

# On Normalized Integral Table Algebras (Fusion Rings)

For further volumes:  
[www.springer.com/series/6253](http://www.springer.com/series/6253)

# Algebra and Applications

---

Volume 16

---

Series Editors:

Alice Fialowski  
*Eötvös Loránd University, Budapest, Hungary*

Eric Friedlander  
*Northwestern University, Evanston, USA*

John Greenlees  
*Sheffield University, Sheffield, UK*

Gerhard Hiß  
*Aachen University, Aachen, Germany*

Ieke Moerdijk  
*Utrecht University, Utrecht, The Netherlands*

Idun Reiten  
*Norwegian University of Science and Technology, Trondheim, Norway*

Christoph Schweigert  
*Hamburg University, Hamburg, Germany*

Mina Teicher  
*Bar-Ilan University, Ramat-Gan, Israel*

Alain Verschoren  
*University of Antwerp, Antwerp, Belgium*

*Algebra and Applications* aims to publish well written and carefully refereed monographs with up-to-date information about progress in all fields of algebra, its classical impact on commutative and noncommutative algebraic and differential geometry, K-theory and algebraic topology, as well as applications in related domains, such as number theory, homotopy and (co)homology theory, physics and discrete mathematics.

Particular emphasis will be put on state-of-the-art topics such as rings of differential operators, Lie algebras and super-algebras, group rings and algebras,  $C^*$ -algebras, Kac-Moody theory, arithmetic algebraic geometry, Hopf algebras and quantum groups, as well as their applications. In addition, *Algebra and Applications* will also publish monographs dedicated to computational aspects of these topics as well as algebraic and geometric methods in computer science.

Zvi Arad • Xu Bangteng • Guiyun Chen •  
Effi Cohen • Arisha Haj Ihia Hussam •  
Mikhail Muzychuk

# On Normalized Integral Table Algebras (Fusion Rings)

Generated by a Faithful Non-real  
Element of Degree 3

Zvi Arad  
Department of Mathematics  
Bar Ilan University  
Ramat Gan, 52900  
Israel  
and  
Netanya Academic College  
1 University Street  
Netanya  
Israel  
[aradtzvi@netanya.ac.il](mailto:aradtzvi@netanya.ac.il)

Xu Bangteng  
Department of Mathematics and Statistics  
Eastern Kentucky University  
Richmond, KY, 40475  
USA  
[bangteng.xu@eku.edu](mailto:bangteng.xu@eku.edu)

Guiyun Chen  
Department of Mathematics  
Southwest University  
Chongqing, 400715  
People's Republic of China  
[gychen@swu.edu.cn](mailto:gychen@swu.edu.cn)

Effi Cohen  
Department of Mathematics  
Bar Ilan University  
Ramat Gan, 52900  
Israel  
[Cohene3@walla.com](mailto:Cohene3@walla.com)

Arisha Haj Ihia Hussam  
Department of Mathematics  
Alqasemi Academic College of Education  
Baqā El-Gharbieh, 30100  
Israel  
[Hussam10@walla.co.il](mailto:Hussam10@walla.co.il)

Mikhail Muzychuk  
Netanya Academic College  
Kibbutz Galuyot 16  
Netanya, 42 365  
Israel  
[muzy@netanya.ac.il](mailto:muzy@netanya.ac.il)

ISSN 1572-5553  
ISBN 978-0-85729-849-2  
DOI 10.1007/978-0-85729-850-8  
Springer London Dordrecht Heidelberg New York

e-ISSN 2192-2950  
e-ISBN 978-0-85729-850-8

British Library Cataloguing in Publication Data  
A catalogue record for this book is available from the British Library

Library of Congress Control Number: 2011934389

Mathematics Subject Classification: 05E10, 05E30, 13A99, 20C15, 20N20, 81T40

© Springer-Verlag London Limited 2011

Apart from any fair dealing for the purposes of research or private study, or criticism or review, as permitted under the Copyright, Designs and Patents Act 1988, this publication may only be reproduced, stored or transmitted, in any form or by any means, with the prior permission in writing of the publishers, or in the case of reprographic reproduction in accordance with the terms of licenses issued by the Copyright Licensing Agency. Enquiries concerning reproduction outside those terms should be sent to the publishers.

The use of registered names, trademarks, etc., in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant laws and regulations and therefore free for general use.

The publisher makes no representation, express or implied, with regard to the accuracy of the information contained in this book and cannot accept any legal responsibility or liability for any errors or omissions that may be made.

*Cover design:* VTeX UAB, Lithuania

Printed on acid-free paper

Springer is part of Springer Science+Business Media ([www.springer.com](http://www.springer.com))

# Preface

Arad and Chen proved in [AC] and [AC1] that a Normalized Integral Table Algebra (fusion ring)  $(A, B)$  generated by a non-real faithful element  $b_3 \in B$  of degree three the non-identity elements of which have minimal degree 3 satisfies the condition  $b_3\bar{b}_3 = 1 + b_8$  where  $b_8 \in B$  is an element of degree 8. They also showed that the general case naturally splits into four main sub-cases:

- (1)  $(A, B) \cong_x (CH(PSL(2, 7), Irr(PSL(2, 7)))$ ;
- (2)  $b_3^2 = b_4 + b_5$  where  $b_4, b_5 \in B$  are elements of degrees 4 and 5;
- (3)  $b_3^2 = \bar{b}_3 + b_6$  where  $b_6 \in B$  is a non-real element of degree 6;
- (4)  $b_3^2 = c_3 + b_6$  where  $c_3, b_6 \in B$  are elements of degrees 3 and 6,  $c_3 \neq b_3, \bar{b}_3$ .

The cases (1), (3) and (4) are considered in Chap. 2. Chapter 3 deals with the case (2). Chapters 4 and 5 analyze the most complicated case—the third one. We developed new original methods for enumerating NITAs in the title. Using the developed technique we settled the above cases almost completely.

# Acknowledgements

Zvi Arad is sincerely grateful to Eastern Kentucky University (EKU) for inviting him to head the Wilson Endowment Chair in the Department of Mathematics and Statistics in Spring 2009. The warm hospitality and friendly atmosphere there enabled him to conduct the joint research of Chap. 3 in the spring semester of 2009.

Effie Cohen submitted Chaps. 4 and 5 of this book in partial fulfillment of the requirements for the Ph.D. degree in the Department of Mathematics, Bar-Ilan University, Ramat Gan, Israel, under the supervision of Prof. Zvi Arad and Prof. Malka Schaps.

The authors thank the anonymous referees for their helpful suggestions and comments.

The author thanks Mrs. Miriam Beller for her great efforts to print the first version of the book and for her helpful remarks.

# Contents

<b>1</b>	<b>Introduction</b> . . . . .	1
	References . . . . .	6
<b>2</b>	<b>Splitting of the Main Problem into Four Sub-cases</b> . . . . .	9
	2.1 Introduction . . . . .	9
	2.2 Two NITA Generated by a Non-real Element of Degree 3 not Derived from a Group and Lemmas . . . . .	11
	2.3 NITA Generated by $b_3$ and Satisfying $b_3^2 = b_4 + b_5$ . . . . .	23
	2.4 General Information on NITA Generated by $b_3$ and Satisfying $b_3^2 = \bar{b}_3 + b_6$ and $b_3^2 = c_3 + b_6$ . . . . .	52
	2.5 NITA Generated by $b_3$ Satisfying $b_3^2 = \bar{b}_3 + b_6$ and $b_6$ Nonreal and $b_{10} \in B$ is Real . . . . .	55
	2.6 NITA Generated by $b_3$ Satisfying $b_3^2 = c_3 + b_6$ , $c_3 \neq b_3, \bar{b}_3, b_6$ Non-real, $(b_3b_8, b_3b_8) = 4$ and $c_3^2 = r_3 + s_6$ . . . . .	82
	2.7 Structure of NITA Generated by $b_3$ and Satisfying $b_3^2 = c_3 + b_6$ , $c_3 \neq b_3, \bar{b}_3$ , $(b_3b_8, b_3b_8) = 3$ and $c_3$ Non-real . . . . .	83
	2.8 Structure of NITA Generated by $b_3$ and Satisfying $b_3^2 = c_3 + b_6$ , $c_3 \neq b_3, \bar{b}_3$ , $(b_3b_8, b_3b_8) = 3$ and $c_3$ Real . . . . .	108
	References . . . . .	150
<b>3</b>	<b>A Proof of a Non-existence of Sub-case (2)</b> . . . . .	151
	3.1 Introduction . . . . .	151
	3.2 Preliminary Results . . . . .	152
	3.3 Case $z = z_3$ . . . . .	158
	3.4 Cases $z = z_4, z = z_5, z = z_6, z = z_7$ , and $z = z_8$ . . . . .	174
	3.5 Case $z = z_9$ . . . . .	180
<b>4</b>	<b>Preliminary Classification of Sub-case (3)</b> . . . . .	187
	4.1 Introduction . . . . .	187
	4.2 Preliminary Results . . . . .	188
	4.3 Case $R_{15} = x_5 + x_{10}$ . . . . .	190
	4.4 Case $R_{15} = x_6 + x_9$ . . . . .	195

4.5	Case $R_{15} = x_7 + x_8$ . . . . .	209
4.6	Case $(b_3x_7, b_3x_7) = 3$ . . . . .	233
4.7	Case $b_3b_{10} = b_{15} + x_5 + y_5 + z_5$ . . . . .	238
<b>5</b>	<b>Finishing the Proofs of the Main Results</b> . . . . .	<b>243</b>
5.1	Introduction . . . . .	243
5.2	Proof of Theorem 5.1 . . . . .	247
5.3	Proof of Theorem 5.2 . . . . .	268
	References . . . . .	271
<b>Index</b>	. . . . .	<b>273</b>