewise-polynomial functions that locally belongs to R_k, we obtain an element of $H(\text{curl}; \Omega)$, hence an element of N_h^k.

We can introduce a natural interpolation operator. Assuming that $\mathbf{z}$ is sufficiently regular, the interpolant $\mathbf{r}_h\mathbf{z} \in N_h^k$ is the unique function in N_h^k that has the same degrees of freedom of $\mathbf{z}$, that is

$$m_e(\mathbf{z} - \mathbf{r}_h\mathbf{z}) = m_f(\mathbf{z} - \mathbf{r}_h\mathbf{z}) = m_h(\mathbf{z} - \mathbf{r}_h\mathbf{z}) = 0$$

for all the edges, faces and tetrahedra of $\mathcal{T}_h$.

We notice that the degrees of freedom $m_e(\mathbf{z})$ are not defined for a general function in $H(\text{curl}; \Omega)$. However they are well-defined if $\mathbf{z} \in (H^s(\Omega))^3$ for some $s > 1/2$ and $\text{curl}\, \mathbf{z} \in (L^p(\Omega))^3$ for some $p > 2$. For the proof of this result see Monk [179] (see also Amrouche et al. [27], where a more general result is proved).

In particular, the following interpolation error estimate holds (see Alonso and Valli [9]):

Theorem A.4. *Let $\mathcal{T}_h$ be a regular family of triangulations of Ω. If $\mathbf{z} \in (H^s(\Omega))^3$ and* $\text{curl}\, \mathbf{z} \in (H^s(\Omega))^3$, $1/2 < s \leq k$, *then there exists a constant $C > 0$, independent of h, such that*

$$\|\mathbf{z} - \mathbf{r}_h\mathbf{z}\|_{0,\Omega} + \|\,\text{curl}(\mathbf{z} - \mathbf{r}_h\mathbf{z})\|_{0,\Omega} \leq C h^s (\|\mathbf{z}\|_{s,\Omega} + \|\,\text{curl}\, \mathbf{z}\|_{s,\Omega}) \, .$$

It is also possible to define an analogous family of curl-conforming finite element spaces when considering a triangulation of Ω consisting of parallelepipeds. For the reference element $\hat{K} = [0,1]^3$ the polynomial space is $\mathbb{Q}_{k-1,k,k} \times \mathbb{Q}_{k,k-1,k} \times \mathbb{Q}_{k,k,k-1}$, with $k \geq 1$, and the degrees of freedom are given on edges $\hat{e}$ with unit tangent $\hat{\boldsymbol{\tau}}$, on faces $\hat{f}$ with unit normal $\hat{\boldsymbol{\nu}}$ and in the interior of $\hat{K}$. In particular we consider the following set of degrees of freedom, unisolvent on $\mathbb{Q}_{k-1,k,k} \times \mathbb{Q}_{k,k-1,k} \times \mathbb{Q}_{k,k,k-1}$:

- edge degrees of freedom

$$m_e(\hat{\mathbf{z}}) := \left\{ \int_{\hat{e}} \hat{\mathbf{z}} \cdot \hat{\boldsymbol{\tau}} \hat{q} \, ds \; \forall \, \hat{q} \in \mathbb{P}_{k-1}(\hat{e}) \text{ for all edges } \hat{e} \text{ of } \hat{K} \right\} ;$$

- face degrees of freedom (for $k \geq 2$)

$$m_f(\hat{\mathbf{z}}) := \Big\{ \textstyle\int_{\hat{f}} \hat{\mathbf{z}} \times \hat{\boldsymbol{\nu}} \cdot \hat{\mathbf{q}} \, dS \; \forall \, \hat{\mathbf{q}} \in \mathbb{Q}_{k-2,k-1}(\hat{f}) \times \mathbb{Q}_{k-1,k-2}(\hat{f}) \\ \text{for all faces } \hat{f} \text{ of } \hat{K} \Big\} ;$$

- volume degrees of freedom (for $k \geq 2$)

$$m_K(\hat{\mathbf{z}}) := \Big\{ \textstyle\int_{\hat{K}} \hat{\mathbf{z}} \cdot \hat{\mathbf{q}} \, dV \\ \forall \, \hat{\mathbf{q}} \in \mathbb{Q}_{k-1,k-2,k-2} \times \mathbb{Q}_{k-2,k-1,k-2} \times \mathbb{Q}_{k-2,k-2,k-1} \Big\} .$$

They are well-defined if $\hat{\mathbf{z}} \in (H^s(\hat{K}))^3$ for some $s > 1/2$ and $\operatorname{curl} \hat{\mathbf{z}} \in (L^p(\hat{K}))^3$ for some $p > 2$. The basis functions on a general element K can be obtained from those on $\hat{K}$ using the transformation $\mathbf{z} \circ T_K = (B_K^T)^{-1} \hat{\mathbf{z}}$. In this way the curl of $\mathbf{z}$ is expressed in terms of the curl of $\hat{\mathbf{z}}$ by

$$\operatorname{curl} \mathbf{z} \circ T_K = \frac{1}{\det(B_K)} B_K \operatorname{curl} \hat{\mathbf{z}} \, .$$

For $k \geq 1$ we can thus consider the following curl-conforming finite element spaces defined on parallelepipeds

$$\widetilde{N}_h^k := \{ \mathbf{z}_h \in H(\operatorname{curl}; \Omega) \mid \mathbf{z}_{h|K} \circ T_K \in \mathbb{Q}_{k-1,k,k} \times \mathbb{Q}_{k,k-1,k} \times \mathbb{Q}_{k,k,k-1} \\ \forall \, K \in \mathcal{T}_h \} .$$

The interpolation error estimate reported in Theorem A.4 still holds.

We notice that, for the curl-conforming finite elements presented here above, when using elements of degree k the interpolation error in the $L^2(\Omega)$-norm is $O(h^k)$. A second family of curl-conforming elements has been introduced by Nédélec in [186], in order to obtain an $O(h^{k+1})$-error estimate in $L^2(\Omega)$.

Let us first consider the case of a tetrahedral mesh. For $k \geq 1$, the discrete functions locally belong to the polynomial space $(\mathbb{P}_k)^3$, and the degrees of freedom, are the following:

- edge degrees of freedom

$$m_e(\mathbf{z}) := \left\{ \int_e \mathbf{z} \cdot \boldsymbol{\tau} \, q \, ds \; \forall \, q \in \mathbb{P}_k(e) \text{ for all edges } e \text{ of } K \right\} ;$$

- face degrees of freedom (for $k \geq 2$)

$$m_f(\mathbf{z}) := \left\{ \int_f \mathbf{z} \cdot \mathbf{q}\, dS \, \forall\, \mathbf{q} \in D_{k-1}(f) \text{ for all faces } f \text{ of } K \right\} ;$$

- volume degrees of freedom (for $k \geq 3$)

$$m_K(\mathbf{z}) := \left\{ \int_K \mathbf{z} \cdot \mathbf{q}\, dV \, \forall\, \mathbf{q} \in D_{k-2} \right\} .$$

Here $D_{k-1}(f)$ is the analogue of D_{k-1} in two dimensions.

This set of degrees of freedom has been proved to be curl-conforming and unisolvent on $(\mathbb{P}_k)^3$. Thus for $k \geq 1$ we can consider the discrete space

$$N^k_{*,h} := \{\mathbf{z}_h \in H(\mathrm{curl}; \Omega) \mid \mathbf{z}_{h|K} \in (\mathbb{P}_k)^3 \quad \forall\, K \in \mathcal{T}_h\} ,$$

and, for a function $\mathbf{z}$ which is regular enough, we can define the interpolant $\mathbf{r}_{*,h}\mathbf{z} \in N^k_{*,h}$.

Concerning the interpolation error the following estimate holds:

Theorem A.5. *Let $\mathcal{T}_h$ be a regular family of triangulations of Ω. If $\mathbf{z} \in (H^{s+1}(\Omega))^3$, $1 \leq s \leq k$, then there exists a constant $C > 0$, independent of h, such that*

$$\|\mathbf{z} - \mathbf{r}_{*,h}\mathbf{z}\|_{0,\Omega} + h\| \mathrm{curl}(\mathbf{z} - \mathbf{r}_{*,h}\mathbf{z})\|_{0,\Omega} \leq C h^{s+1} |\mathbf{z}|_{s+1,\Omega} .$$

Comparing the interpolation errors in N^k_h and $N^k_{*,h}$ we see that $L^2(\Omega)$-norms of the curl are of the same order with respect to h, while the $L^2(\Omega)$-norms of the fields are $O(h^s)$ for N^k_h and $O(h^{s+1})$ for $N^k_{*,h}$. On the other hand, the number of degrees of freedom of $N^k_{*,h}$ is greater than that of N^k_h.

It is also possible to define a second family of Nédélec curl-conforming finite elements when considering a triangulation of Ω consisting of parallelepipeds. For the reference element $\hat{K} = [0,1]^3$ and $k \geq 1$ the polynomial space is $(\mathbb{Q}_k)^3$ and the degrees of freedom, unisolvent on $(\mathbb{Q}_k)^3$, are given by:

- edge degrees of freedom

$$m_e(\hat{\mathbf{z}}) := \left\{ \int_{\hat{e}} \hat{\mathbf{z}} \cdot \hat{\boldsymbol{\tau}}\, \hat{q}\, ds \, \forall\, \hat{q} \in \mathbb{P}_k(\hat{e}) \text{ for all edges } \hat{e} \text{ of } \hat{K} \right\} ;$$

- face degrees of freedom (for $k \geq 2$)

$$m_f(\hat{\mathbf{z}}) := \left\{ \int_{\hat{f}} \hat{\mathbf{z}} \cdot \hat{\mathbf{q}}\, dS \, \forall\, \hat{\mathbf{q}} \in \mathbb{Q}_{k,k-2}(\hat{f}) \times \mathbb{Q}_{k-2,k}(\hat{f}) \text{ for all faces } \hat{f} \text{ of } \hat{K} \right\};$$

- volume degrees of freedom (for $k \geq 2$)

$$m_K(\hat{\mathbf{z}}) := \left\{ \int_{\hat{K}} \hat{\mathbf{z}} \cdot \hat{\mathbf{q}}\, dV \, \forall\, \hat{\mathbf{q}} \in \mathbb{Q}_{k,k-2,k-2} \times \mathbb{Q}_{k-2,k,k-2} \times \mathbb{Q}_{k-2,k-2,k} \right\} .$$

The corresponding discrete space is given by

$$\widetilde{N}^k_{*,h} := \{\mathbf{z}_h \in H(\mathrm{curl}; \Omega) \mid \mathbf{z}_{h|K} \circ T_K \in (\mathbb{Q}_k)^3 \quad \forall\, K \in \mathcal{T}_h\} .$$

A.3 Orthogonal decomposition results

We prove in this section some orthogonal decomposition results that are useful for splitting the magnetic field $\mathbf{H}_I$ or the electric field $\mathbf{E}_I$ into the sum of suitable terms (for a more detailed presentation, see also, e.g., Dautray and Lions [95], Saranen [218], [219], Auchmuty [29], Cantarella et al. [73]).

Here the geometrical assumptions on Ω, Ω_C and Ω_I are those of Section 1.3. Moreover, we again assume that the matrix $\boldsymbol{\mu}$ is symmetric and uniformly positive definite in Ω, with entries belonging to $L^\infty(\Omega)$ and that the matrix $\boldsymbol{\varepsilon}_I$ is symmetric and uniformly positive definite in Ω_I, with entries belonging to $L^\infty(\Omega_I)$. Finally, the spaces of harmonic fields are introduced in Section 1.4 (see also Section A.4).

A.3.1 First decomposition result

Let us start by introducing the scalar product

$$(\mathbf{w}_I, \mathbf{z}_I)_{\varepsilon_I,\Omega_I} := \int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{w}_I \cdot \overline{\mathbf{z}_I} \,,$$

where $\overline{\mathbf{z}_I}$ indicates the complex conjugate of $\mathbf{z}_I$, and by denoting with the symbol $\perp^{\varepsilon_I}$ the orthogonality with respect to this scalar product. Instead, $\perp$ denotes the orthogonality with respect to the standard $L^2(\Omega_I)$-scalar product. We have the following theorem:

Theorem A.6. *Any vector function $\mathbf{z}_I \in (L^2(\Omega_I))^3$ can be written as*

$$\mathbf{z}_I = \boldsymbol{\varepsilon}_I^{-1} \operatorname{curl} \mathbf{q}_I + \operatorname{grad} \varphi_I + \mathbf{h}_I \,, \tag{A.12}$$

where $\mathbf{q}_I \in H_{0,\partial\Omega}(\mathrm{curl};\Omega_I) \cap H^0_{0,\Gamma}(\mathrm{div};\Omega_I) \cap \mathcal{H}(\partial\Omega,\Gamma;\Omega_I)^\perp$, $\varphi_I \in H^1_{0,\Gamma}(\Omega_I)$ and $\mathbf{h}_I \in \mathcal{H}_{\varepsilon_I}(\Gamma,\partial\Omega;\Omega_I)$, and each term of the decomposition (A.12) *is orthogonal to the others, with respect to the scalar product $(\cdot,\cdot)_{\varepsilon_I,\Omega_I}$.*

Moreover, if $\operatorname{curl} \mathbf{z}_I = \mathbf{0}$ in Ω_I and $\mathbf{z}_I \times \mathbf{n}_I = \mathbf{0}$ on Γ it follows $\mathbf{q}_I = \mathbf{0}$, if $\mathrm{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I) = 0$ in Ω_I and $\boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$ one has $\varphi_I = 0$, and if $\mathbf{z}_I \perp^{\varepsilon_I} \mathcal{H}_{\varepsilon_I}(\Gamma,\partial\Omega;\Omega_I)$ one finds $\mathbf{h}_I = \mathbf{0}$.

Proof. To prove this result, let us start showing how $\mathbf{q}_I$, φ_I and $\mathbf{h}_I$ can be determined in terms of $\mathbf{z}_I$.

First of all, setting

$$Y_I := \left\{ \mathbf{p}_I \in H_{0,\partial\Omega}(\mathrm{curl};\Omega_I) \cap H_{0,\Gamma}(\mathrm{div};\Omega_I) \,|\, \mathbf{p}_I \perp \mathcal{H}(\partial\Omega,\Gamma;\Omega_I) \right\} , \tag{A.13}$$

the vector field $\mathbf{q}_I \in Y_I$ is the solution to

$$\int_{\Omega_I} (\boldsymbol{\varepsilon}_I^{-1} \operatorname{curl} \mathbf{q}_I \cdot \operatorname{curl} \overline{\mathbf{p}_I} + \mathrm{div}\, \mathbf{q}_I \, \mathrm{div}\, \overline{\mathbf{p}_I}) = \int_{\Omega_I} \mathbf{z}_I \cdot \operatorname{curl} \overline{\mathbf{p}_I} \tag{A.14}$$

for all $\mathbf{p}_I \in Y_I$. Concerning the solvability of this problem, note that using the compactness of Y_I in $L^2(\Omega_I)$ (see, e.g., Fernandes and Gilardi [104]), the following Poincaré-like inequality is easily proved

$$\int_{\Omega_I} |\mathbf{p}_I|^2 \leq C \int_{\Omega_I} (|\operatorname{curl} \mathbf{p}_I|^2 + |\operatorname{div} \mathbf{p}_I|^2) \qquad \forall\, \mathbf{p}_I \in Y_I \,. \tag{A.15}$$

Therefore, the existence of a unique solution $\mathbf{q}_I$ to (A.14) is a consequence of the Lax–Milgram lemma. It can be also verified that, if $\operatorname{curl} \mathbf{z}_I = \mathbf{0}$ in Ω_I and $\mathbf{z}_I \times \mathbf{n}_I = \mathbf{0}$ on Γ, then the right-hand side of (A.14) vanishes, thus $\mathbf{q}_I = \mathbf{0}$.

Since equation (A.14) is trivially satisfied for each test function belonging to $\mathcal{H}(\partial\Omega, \Gamma; \Omega_I)$, there we can select the test functions $\mathbf{p}_I$ not only in Y_I but also in $H_{0,\partial\Omega}(\text{curl}; \Omega_I) \cap H_{0,\Gamma}(\text{div}; \Omega_I)$, namely, without imposing the orthogonality constraint.

We also see that $\operatorname{div} \mathbf{q}_I = 0$ in Ω_I, as in (A.14) we can choose $\mathbf{p}_I = \operatorname{grad} v_I$, where the function $v_I \in H^1(\Omega_I)$ satisfies $\Delta v_I = \operatorname{div} \mathbf{q}_I$ in Ω_I, $v_I = 0$ on $\partial\Omega$ and $\operatorname{grad} v_I \cdot \mathbf{n}_I = 0$ on Γ, thus obtaining $\int_{\Omega_I} |\operatorname{div} \mathbf{q}_I|^2 = 0$.

We have therefore shown that $\mathbf{q}_I$ satisfies

$$\int_{\Omega_I} \varepsilon_I^{-1} \operatorname{curl} \mathbf{q}_I \cdot \operatorname{curl} \overline{\mathbf{p}_I} = \int_{\Omega_I} \mathbf{z}_I \cdot \operatorname{curl} \overline{\mathbf{p}_I} \tag{A.16}$$

for all $\mathbf{p}_I \in H_{0,\partial\Omega}(\text{curl}; \Omega_I) \cap H_{0,\Gamma}(\text{div}; \Omega_I)$. We can indeed prove something more, namely, that equation (A.16) is satisfied for each test function $\mathbf{p}_I^* \in H_{0,\partial\Omega}(\text{curl}; \Omega_I)$. In fact, denoting by $v_I^* \in H^1(\Omega_I)$ the solution of $\Delta v_I^* = \operatorname{div} \mathbf{p}_I^*$ in Ω_I, $v_I^* = 0$ on $\partial\Omega$ and $\operatorname{grad} v_I^* \cdot \mathbf{n}_I = \mathbf{p}_I^* \cdot \mathbf{n}_I$ on Γ, we have $\mathbf{p}_I = (\mathbf{p}_I^* - \operatorname{grad} v_I^*) \in H_{0,\partial\Omega}(\text{curl}; \Omega_I) \cap H_{0,\Gamma}(\text{div}; \Omega_I)$, and the result follows from the fact that $\operatorname{curl} \mathbf{p}_I^* = \operatorname{curl} \mathbf{p}_I$.

Hence, choosing in (A.16) a test function $\mathbf{p}_I^* \in (C_0^\infty(\Omega_I))^3$, we find by integration by parts that

$$\operatorname{curl}(\varepsilon_I^{-1} \operatorname{curl} \mathbf{q}_I - \mathbf{z}_I) = \mathbf{0} \qquad \text{in } \Omega_I \,;$$

finally, repeating the same computation for $\mathbf{p}_I^* \in H_{0,\partial\Omega}(\text{curl}; \Omega_I)$ gives

$$(\varepsilon_I^{-1} \operatorname{curl} \mathbf{q}_I - \mathbf{z}_I) \times \mathbf{n}_I = \mathbf{0} \qquad \text{on } \Gamma \,.$$

The function $\varphi_I \in H^1_{0,\Gamma}(\Omega_I)$ is such that

$$\int_{\Omega_I} \varepsilon_I \operatorname{grad} \varphi_I \cdot \operatorname{grad} \overline{\eta_I} = \int_{\Omega_I} \varepsilon_I \mathbf{z}_I \cdot \operatorname{grad} \overline{\eta_I} \tag{A.17}$$

for all $\eta_I \in H^1_{0,\Gamma}(\Omega_I)$, and also this problem is uniquely solvable by the Lax–Milgram lemma, as the Poincaré inequality

$$\int_{\Omega_I} |\eta_I|^2 \leq C \int_{\Omega_I} |\operatorname{grad} \eta_I|^2 \tag{A.18}$$

holds in $H^1_{0,\Gamma}(\Omega_I)$ (see, e.g., Dautray and Lions [94], Chap. IV, Sect. 7, Rem. 4). It is readily seen that, if $\operatorname{div}(\varepsilon_I \mathbf{z}_I) = 0$ in Ω_I and $\varepsilon_I \mathbf{z}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$, then the right-hand side of (A.17) vanishes, and consequently $\varphi_I = 0$.

Then, selecting $\eta_I \in C_0^\infty(\Omega_I)$, an integration by parts in (A.17) yields

$$\operatorname{div}[\boldsymbol{\varepsilon}_I(\operatorname{grad}\varphi_I - \mathbf{z}_I)] = 0 \qquad \text{in } \Omega_I ,$$

and the choice $\eta_I \in H^1_{0,\Gamma}(\Omega_I)$ gives

$$\boldsymbol{\varepsilon}_I(\operatorname{grad}\varphi_I - \mathbf{z}_I) \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega .$$

Finally, $\mathbf{h}_I \in \mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$ satisfies

$$\int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{h}_I \cdot \operatorname{grad} w_{j,I} = \int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \operatorname{grad} w_{j,I} , \; \int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{h}_I \cdot \boldsymbol{\pi}_{k,I} = \int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \boldsymbol{\pi}_{k,I}$$

for each $j = 1, \ldots, p_\Gamma$ and $k = 1, \ldots, n_{\partial\Omega}$, the harmonic vector fields $\operatorname{grad} w_{j,I}$ and $\boldsymbol{\pi}_{k,I}$ being the basis functions of the space $\mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$. In other words, $\mathbf{h}_I$ can be written as

$$\mathbf{h}_I = \sum_{j=1}^{p_\Gamma} c_{I,j} \operatorname{grad} w_{j,I} + \sum_{k=1}^{n_{\partial\Omega}} d_{k,I} \boldsymbol{\pi}_{k,I} ,$$

where $(c_{I,j}, d_{k,I})$ are the solution of the linear system

$$A^\dagger \begin{pmatrix} c_{I,j} \\ d_{I,k} \end{pmatrix} = \begin{pmatrix} \int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \operatorname{grad} w_{g,I} \\ \int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{z}_I \cdot \boldsymbol{\pi}_{i,I} \end{pmatrix} , \tag{A.19}$$

$g = 1, \ldots, p_\Gamma$, $i = 1, \ldots, n_{\partial\Omega}$, where $A^\dagger := \begin{pmatrix} D^\dagger & B^\dagger \\ (B^\dagger)^T & C^\dagger \end{pmatrix}$ with

$$\begin{aligned} D^\dagger_{gj} &:= \textstyle\int_{\Omega_I} \boldsymbol{\varepsilon}_I \operatorname{grad} w_{j,I} \cdot \operatorname{grad} w_{g,I} \\ B^\dagger_{gk} &:= \textstyle\int_{\Omega_I} \boldsymbol{\varepsilon}_I \boldsymbol{\pi}_{k,I} \cdot \operatorname{grad} w_{g,I} \\ C^\dagger_{ik} &:= \textstyle\int_{\Omega_I} \boldsymbol{\varepsilon}_I \boldsymbol{\pi}_{k,I} \cdot \boldsymbol{\pi}_{i,I} . \end{aligned}$$

It is easily proved that the matrix $A^\dagger$ is symmetric and positive definite, as the matrix $\boldsymbol{\varepsilon}_I(\mathbf{x})$ is symmetric and positive definite, uniformly with respect to $\mathbf{x}$, and the functions $\boldsymbol{\pi}_{k,I}$ and $\operatorname{grad} w_{j,I}$ are linearly independent. Therefore (A.17) is uniquely solvable; its right-hand side vanishes if $\mathbf{z}_I \perp^{\varepsilon_I} \mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$, so that in that case one has $\mathbf{h}_I = \mathbf{0}$.

The three terms $\boldsymbol{\varepsilon}_I^{-1} \operatorname{curl} \mathbf{q}_I$, $\operatorname{grad}\varphi_I$ and $\mathbf{h}_I$ are orthogonal with respect to the scalar product $(\cdot,\cdot)_{\varepsilon_I,\Omega_I}$: in fact

$$\int_{\Omega_I} \boldsymbol{\varepsilon}_I(\boldsymbol{\varepsilon}_I^{-1} \operatorname{curl} \mathbf{q}_I) \cdot \operatorname{grad} \overline{\varphi_I} = \int_{\partial\Omega\cup\Gamma} \operatorname{curl} \mathbf{q}_I \cdot \mathbf{n}_I \, \overline{\varphi_I} = 0 ,$$

as $\operatorname{curl} \mathbf{q}_I \cdot \mathbf{n} = \operatorname{div}_\tau(\mathbf{q}_I \times \mathbf{n}) = \mathbf{0}$ on $\partial\Omega$ and $\varphi_I = 0$ on Γ. Then

$$\int_{\Omega_I} \boldsymbol{\varepsilon}_I \mathbf{h}_I \cdot \operatorname{grad} \overline{\varphi_I} = \int_{\partial\Omega\cup\Gamma} \boldsymbol{\varepsilon}_I \mathbf{h}_I \cdot \mathbf{n}_I \, \overline{\varphi_I} = 0 ,$$

as $\operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{h}_I) = 0$ in Ω_I, $\boldsymbol{\varepsilon}_I \mathbf{h}_I \cdot \mathbf{n}_I = 0$ on $\partial\Omega$ and and $\varphi_I = 0$ on Γ. Finally,

$$\int_{\Omega_I} \boldsymbol{\varepsilon}_I(\boldsymbol{\varepsilon}_I^{-1} \operatorname{curl} \mathbf{q}_I) \cdot \overline{\mathbf{h}_I} = \int_{\partial\Omega\cup\Gamma} \mathbf{n}_I \times \mathbf{q}_I \cdot \overline{\mathbf{h}_I} = 0 ,$$

as $\operatorname{curl} \mathbf{h}_I = \mathbf{0}$ in Ω_I, $\mathbf{h}_I \times \mathbf{n}_I = \mathbf{0}$ on Γ and $\mathbf{q}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.

The decomposition result (A.12) is then straightforwardly verified. In fact, let us set $\mathbf{U}_I := \mathbf{z}_I - \boldsymbol{\varepsilon}_I^{-1} \operatorname{curl} \mathbf{q}_I - \operatorname{grad} \varphi_I - \mathbf{h}_I$. Since $\operatorname{curl} \mathbf{q}_I \cdot \mathbf{n} = \operatorname{div}_\tau(\mathbf{q}_I \times \mathbf{n}) = 0$ on $\partial\Omega$ and $\operatorname{grad} \varphi_I \times \mathbf{n}_I = \mathbf{0}$ on Γ, from the results proved above we verify at once that $\mathbf{U}_I$ belongs to $\mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$. On the other hand, by construction $\mathbf{h}_I$ is the orthogonal projection of $\mathbf{z}_I$ on $\mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$ with respect to the scalar product $(\cdot,\cdot)_{\varepsilon_I,\Omega_I}$, hence $\mathbf{U}_I$ is orthogonal to $\mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$ with respect to the scalar product $(\cdot,\cdot)_{\varepsilon_I,\Omega_I}$, and the conclusion $\mathbf{U}_I = \mathbf{0}$ follows at once. □

A.3.2 Second decomposition result

Let us define the scalar product

$$(\mathbf{u}_I, \mathbf{v}_I)_{\mu_I,\Omega_I} := \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{u}_I \cdot \mathbf{v}_I \tag{A.20}$$

and denote by the symbol $\perp^{\mu_I}$ the orthogonality with respect to this scalar product. By interchanging the role of Γ and $\partial\Omega$ and by replacing $\boldsymbol{\varepsilon}_I$ with $\boldsymbol{\mu}_I$, by proceeding as in Section A.3.1 it is easy to obtain the following theorem, whose proof is presented for the ease of the reader.

Theorem A.7. *Any given vector function* $\mathbf{v}_I \in (L^2(\Omega_I))^3$ *can be decomposed into the following sum*

$$\mathbf{v}_I = \boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{Q}_I + \operatorname{grad} \chi_I + \sum_{r=1}^{p_{\partial\Omega}} a_{I,r} \operatorname{grad} z_{r,I} + \sum_{l=1}^{n_\Gamma} b_{I,l} \boldsymbol{\rho}_{l,I} \ , \tag{A.21}$$

where $\mathbf{Q}_I \in H_{0,\Gamma}(\mathrm{curl};\Omega_I) \cap H^0_{0,\partial\Omega}(\mathrm{div};\Omega_I) \cap \mathcal{H}(\Gamma,\partial\Omega;\Omega_I)^\perp$ *are introduced in* (A.22), $\chi_I \in H^1_{0,\partial\Omega}(\Omega_I)$ *in* (A.23), *and* $a_{I,r}$, $b_{I,l}$, $r = 1,\ldots,p_{\partial\Omega}$, $l = 1,\ldots,n_\Gamma$, *in* (A.24). *Setting* $\mathbf{k}_I := \sum_{r=1}^{p_{\partial\Omega}} a_{I,r} \operatorname{grad} z_{r,I} + \sum_{l=1}^{n_\Gamma} b_{I,l}\boldsymbol{\rho}_{l,I}$, *each of the three terms* $\boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{Q}_I$, $\operatorname{grad} \chi_I$ *and* $\mathbf{k}_I$ *of the decomposition* (A.21) *is orthogonal to the others, with respect to the scalar product* $(\cdot,\cdot)_{\mu_I,\Omega_I}$.

Moreover, if $\operatorname{curl} \mathbf{v}_I = \mathbf{0}$ *in* Ω_I *and* $\mathbf{v}_I \times \mathbf{n} = \mathbf{0}$ *on* $\partial\Omega$ *it follows* $\mathbf{Q}_I = \mathbf{0}$, *if* $\operatorname{div}(\boldsymbol{\mu}_I \mathbf{v}_I) = 0$ *in* Ω_I *and* $\boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I = 0$ *on* Γ *one has* $\chi_I = 0$, *and if* $\mathbf{v}_I \perp^{\mu_I} \mathcal{H}_{\mu_I}(\partial\Omega,\Gamma;\Omega_I)$ *one finds* $a_{I,r} = 0$ *and* $b_{I,l} = 0$, $r = 1,\ldots,p_{\partial\Omega}$, $l = 1,\ldots,n_\Gamma$.

Proof. For the sake of variety with respect to the proof of Theorem A.6, let us present the result by resorting to the strong formulations. The vector function $\mathbf{Q}_I \in H(\mathrm{curl};\Omega_I) \cap H(\mathrm{div};\Omega_I)$ is the solution to

$$\begin{cases} \operatorname{curl}(\boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{Q}_I) = \operatorname{curl} \mathbf{v}_I & \text{in } \Omega_I \\ \operatorname{div} \mathbf{Q}_I = 0 & \text{in } \Omega_I \\ \mathbf{Q}_I \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\ \mathbf{Q}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ (\boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{Q}_I) \times \mathbf{n} = \mathbf{v}_I \times \mathbf{n} & \text{on } \partial\Omega \\ \mathbf{Q}_I \perp \mathcal{H}(\Gamma,\partial\Omega;\Omega_I) \ . \end{cases} \tag{A.22}$$

The existence and uniqueness of the solution $\mathbf{Q}_I$ can be proved by proceeding as was done for problem (A.14). It is readily verified that one has $\mathbf{Q}_I = \mathbf{0}$ if $\operatorname{curl} \mathbf{v}_I = \mathbf{0}$ in Ω_I and $\mathbf{v}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.

The scalar function $\chi_I \in H^1(\Omega_I)$ is the solution to the elliptic mixed boundary value problem

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}_I \operatorname{grad} \chi_I) = \operatorname{div}(\boldsymbol{\mu}_I \mathbf{v}_I) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \operatorname{grad} \chi_I \cdot \mathbf{n}_I = \boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I & \text{on } \Gamma \\ \chi_I = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{A.23}$$

The existence and uniqueness of the solution χ_I is well-known from the classical theory on elliptic boundary value problems. Clearly, if $\operatorname{div}(\boldsymbol{\mu}_I \mathbf{v}_I) = 0$ in Ω_I and $\boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I = 0$ on Γ it follows $\chi_I = 0$ in Ω_I.

Finally, the vector $(a_{I,r}, b_{I,l})$, $r = 1, \ldots, p_{\partial\Omega}$, $l = 1, \ldots, n_\Gamma$, is the solution of the linear system

$$A \begin{pmatrix} a_{I,r} \\ b_{I,l} \end{pmatrix} = \begin{pmatrix} \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{v}_I \cdot \operatorname{grad} z_{s,I} \\ \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{v}_I \cdot \boldsymbol{\rho}_{m,I} \end{pmatrix} , \tag{A.24}$$

$s = 1, \ldots, p_{\partial\Omega}$, $m = 1, \ldots, n_\Gamma$, where $A := \begin{pmatrix} D & B \\ B^T & C \end{pmatrix}$ with

$$\begin{aligned} D_{sr} &:= \int_{\Omega_I} \boldsymbol{\mu}_I \operatorname{grad} z_{r,I} \cdot \operatorname{grad} z_{s,I} \\ B_{sl} &:= \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}_{l,I} \cdot \operatorname{grad} z_{s,I} \\ C_{ml} &:= \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}_{l,I} \cdot \boldsymbol{\rho}_{m,I} \,, \end{aligned} \tag{A.25}$$

and the harmonic vector fields $\operatorname{grad} z_{r,I}$ and $\boldsymbol{\rho}_{l,I}$ are the basis functions of the space $\mathcal{H}_{\mu_I}(\partial\Omega, \Gamma; \Omega_I)$. As in the preceeding Theorem A.6, it is easily proved that the matrix A is symmetric and positive definite, and that $a_{I,r} = 0$, $b_{I,l} = 0$ for $r = 1, \ldots, p_{\partial\Omega}$, $l = 1, \ldots, n_\Gamma$ if $\mathbf{v}_I \perp^{\mu_I} \mathcal{H}_{\mu_I}(\partial\Omega, \Gamma; \Omega_I)$.

The verification that the three terms in the decomposition are orthogonal with respect to the scalar product $(\cdot, \cdot)_{\mu_I, \Omega_I}$ is readily carried out by proceeding as in Theorem A.6. Moreover, defining

$$\mathbf{V}_I := \mathbf{v}_I - \boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{Q}_I - \operatorname{grad} \chi_I - \sum_{r=1}^{p_{\partial\Omega}} a_{I,r} \operatorname{grad} z_{r,I} - \sum_{l=1}^{n_\Gamma} b_{I,l} \boldsymbol{\rho}_{l,I} \,,$$

it can be directly verified that

$$\mathbf{V}_I \in \mathcal{H}_{\mu_I}(\partial\Omega, \Gamma; \Omega_I) \ , \ \ \mathbf{V}_I \perp^{\mu_I} \mathcal{H}_{\mu_I}(\partial\Omega, \Gamma; \Omega_I) \,,$$

and the thesis follows. □

A.3.3 Third decomposition result

Another decomposition result, based on different harmonic fields, is the following one

Theorem A.8. *Any vector function* $\mathbf{v}_I \in (L^2(\Omega_I))^3$ *can be decomposed into the following sum*

$$\mathbf{v}_I = \boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{Q}_I^* + \operatorname{grad} \chi_I^* + \sum_{\alpha=1}^{n_{\Omega_I}} \theta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \ , \tag{A.26}$$

where $\mathbf{Q}_I^* \in H_0(\text{curl}; \Omega_I) \cap H^0(\text{div}; \Omega_I) \cap \mathcal{H}(e; \Omega_I)^\perp$ *is introduced in* (A.27)*,* $\chi_I^* \in H^1(\Omega_I)/\mathbb{C}$ *in* (A.28) *and* $\theta_{I,\alpha}^*$*,* $\alpha = 1, \ldots, n_{\Omega_I}$*, in* (A.29)*, and each term of the decomposition* (A.26) *is orthogonal to the others, with respect to the scalar product* $(\cdot, \cdot)_{\mu_I, \Omega_I}$.

Moreover, if $\operatorname{curl} \mathbf{v}_I = \mathbf{0}$ *in* Ω_I *it follows* $\mathbf{Q}_I^* = \mathbf{0}$*, if* $\operatorname{div}(\boldsymbol{\mu}_I \mathbf{v}_I) = 0$ *in* Ω_I *and* $\boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I = 0$ *on* $\Gamma \cup \partial\Omega$ *one has* $\operatorname{grad} \chi_I^* = \mathbf{0}$*, and if* $\mathbf{v}_I \perp^{\mu_I} \mathcal{H}_{\mu_I}(m; \Omega_I)$ *one finds* $\theta_{I,\alpha}^* = 0$*,* $\alpha = 1, \ldots, n_{\Omega_I}$.

Proof. Since the proof is similar to that of Theorems A.6 and A.7, let us just sketch it. For any vector function $\mathbf{v}_I \in (L^2(\Omega_I))^3$, construct the solution $\mathbf{Q}_I^* \in H(\text{curl}; \Omega_I) \cap H(\text{div}; \Omega_I)$ to

$$\begin{cases} \operatorname{curl}(\boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{Q}_I^*) = \operatorname{curl} \mathbf{v}_I & \text{in } \Omega_I \\ \operatorname{div} \mathbf{Q}_I^* = 0 & \text{in } \Omega_I \\ \mathbf{Q}_I^* \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \cup \partial\Omega \\ \mathbf{Q}_I^* \perp \mathcal{H}(e; \Omega_I) \ . \end{cases} \tag{A.27}$$

The existence and uniqueness of the solution $\mathbf{Q}_I^*$ can be proved as done for problem (A.14). Note that $\mathbf{Q}_I^* = \mathbf{0}$ if $\operatorname{curl} \mathbf{v}_I = \mathbf{0}$ in Ω_I.

The scalar function $\chi_I^* \in H^1(\Omega_I)/\mathbb{C}$ is the solution to the elliptic Neumann boundary value problem

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}_I \operatorname{grad} \chi_I^*) = \operatorname{div}(\boldsymbol{\mu}_I \mathbf{v}_I) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \operatorname{grad} \chi_I^* \cdot \mathbf{n}_I = \boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I & \text{on } \Gamma \cup \partial\Omega \ . \end{cases} \tag{A.28}$$

It is clear that $\operatorname{grad} \chi_I^* = \mathbf{0}$ in Ω_I provided that $\operatorname{div}(\boldsymbol{\mu}_I \mathbf{v}_I) = 0$ in Ω_I and $\boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I = 0$ on $\Gamma \cup \partial\Omega$.

Finally, the vector $\theta_{I,\alpha}^*$, $\alpha = 1, \ldots, n_{\Omega_I}$, is the solution of the linear system

$$\sum_{\alpha=1}^{n_{\Omega_I}} A_{\beta\alpha}^* \theta_{I,\alpha}^* = \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{v}_I \cdot \boldsymbol{\rho}_{\beta,I}^* \ , \ \ \beta = 1, \ldots, n_{\Omega_I} \ , \tag{A.29}$$

where

$$A_{\beta\alpha}^* := \textstyle\int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}_{\alpha,I}^* \cdot \boldsymbol{\rho}_{\beta,I}^* \ , \tag{A.30}$$

and the harmonic vector fields $\boldsymbol{\rho}_{\alpha,I}^*$ are the basis functions of the space $\mathcal{H}_{\mu_I}(m; \Omega_I)$. Hence $\theta_{I,\alpha}^* = 0$ for $\alpha = 1, \ldots, n_{\Omega_I}$ when $\mathbf{v}_I \perp^{\mu_I} \mathcal{H}_{\mu_I}(m; \Omega_I)$.

The proof of the theorem is now easily done by following the same procedure used in Theorems A.6 and A.7. □

A.4 More on harmonic fields

In this section we give an explicit construction of the basis functions of the spaces of harmonic fields presented in Section 1.4. Useful references about this topic are, e.g., the papers by Foias and Temam [106], Picard [192], Amrouche et al. [27], Fernandes and Gilardi [104], Hiptmair [126], Cantarella et al. [73], Auchmuty and Alexander [30], [31], and the books by Bossavit [59], Gross and Kotiuga [115]. The most complete results are in Ghiloni [110].

Let us start from the space $\mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I)$, defined as

$$\begin{aligned}\mathcal{H}_{\varepsilon_I}(\Gamma, \partial\Omega; \Omega_I) := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 \mid \operatorname{curl} \mathbf{v}_I = \mathbf{0}, \operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{v}_I) = 0,\\ \mathbf{v}_I \times \mathbf{n}_I = \mathbf{0} \text{ on } \Gamma, \boldsymbol{\varepsilon}_I \mathbf{v}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} .\end{aligned}$$

The determination of the basis functions $\operatorname{grad} w_{j,I}$, $j = 1, \dots, p_\Gamma$, is easily done as follows: $w_{j,I} \in H^1(\Omega_I)$ is the solution of the elliptic problem

$$\begin{cases} \operatorname{div}(\boldsymbol{\varepsilon}_I \operatorname{grad} w_{j,I}) = 0 & \text{in } \Omega_I \\ \boldsymbol{\varepsilon}_I \operatorname{grad} w_{j,I} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ w_{j,I} = 0 & \text{on } \Gamma \setminus \Gamma_j \\ w_{j,I} = 1 & \text{on } \Gamma_j \ . \end{cases} \tag{A.31}$$

Instead, the basis functions $\boldsymbol{\pi}_{k,I}$, $k = 1, \dots, n_{\partial\Omega}$, need some preliminary notation. It is known that in Ω_I there exist $n_{\partial\Omega}$ connected orientable Lipschitz surfaces Σ_k, with $\partial\Sigma_k \subset \partial\Omega$, such that every curl-free vector field in Ω_I with vanishing tangential component on Γ has a global potential in $\Omega_I \setminus \cup_k \Sigma_k$. These surfaces, usually called Seifert surfaces, are "cutting" surfaces: each one of them "cuts" a Γ-independent non-bounding cycle in Ω_I (for notation, see Section 1.4). We can now introduce the functions $q_{k,I} \in H^1(\Omega_I \setminus \Sigma_k)$, solutions to

$$\begin{cases} \operatorname{div}(\boldsymbol{\varepsilon}_I \operatorname{grad} q_{k,I}) = 0 & \text{in } \Omega_I \setminus \Sigma_k \\ \boldsymbol{\varepsilon}_I \operatorname{grad} q_{k,I} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \setminus \partial\Sigma_k \\ q_{k,I} = 0 & \text{on } \Gamma \\ [\boldsymbol{\varepsilon}_I \operatorname{grad} q_{k,I} \cdot \mathbf{n}_\Sigma]_{\Sigma_k} = 0 & \\ [q_{k,I}]_{\Sigma_k} = 1 \ , & \end{cases} \tag{A.32}$$

having denoted by $[\,\cdot\,]_{\Sigma_k}$ the jump across the surface Σ_k and by $\mathbf{n}_\Sigma$ the unit normal vector on Σ_k. We finally define $\boldsymbol{\pi}_{k,I}$ as the $(L^2(\Omega_I))^3$-extension of $\operatorname{grad} q_{k,I}$ (computed in $\Omega_I \setminus \Sigma_k$).

The basis functions for the other harmonic spaces can be defined in a similar way: let us go on with $\mathcal{H}_{\mu_I}(\partial\Omega, \Gamma; \Omega_I)$, defined as

$$\begin{aligned}\mathcal{H}_{\mu_I}(\partial\Omega, \Gamma; \Omega_I) := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 \mid \operatorname{curl} \mathbf{v}_I = \mathbf{0}, \operatorname{div}(\boldsymbol{\mu}_I \mathbf{v}_I) = 0,\\ \mathbf{v}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega, \boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I = 0 \text{ on } \Gamma\} .\end{aligned}$$

The basis functions grad $z_{r,I}$, $r = 1, \dots, p_{\partial\Omega}$, are the gradients of the solutions $z_{r,I} \in H^1(\Omega_I)$ to

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}_I \operatorname{grad} z_{r,I}) = 0 & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \operatorname{grad} z_{r,I} \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \\ z_{r,I} = 0 & \text{on } \partial\Omega \setminus (\partial\Omega)_r \\ z_{r,I} = 1 & \text{on } (\partial\Omega)_r \ . \end{cases} \tag{A.33}$$

Moreover, similarly to the preceding case, in Ω_I there exist n_Γ connected orientable Lipschitz surfaces Ξ_l, $l = 1, \dots, n_\Gamma$, with $\partial\Xi_l \subset \Gamma$, such that every curl-free vector field in Ω_I with vanishing tangential component on $\partial\Omega$ has a global potential in $\Omega_I \setminus \cup_l \Xi_l$. These "cutting" surfaces "cuts" the $\partial\Omega$-independent non-bounding cycles in Ω_I. Then introduce the functions $p_{l,I} \in H^1(\Omega_I \setminus \Xi_l)$, solutions to

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}_I \operatorname{grad} p_{l,I}) = 0 & \text{in } \Omega_I \setminus \Xi_l \\ \boldsymbol{\mu}_I \operatorname{grad} p_{l,I} \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \setminus \partial\Xi_l \\ p_{l,I} = 0 & \text{on } \partial\Omega \\ [\boldsymbol{\mu}_I \operatorname{grad} p_{l,I} \cdot \mathbf{n}_\Xi]_{\Xi_l} = 0 & \\ [p_{l,I}]_{\Xi_l} = 1 \ , & \end{cases} \tag{A.34}$$

having denoted by $[\,\cdot\,]_{\Xi_l}$ the jump across the surface Ξ_l and by $\mathbf{n}_\Xi$ the unit normal vector on Ξ_l. The basis functions $\boldsymbol{\rho}_{l,I}$ are the $(L^2(\Omega_I))^3$-extension of grad $p_{l,I}$ (computed in $\Omega_I \setminus \Xi_l$).

For the space $\mathcal{H}_{\varepsilon_I}(e; \Omega_I)$, defined as

$$\mathcal{H}_{\varepsilon_I}(e; \Omega_I) := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 \mid \operatorname{curl} \mathbf{v}_I = \mathbf{0}, \operatorname{div}(\boldsymbol{\varepsilon}_I \mathbf{v}_I) = 0, \\ \mathbf{v}_I \times \mathbf{n}_I = \mathbf{0} \text{ on } \Gamma \cup \partial\Omega\} \ ,$$

the basis functions are grad $w^*_{\gamma,I}$, $\gamma = 0, \dots, p_{\partial\Omega} + p_\Gamma$, where $w^*_{\gamma,I} \in H^1(\Omega_I)$ is the solution to

$$\begin{cases} \operatorname{div}(\boldsymbol{\varepsilon}_I \operatorname{grad} w^*_{\gamma,I}) = 0 & \text{in } \Omega_I \\ w^*_{\gamma,I} = 0 & \text{on } (\partial\Omega \cup \Gamma) \setminus \Theta_\gamma \\ w^*_{\gamma,I} = 1 & \text{on } \Theta_\gamma \ . \end{cases} \tag{A.35}$$

Here the surfaces Θ_γ are defined as $\Theta_\gamma := (\partial\Omega)_\gamma$ for $\gamma = 0, \dots, p_{\partial\Omega}$ and $\Theta_\gamma := \Gamma_{\gamma - p_{\partial\Omega}}$ for $\gamma = p_{\partial\Omega} + 1, \dots, p_{\partial\Omega} + p_\Gamma$.

When considering the space $\mathcal{H}_{\mu_I}(m; \Omega_I)$, defined as

$$\mathcal{H}_{\mu_I}(m; \Omega_I) := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 \mid \operatorname{curl} \mathbf{v}_I = \mathbf{0}, \operatorname{div}(\boldsymbol{\mu}_I \mathbf{v}_I) = 0, \\ \boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n} = 0 \text{ on } \Gamma \cup \partial\Omega\} \ ,$$

one has first to introduce the "cutting" surfaces $\Xi^*_\alpha \subset \Omega_I$, $\alpha = 1, \dots, n_{\Omega_I}$, each one "cutting" an independent non-bounding cycle in Ω_I. They are connected orientable Lipschitz surfaces with $\partial\Xi^*_\alpha \subset \partial\Omega \cup \Gamma$, such that every curl-free vector field in Ω_I has a global potential in $\Omega_I \setminus \cup_\alpha \Xi^*_\alpha$. The basis functions $\boldsymbol{\rho}^*_{\alpha,I}$ are the $(L^2(\Omega_I))^3$-extension of grad $p^*_{\alpha,I}$, where $p^*_{\alpha,I} \in H^1(\Omega_I \setminus \Xi^*_\alpha)$ is the solution, determined up to an additive constant, to

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}_I \operatorname{grad} p^*_{\alpha,I}) = 0 & \text{in } \Omega_I \setminus \Xi^*_\alpha \\ \boldsymbol{\mu}_I \operatorname{grad} p^*_{\alpha,I} \cdot \mathbf{n}_I = 0 & \text{on } (\partial\Omega \cup \Gamma) \setminus \partial\Xi^*_\alpha \\ \left[\boldsymbol{\mu}_I \operatorname{grad} p^*_{\alpha,I} \cdot \mathbf{n}_{\Xi^*}\right]_{\Xi^*_\alpha} = 0 & \\ \left[p^*_{\alpha,I}\right]_{\Xi^*_\alpha} = 1 \ , & \end{cases} \tag{A.36}$$

having denoted by $[\,\cdot\,]_{\Xi^*_\alpha}$ the jump across the surface Ξ^*_α and by $\mathbf{n}_{\Xi^*}$ the unit normal vector on Ξ^*_α.

The basis functions $\operatorname{grad} \hat{z}_r$, $r = 1, \ldots, p_{\partial\Omega}$, of the space

$$\mathcal{H}(e;\Omega) := \{\mathbf{v} \in (L^2(\Omega))^3 \mid \operatorname{curl} \mathbf{v} = \mathbf{0}, \operatorname{div} \mathbf{v} = 0, \\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\} \ ,$$

and $\hat{\boldsymbol{\pi}}_t$, $t = 1, \ldots, n_\Omega$, of the space

$$\mathcal{H}(m;\Omega) := \{\mathbf{v} \in (L^2(\Omega))^3 \mid \operatorname{curl} \mathbf{v} = \mathbf{0}, \operatorname{div} \mathbf{v} = 0, \\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \ ,$$

are determined in a similar way to those of the spaces $\mathcal{H}_{\varepsilon_I}(e;\Omega_I)$ and $\mathcal{H}_{\mu_I}(m;\Omega_I)$, respectively.

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