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# Stability Theory of Switched Dynamical Systems

 Springer

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*In memory of Professor Wei-Bing Gao*

# Preface

Switched control systems have attracted much interest from the control community not only because of their inherent complexity, but also due to the practical importance with a wide range of their applications in nature, engineering, and social sciences. Switched systems are necessary because various natural, social, and engineering systems cannot be described simply by a single model, and many systems exhibit switching between several models depending on various environments. Natural biological systems switch strategies in accordance to environmental changes for survival. Switched behaviors have also been exhibited in a number of social systems. To achieve an improved performance, switching has been extensively utilized/exploited in many engineering systems such as electronics, power systems, and traffic control, among others.

Theoretical investigation and examination of switched control systems are academically more challenging due to their rich, diverse, and complex dynamics. Switching makes those systems much more complicated than standard systems. Many more complicated behaviors/dynamics and fundamentally new properties, which standard systems do not have, have been demonstrated on switched systems. From the viewpoint of control system design, switching brings an additional degree of freedom in control system design. Switching laws, in addition to control laws, may be utilized to manipulate switched systems to achieve a better performance of a system. This can be seen as an added advantage for control design to attain certain control purposes.

On the one hand, switching could be induced by any unpredictable sudden change in system dynamics/structures, such as a sudden change of a system structure due to the failure of a component/subsystems, or the accidental activation of any subsystems. On the other hand, the switching is introduced artificially to effectively control highly complex nonlinear systems under the umbrella of the so-called hybrid control. In both cases, an essential feature is the interaction between the continuous system dynamics and the discrete switching dynamics. Such switched dynamical systems typically consist of sets of subsystems and switching signals that coordinate the switching among the subsystems.

In this book, we investigate the stability issues under various switching mechanisms. For a controlled switching, the switching signal is a design variable just as

the control input in the conventional systems. It is measurable and can be freely assigned. In this case, the stability is in fact a kind of stabilization by stabilizing switching design. For an arbitrary switching, the switching signal is blind and uncontrolled, and the stability is in fact a kind of robustness against the switching perturbations. Besides the two extreme cases, the switching signal could be constrained in that it is neither controlled nor free arbitrarily. In other words, partial information is known about the switching mechanism. Typical constrained switchings include (i) autonomous switching, where the switching signal is generated autonomously with a preassigned state-space-partition-based switching mechanism; (ii) dwell-time switching, where the minimum duration on each subsystem is known and positive; and (iii) random switching with a known stochastic distribution. Switched systems under various constrained switching might behave in a rich, diverse and complex manner.

The objective of this book is to present in a systematic manner the stability theory of switched dynamical systems under different switching mechanisms. By bringing forward fresh new concepts, novel methods, and innovative tools into the exploration of various switching schemes, we are to provide a state-of-the-art and comprehensive systematic treatment of the stability issues for switched dynamical systems.

The book is organized in five chapters. Except for Chap. 1 that briefly introduces the problem formations and the organization of the book, subsequent chapters exploit several important topics in detail in a timely manner.

In Chap. 2, we focus on the guaranteed stability analysis of switched dynamical systems under arbitrary switching. As global uniform asymptotic stability is equivalent to the existence of a common Lyapunov function of the subsystems, the Lyapunov approach plays a dominant role in the stability analysis. For switched linear systems, due to the fact that quadratic Lyapunov candidates are insufficient for coping with stability, emphasis is laid on the sets of functions which are universal in the sense that each asymptotically stable system admits a Lyapunov function from the function set. Both piecewise linear functions and piecewise quadratic functions are proven to be universal, and their connections to algebraic stability criteria are also established. We also pay much attention to the algebraic theory of discrete-time switched linear systems, where the stability is elegantly characterized by the spectral radius of the matrix set, which generalizes the standard matrix spectral theory. While determining the spectral radius has been proven to be *NP*-hard, we introduce the homogeneous polynomials to serve as common Lyapunov functions and utilize the sum-of-squares technique and the semi-definite programming to approximate the spectral radius. Finally, the more subtle issue of marginal stability is carefully examined, and its connection to the common weak Lyapunov function is established. We reveal that marginal stability admits a block triangular decomposition with clear spectral information, and this leads to an invariant set viewpoint for characterizing marginal stability and marginal instability.

Chapter 3 presents stability theory for switched dynamical systems under constrained switching. There are three types of constrained switching addressed in this chapter. The first type of constrained switching is the random switching with a pre-assigned jump distribution. When the subsystems are linear and the switching is

governed by a Markov process, the switched linear system is known to be a jump linear system. We introduce various stability concepts and their criteria, and establish the connections to the guaranteed stability criteria in Chap. 2. The second is the piecewise affine systems where the state space is partitioned into a set of polyhedral cells, each relating to a subsystem, and hence the switching is totally autonomous. The piecewise quadratic Lyapunov approach, the surface Lyapunov approach, and the transition graph approach are introduced. The pros and cons of the approaches are compared and discussed. The third type of constrained switching is the dwell-time switching, where the switching duration between any two consecutive switches admits a positive lower bound. We address both the stability analysis, where the dwell time is preassigned, and the stabilizing switching design, where the minimum or maximum dwell time is to be designed. The design captures the capability and the limitation of the switching mechanism.

Chapter 4 is devoted to the stabilizing switching design for switched dynamical systems under controlled switching. It is proven that a switched Lyapunov function exists if the system is globally asymptotically stabilizable. However, counterexamples exhibit that even stabilizable planar switched linear systems may not admit any convex switched Lyapunov function. To overcome the intrinsic difficulty, we introduce a class of nonconvex functions known as min functions that are piecewise quadratic and prove that each stabilizable switched linear system admits a min function as a switched Lyapunov function. To further address the stabilizability and robustness of switched linear system, we propose a pathwise state-feedback switching strategy, which accounts to concatenating a finite number of switching paths based on appropriate partitions of the state space. By aggregating the overall system into a discrete-time piecewise linear system, we are able to prove that the switching strategy exponentially stabilizes the original switched linear system whenever it is asymptotically stabilizable. We develop a computational procedure to calculate a stabilizing pathwise state-feedback switching law for an asymptotically stabilizable switched linear system. To further investigate the robustness of the pathwise state-feedback switching strategy, we define a (relative) distance between two switching signals and prove that the closed-loop system is robust against structural/unstructural/switching perturbations.

With the stability theory presented in the previous chapters, we further exploit its connections and implications to several fundamental control problems in Chap. 5. For absolute stability of Lur'e systems, an elegant connection to the guaranteed stability of switched linear systems is established. Utilizing this connection, computational algorithms are presented to verify absolute stability for planar Lur'e systems. Another implication of the guaranteed stability criteria is the consensus analysis of multiagent systems with dynamic neighbors, and exponential agreement is reached if the graph is always strongly connected. For an intelligent system with linear local controllers and a fuzzy rule, it is naturally converted into a piecewise linear system, and hence the stability analysis can be conducted by means of the stability criteria presented in Chap. 3. This brings a new design method and a fresh observation to the fuzzy control problem. For a SISO linear process with unknown parameters, an adaptive control framework is established based on appropriate partitions of the parameter space and proper stabilizing switching strategy among the local controllers



which are designed to stabilize the system in a local sense. Finally, for controllable switched linear systems, a multilinear feedback design approach is proposed to tackle the stabilization problem. The main idea is to associate a set of candidate linear controllers with each subsystem, such that the extended switched system is stabilizable. By utilizing the pathwise state-feedback switching design diagram, the problem of stabilization is solved in a constructive manner.

The book is primarily intended for researchers and engineers in the system and control community. It can also serve as complementary reading for nonlinear system theory at the postgraduate level.

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# List of Symbols

$a \stackrel{\text{def}}{=} b$	Defines $a$ to be $b$
iff	If and only if
s.t.	Such that
w.r.t.	With respect to
a.s.	Almost surely
$\mathbf{R}$	The field of real numbers
$\mathbf{C}$	The field of complex numbers
$\mathbf{N}$	The set of integers
$\mathbf{R}^+$	The set of positive real numbers
$\mathbf{R}_+$	The set of nonnegative real numbers
$\mathbf{N}^+$	The set of natural numbers
$\mathbf{N}_+$	The set of nonnegative integers
$\mathbf{R}^n$	The set of $n$ -dimensional real vectors
$\mathbf{R}^{n \times m}$	The set of $n \times m$ -dimensional real matrices
$z^*$	The conjugate transpose of complex number $z$
$\sqrt{-1}$	The imaginary unit
$\lfloor a \rfloor$	The largest integer less than or equal to $a$
$\lceil a \rceil$	The smallest integer larger than or equal to $a$
$I$ ( $I_n$ )	The identity matrix (of dimension $n \times n$ )
$x^T$ or $A^T$	The transpose of vector $x$ or matrix $A$
$A^{-1}$	The inverse of matrix $A$
$A^+$	The Moore-Penrose pseudo-inverse of matrix $A$
$[A_1, A_2]$	The matrix commutator $A_1A_2 - A_2A_1$
$A \otimes B$	Kronecker product of matrices $A$ and $B$
$A \oplus B$	Kronecker sum of matrices $A$ and $B$
$P > 0$ ( $P \geq 0$ )	Matrix $P$ is Hermitian and positive (semi-)definite
$P < 0$ ( $P \leq 0$ )	Matrix $P$ is Hermitian and negative (semi-)definite
$x \succ y$ ( $x \succeq y$ )	$x$ is entrywise greater than (not less than) $y$
$\det A$	The determinant of a matrix $A$
$\text{tr}(A)$	The trace of a matrix $A$
$\text{disc}(A)$	The discriminant of a matrix $A$

$\lambda(A)$	The set of eigenvalues of $A$
$\lambda_{\max}(A)$	The maximum eigenvalue of a real symmetric matrix $A$
$\lambda_{\min}(A)$	The minimum eigenvalue of a real symmetric matrix $A$
$\rho(\mathbf{A})$	The spectral radius of a matrix set $\mathbf{A}$
$x_i, x(i)$	The $i$ th element of a vector $x$
$A(i, j), A_{ij}$	The $ij$ th element of a matrix $A$
$ p $	The length of a switching path $p$
$ x $	The norm of a vector $x$
$\ A\ $	The induced norm of matrix $A$
$ x _p$	The $\ell_p$ norm of vector $x$
$\ A\ _p$	The induced $l_p$ -norm of a matrix $A$
$\mu_{ \cdot }$	The matrix measure induced by a norm $ \cdot $
$\text{Im } A$	The image of an operator/matrix $A$
$E\xi^\delta$	The $\delta$ th-order moment of a random variable $\xi$
$S = \{x, y, \dots\}$	The set $S$ with quantities $x, y$ , etc.
$\mu S$	The set $\{\mu x : x \in S\}$
$\#S$	The cardinality of a set $S$
$\max S$	The maximum element of a set $S$
$\min S$	The minimum element of a set $S$
$\sup S$	The smallest number that is larger than or equal to each element of a set $S$
$\inf S$	The largest number that is smaller than or equal to each element of a set $S$
$S_1 - S_2$	The set $\{s \in S_1 : s \notin S_2\}$
$\arg \max S$	The index of the maximum element of an ordered set $S$
$\arg \min S$	The index of the minimum element of an ordered set $S$
$\emptyset$	The empty set
$\text{meas } \Omega$	The Lebesgue measure of a set $\Omega$ in $\mathbf{R}^n$
$\Omega^o$	The interior of a set $\Omega$
$\text{co } \Omega$	The convex hull of a set $\Omega$
$\mathbf{B}_r$	The ball centered at the origin with radius $r$
$\mathbf{H}_r$	The sphere centered at the origin with radius $r$
$\text{mod}(a, b)$	The remainder of $a$ divided by $b$
$\text{sgn}(\cdot)$	The signum function
$\text{sat}(\cdot)$	The saturation function with unit limits
$\lim_{s \uparrow t} f(s)$	The limit from the left of a function $f(\cdot)$ at $t$
$\lim_{s \downarrow t} f(s)$	The limit from the right of a function $f(\cdot)$ at $t$
$\mathcal{C}^k$	The set of functions with continuous $k$ th-order derivative
$\mathcal{D}^+ V$	The upper Dini derivative of a function $V$
class $\mathcal{K}$	The set of continuous and strictly increasing functions that vanish at zero
class $\mathcal{K}_\infty$	The set of unbounded class $\mathcal{K}$ functions
class $\mathcal{KL}$	The set of functions $\beta: \mathbf{R}_+ \times \mathbf{R}_+ \mapsto \mathbf{R}_+$ with $\beta(\cdot, t) \in \mathcal{K} \forall t \geq 0$ and $\lim_{t \rightarrow \infty} \beta(r, t) = 0 \forall r \geq 0$
$\text{MF}_\Gamma$	The Minkowski function of a region $\Gamma$

$M$	The index set $\{1, 2, \dots, m\}$ of the discrete state
$\mathcal{T}$	The time space
$\mathcal{T}_s$	The set $\{t \in \mathcal{T} : t \geq s\}$
$[a, b)$	The time interval $\{t \in \mathcal{T} : a \leq t < b\}$
$\sigma$	The switching signal of the switched system
$\hat{i}$	The constant switching signal $\sigma(t) = i \forall t$
$\mathcal{S}_{[a,b)}$	The set of well-defined switching paths over $[a, b)$
$\mathcal{S}_{[t_0, \infty)}$ or $\mathcal{S}$	The set of well-defined switching signals over $[t_0, \infty)$
$\Psi$	The infinitesimal matrix of a Markov process
$\phi(t; t_0, x_0, \sigma)$	The solution of the switched system
$\Phi(t_1, t_2, \sigma)$	The state transition matrix of the switched linear system