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A Course in Commutative Banach Algebras

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Dedicated to my wife Ursula

Preface

Banach algebras are Banach spaces equipped with a continuous multiplication. In rough terms, there are three types of them: algebras of bounded linear operators on Banach spaces with composition and the operator norm, algebras consisting of bounded continuous functions on topological spaces with pointwise product and the uniform norm, and algebras of integrable functions on locally compact groups with convolution as multiplication. These all play a key role in modern analysis. Much of operator theory is best approached from a Banach algebra point of view and many questions in complex analysis (such as approximation by polynomials or rational functions in specific domains) are best understood within the framework of Banach algebras. Also, the study of a locally compact Abelian group is closely related to the study of the group algebra $L^1(G)$.

There exist a rich literature and excellent texts on each single class of Banach algebras, notably on uniform algebras and on operator algebras. This work is intended as a textbook which provides a thorough introduction to the theory of commutative Banach algebras and stresses the applications to commutative harmonic analysis while also touching on uniform algebras. In this sense and purpose the book resembles Larsen's classical text [75] which shares many themes and has been a valuable resource. However, for advanced graduate students and researchers I have covered several topics which have not been published in books before, including some journal articles.

The reader is expected to have some basic knowledge of functional analysis, point set topology, complex analysis, measure theory, and locally compact groups. However, many of the prerequisites are collected (without proofs) in the appendix. Here the reader may also find (including proofs) some facts about convolution of functions on locally compact groups, the Pontryagin duality theorem and some of its consequences, and a description of the closed sets in the coset ring of an Abelian topological group.

The book is divided into five chapters, the contents of which can be described as follows. The first chapter introduces the basic concepts and constructions and provides a comprehensive treatment of the spectrum of a Banach algebra element.

Chapter 2 begins with Gelfand's fundamental theorem on representing a commutative Banach algebra A as an algebra of continuous functions on a locally compact Hausdorff space, the structure space $\Delta(A)$ of A, which is defined to be the set of all homomorphisms from A onto $\mathbb{C}$, equipped with the w^*-topology. This Gelfand homomorphism turns out to be an isometric isomorphism onto $C_0(\Delta(A))$ if and only if A is a commutative C^*-algebra. Applications of this basic result include proofs for the existence of the Stone-Čech compactification of a completely regular topological space and of the Bohr compactification of a locally compact Abelian group. The structure space of a finitely generated algebra identifies canonically with the joint spectrum of the set of generators and this leads to a description of the Gelfand representation of several uniform algebras, such as the closure of algebras of polynomial and of rational functions on compact subsets of $\mathbb{C}^n$. Following our intention to emphasize the connection with commutative harmonic analysis, we extensively study the Gelfand representation of algebras associated with locally compact groups. This concerns, in the first place, the convolution algebra $L^1(G)$ of integrable functions on a locally compact Abelian group, but also weighted algebras $L^1(G, \omega)$ and Fourier algebras. Chapter 2 concludes with determining the structure spaces of tensor products of two commutative Banach algebras and a discussion of semisimplicity of the projective tensor product.

In Chapter 3 we focus on some important problems which evolve from the Gelfand representation theory and concern the structure space $\Delta(A)$ and the structure of A itself. The new tools required are holomorphic functional calculi for Banach algebra elements. These are developed in Section 3.1 and subsequently applied to study the topological group of invertible elements of a unital commutative Banach algebra A and the problem of which elements of $\Delta(A)$ extend to elements of $\Delta(B)$ for any commutative Banach algebra B containing A as a closed subalgebra. This latter question is linked with the Shilov boundary which we investigate thoroughly. One of the major highlights in the theory of commutative Banach algebras is Shilov's idempotent theorem. This rests on the multivariable holomorphic functional calculus and is established in Section 3.5, followed by several applications that illustrate the power of the idempotent theorem.

The concept of regularity and its role in ideal theory is the main subject of Chapter 4. The relevance of regularity is due to the fact that it is equivalent to coincidence of the Gelfand topology and the hull-kernel topology on $\Delta(A)$. We show the existence of a greatest regular subalgebra of any commutative Banach algebra and study permanence properties of regularity. One of the most profound results in commutative harmonic analysis is regularity of the group algebra $L^1(G)$. To prove this, we have chosen an approach which is based on the Gelfand theory of commutative C^*-algebras. Recently, certain properties related to, but weaker than, regularity have been investigated. We give a detailed account and comparison of these so-called spectral extension properties and the unique uniform norm property. Finally, we establish Domar's result which asserts that $L^1(G, \omega)$ is regular whenever the weight ω is nonquasianalytic.

The last chapter is devoted to ideal theory of regular semisimple commutative Banach algebras and to spectral synthesis problems in particular. The basic notions are that of a spectral set and of a Ditkin set in $\Delta(A)$. It is customary to say that spectral synthesis holds for the algebra A if every closed subset of $\Delta(A)$ is a spectral set (equivalently, every closed ideal of A is the intersection of the maximal ideals containing it). In Section 5.2 we present a number of results on generating spectral sets and Ditkin sets, some of which cannot be found elsewhere in this generality. Subsequently, these results are applied to $L^1(G)$. In this context we point out that a famous theorem of Malliavin states that spectral synthesis fails to hold for $L^1(G)$ whenever G is a noncompact locally compact Abelian group. We also present a complete description of all the closed ideals in $L^1(G)$ with bounded approximate identities. Spectral synthesis also fails for the algebra $C^n[0,1]$ of n-times continuously differentiable functions on the interval $[0,1]$ and even for a certain algebra with discrete structure space, the Mirkil algebra. Both of these algebras are discussed in detail: $C^n[0,1]$ because it nevertheless admits a satisfactory ideal structure and the Mirkil algebra because it serves as a counterexample to several conjectures in spectral synthesis.

Each chapter is accompanied by an extensive set of exercises, ranging from simple and straightforward applications of concepts and results developed in the chapter in question to more sophisticated supplements to the theory. These exercises add numerous examples to those already given in the text. In several cases hints are provided, and the reader is strongly encouraged to solve and work out as many of these exercises as possible.

There are various options for using the material as a text for courses, depending on the instructor's intention and inclination. Any one-semester course, however, has to cover Sections 1.1 and 1.2 and Sections 2.1 to 2.4, and might then continue with

- Sections 2.5 and 2.6 and the Shilov boundary if the main emphasis is on uniform algebras,
- Sections 1.5 and 2.11 and the corresponding passages of Chapters 3, 4 and 5 when concentrating on projective tensor products,
- Selected topics from Chapter 3 if the focus is on general Banach algebras rather than group algebras or uniform algebras,
- Sections 2.7 and 4.4 and, if time permits, parts of Chapter 5 whenever applications in commutative harmonic analysis is the preferred objective.

Major portions of the book grew out of graduate courses taught at the University of Heidelberg, the Technical University of Munich and the University of Paderborn.

I owe a great deal to two colleagues and friends. Robert J. Archbold and Ali Ülger have both taken up the onerous burden of reading substantial parts of the text and made many helpful suggestions for improvement. I am also

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