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Editor

Random Matrices, Random Processes and Integrable Systems

Springer

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ISBN 978-1-4419-9513-1 e-ISBN 978-1-4419-9514-8
DOI 10.1007/978-1-4419-9514-8
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011925846

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

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This volume is intended as an introduction and overview of the three domains featured in the title, with emphasis on the remarkable links between them. It has its origins in an intensive series of advanced courses given by the authors at the Centre de Recherches Mathématiques in Montréal in the summer of 2005. Since then, it has grown considerably beyond the material originally presented, and been extended in content and scope. The original courses were interlaced in schedule with an extended research workshop, dealing with related topics, whose proceedings, in enhanced form, have since been published as a special issue of Journal of Physics A [1].

The participants at the two events included a large number of graduate students and postdoctoral fellows, as well as many of the researchers who had made pioneering contributions to these domains, The original content of the lecture series, given by several of the subject's leading contributors, has since been considerably developed and polished, to the point where it may be viewed as both a research survey and pedagogical monograph providing a panoramic view of this very rich and still rapidly developing domain.

In Part I, we have combined the introductory chapters by Pierre van Moerbeke, covering nearly all the topics occurring in the rest of the volume, together with the further, more detailed chapters linking random matrices and integrable systems, concerning mainly their joint work, written by Mark Adler.

Van Moerbeke's part consists of nine chapters. The first of these concerns random permutations, random words and percolation, linked with random partitions through the Robinson–Schensted–Knuth (RSK) correspondence. This includes an introduction to the Ulam problem (concerning the longest increasing subsequence of a random permutation), the Plancherel measure on partitions, the relation to non-intersecting random walkers, as well as queueing problems and polynuclear growth. He then discusses Toeplitz determinants, infinite wedge product representations, and the non-positive generalized probability measure of Borodin–Okounkov–Olshanski, expressed as a matrix integral over $\mathrm{U}(n)$. Several examples follow, including the Poissonized Plancherel measure, bringing in the use of Fredholm determinants of integral operators with a variety of kernels (Bessel, Charlier and Meixner polynomial type kernels), with applications to the distributions arising in the above random processes.

There follows a discussion of limit theorems, such as the Vershik–Kerov limiting shape of a random partition and the Tracy-Widom distribution for the longest increasing subsequences, as well as geometrically distributed percolation problems. Next, it is shown that the multiple (N-fold) integrals obtained upon reducing $\mathrm{U}(n)$ invariant Hermitian matrix models with arbitrary exponential trace invariant series deformations are tau functions of the KP (Kadomtsev–Petviashvili) integrable hierarchy, as well as satisfying the usual Toda lattice equations for varying N's, and the Hirota bilinear relations. Next, Virasoro algebra constraints are deduced for the multiple integrals defining the generalized β-type integrals. There is also a review of the basic finite N Hermitian matrix model results, including the form of the reduced integrals over the eigenvalues, computation of the determinantal form of the correlation functions in terms of suitable (Christoffel–Darboux) correlation kernels, and Fredholm integral expressions for the gap probabilities. The PDEs satisfied by these Fredholm integrals when the endpoints of the support intervals are varied are derived, for a variety of limiting kernels.

In the subsequent chapters, by Mark Adler, further links between random matrix theory and integrable models are developed, using vertex operator constructions. A soliton-like tau function is constructed using a Fredholm determinant and shown to satisfy Virasoro constraints. For gap probabilities, these are used as a vehicle to deduce differential equations that they must satisfy. There are also a number of lattice systems that are constructed using as phase space variables that are defined as N-fold matrix-like integrals. Exponential trace series deformations of 2-matrix integrals are shown to satisfy the equations of the 2-Toda hierarchy, and bilinear identities. Using the Virasoro constraints, PDEs for the gap probabilities are also deduced.

There follows a discussion of the Dyson diffusion process and its relation to random matrices, and chains of random matrices, as well as the bulk and edge scaling limits (sine and Airy processes). Equations are derived for these processes similar to those for the gap probabilities, with respect to the edges of the windows where the nonintersecting random paths are excluded, as well as asymptotic expansions. The GUE with external source and its relation to conditioned non-intersecting Brownian motion, as developed by Aptekarev, Bleher and Kuijlaars is developed, together with its relation to the Riemann–Hilbert problem for multiple orthogonal polynomials. (See Bleher's chapters for further details.) Finally there is a derivation of PDEs for the Pearcey process again through the introduction of integrable deformations of the measure.

The second part of this monograph is mainly concerned with the spectral theory of random matrices, but ideas and methods from the theory of integrable systems plays a prominent role. The introductory chapter, by Harold Widom, begins with a review of basic operator theory definitions and results that are required for applications to random matrices. Then derivations are given for spacing distributions between consecutive eigenvalues, in term of gap probabilities. Using operatorial methods, these are expressed as Fredholm determinants, in suitable scaling limits, of integral operators with integrable kernels of sine type (for the bulk) and Airy type (leading to theTracy-Widom distributions) for the edge. All three cases, orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) ensembles are treated. Finally, differential equations for distribution functions are derived, in particular, equations of Painlevé type.

In his series of chapters, Pavel Bleher gives a detailed survey of the use of Riemann–Hilbert methods for the study of the asymptotics of spectral distributions of random matrices. First, unitary ensembles with polynomial potentials are treated, and their relation to orthogonal polynomials and the associated Christoffel–Darboux kernels determining the correlation functions at finite N, as well as the string equations determining the recurrence coefficients in the asymptotic $1/N$ series. The Riemann–Hilbert characterization of the orthogonal polynomials is then introduced, and it is shown that the equilibrium measure is supported on a finite union of intervals coinciding with the cuts defining a hyperelliptic algebraic curve. In particular, the Wigner semicircle law is derived for the Gaussian case, and the case of quartic potentials is treated in detail. There follows the treatment of scaled large N asymptotics of orthogonal polynomials, using the Riemann–Hilbert approach and the method of nonlinear steepest descent of Deift, Kricherbauer, Mclaughlin, Venakides Zhou (DKMVZ). A solution of the model Riemann–Hilbert problem is given in terms of Riemann theta functions. The Airy parametrix at the end points is constructed to complete the study of uniform asymptotics, and an indication of the proof of sine kernel universality in the bulk and Airy kernel universality at the edge.

Next, the double scaling limit for the critical point of the even quartic potential case is treated, and its relation to the Hastings–Mcleod solution of the P_{II} Painlevé equation derived. More generally, the asymptotics of the free energy in the one cut case is studied (assuming a special regularity property of the potential). Analyticity in a parameter defining the potential is examined for the q-cut case for the density function and the free energy. The quartic deviation in the free energy from the Gaussian case is expressed in terms of the first two terms of the large N asymptotic expansion, and related to the Tracy–Widom distribution with an error estimate.

Random matrix models with exponential external coupling are then analyzed in terms of multiple orthogonal polynomials, with emphasis on the case of two distinct eigenvalues in the externally coupled matrix. Correlation functions, at finite N, are expressed in determinantal form in terms of an analog of the Christoffel–Darboux kernel. The higher rank Riemann–Hilbert characterization of such multiple orthogonal polynomials is given, and the differential equations and recursion relations for these expressed in terms of these matrices. The relation to Brownian bridges is explained, and, finally, the Pearcey kernel is derived in the double scaling limit using the nonlinear steepest descent method.

Alexander Its, in his series, focuses upon the large N asymptotics of the spectra of random matrices. The reduced N-fold integral representation of the partition function of Hermitian matrix models is recalled, and the expression of eigenvalue correlation functions in terms of the Christoffel–Darboux kernel of the associated orthogonal polynomials. An introduction to the Its–Kitaev–Fokas Riemann–Hilbert approach to orthogonal polynomials is given, and a proof is given of its unique solvability under certain assumptions.

The asymptotic analysis, along the lines of the DKMVZ method, is then recalled, based on the introduction of the g-function (essentially, the log-Coulomb energy of the equilibrium distribution) to transform the exact RH problem into one which, in leading order, has only jump discontinuities, and hence may be solved exactly. A detailed analysis is then given for the case of even quartic potentials. This suffices to deduce the sine kernel asymptotic form for the correlation kernel in the bulk. The construction of the Airy parametrix at the the end points of the cuts is then discussed and an asymptotic solution given with uniform estimates in each region.

Bertrand Eynard reviews the relationship between convergent matrix integrals and formal matrix integrals, serving as generating functions for the combinatorics of maps. Essentially, the formal model is obtained by treating the integrand as a perturbation series about the Gaussian measure and interchanging the orders of integration and summation, without regard to convergence. He indicates the derivation of the loop equations relating the expectation values of invariant polynomials of various degrees as Dyson–Schwinger equations. This is illustrated by various examples, including 1 and 2 matrix models, as well as chains of matrices, the $\mathrm{O}(N)$ chain model and certain statistical models such as the Potts model. He ends with a number of interesting conjectures about the solution of the loop equations. In the current literature, this has led to a very remarkable program in which many results on random matrices, solvable statistical models, combinatorial, topological and representation theoretical generating functions may be included as part of a general scheme, based on the properties of Riemann surfaces, and their deformations.

The contribution of Momar Dieng and Craig Tracy deals in part with the earliest appearance of random matrices, due to Wishart, in the theory of mutivariate statistics, the so-called Wishart distribution. They present Johnstone's result relating the largest component variance to the F_1 Tracy–Widom distribution, as well as Soshnikov's generalization to lower components. The expression of the F_1, F_2 and F_4 distributions for the edge distributions in GOE, GUE and GSE respectively in terms of the Hastings–McLeod solution of the P_{II} Painlevé equation is recalled. There follow a discussion of the recurrence relations of Dieng which enter in the computation of the mth largest eigenvalues in GOE and GSE.

A derivation of the Airy kernel for the edge scaling limit of GUE from Plancherel–Rotach asymptotics of the Hermite polynomials is given, as well as the P_I equation that determines the Fredholm determinant of the Airy kernel integral operator supported on a semi-infinite interval. The computation of the mth largest eigenvalue distribution in the GSE and GOE is indicated, together with an interlacing property identifying the first sequence with the even terms of the second. Finally, numerical results are given comparing these distributions with empirical data.

This volume is a masterly combined effort by several of the leading contributors to this remarkable domain, covering a range of topics and applications that no individual author could hope to encompass. For the reader wishing to have a representative view of the fascinating ongoing developments in this domain, as well as a reliable account of the, many results that are by now classically established, it should provide an excellent reference and entry point to the subject.

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