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The Parabolic Anderson Model

Random Walk in Random Potential

Birkhäuser

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Preface

This is a survey of the parabolic Anderson model (PAM), the Cauchy problem for the heat equation with random potential. This model and many variants and related models are studied for decades by many authors from various points of view and with various intentions. The PAM has rich and deep connections with questions on random motions in random potential, trapping of random paths, branching processes in random medium, spectra of random operators, Anderson localisation and more. We are mainly interested in the long-time behaviour of the solution of the PAM, which shows interesting behaviours like intermittency, mass concentration, ageing, Poisson process convergence of eigenvalues and eigenfunction localisation centres and more. Its mathematical investigations require combinations of tools from various parts of probability and analysis, like spectral theory of random operators, large deviations or extreme value statistics.

The research on the PAM and its variants has a high intensity since 1990 and continues to have. I felt that a survey text should be very useful, at a point at which most of the understanding of the basic model has been rigorously derived, and a variety of variants and additional features, like random environments and time-dependent potentials, and a number of related questions, like critically rescaled potentials, transition from concentrated to homogenised behaviour, spatial branching processes in random environment and Anderson localisation, receive an increasing interest.

The focus of this book is characterised by the intersection of a number of features, whose most important ones are the following:

- The solution to the PAM admits explicit formulas (Feynman-Kac formula and eigenvalue expansion),
- Its large-time behaviour can be investigated with the help of large-deviations theory,
- The arising variational formulas admit a deeper study and
- There are deep connections with the spectral theory for a prominent random Schrödinger operator, the Anderson operator.

All these aspects are more or less closely connected with the main property of the PAM, the intermittency, a concentration property of the main part of the solution in small islands. Intermittency is one of the leading ideas in this book and is almost ubiquitous.

For this reason, such important topics as directed polymers in random environment, PAM with drift and PAM with certain types of time-dependent potentials do not receive the space that they otherwise should have; they are just outside of the scope of this book. Actually, this text ends at a point where it is getting really interesting, as the stochastic heat equation and the KPZ equation come into play (however, the account on PAM with time-dependent potential given in Chap. 8 is quite comprehensive in a sense).

My intention was to provide a concise, but fairly complete, survey of the heuristic understanding of the PAM on one hand and of the state of the art of its mathematical treatment on the other hand. The goal is to quickly guide the reader to a good understanding of the essentials. I tried to give illuminating and nontechnical explanations, and I sometimes decided to provide simplified versions of the main theorems, many of which are embedded in the running text. Where some background is needed, the underlying theory is summarised in a most compact way, just at a length that is necessary to understand the fundamentals' all important connections.

There are a lot of precise references given to the first-hand literature, and many side remarks hint at deeper results and open problems that emanate from the material. I also found it useful to isolate the essentials of proof methods from the original papers, if time has shown that they are useful and can be adapted to several situations; not only Chap. 4 is devoted to this but also a number of remarks that are scattered over the text.

Originally, the text was meant to address experienced researchers, but in the course of writing, I felt that it would be desirable to attract also newcomers and young researchers to this field; therefore I added also explanations of terms, concepts, jargons and methods that are known to the community of the PAM and neighbouring fields. I hope that I found a style that is understandable and encouraging for all mathematically interested people from advanced undergraduates onwards.

In an appendix, I enumerated some open research directions that lie within the scope of this book or at its outer boundary. Certainly their choice relies on my personal taste, but I think that they each give rise to exciting new research, and hopefully they attract new people to the field.

Let me express my sincere thanks to my former PhD student Tilman Wolff, who helped me at an early stage in collecting some material, and to my current PhD student Franziska Flegel, who produced instrumental illustrations.

Berlin, Germany Wolfgang König
March 2016

Some Remarks on Notation

For describing asymptotic assertions, I will use the symbol '$\sim$' to denote asymptotic equivalence, i.e. that the quotient of the two sides converges to one, '$\asymp$' for asymptotic comparability, i.e. that the ratio stays bounded and bounded away from zero, and $\ll$ and $\gg$ for asymptotic negligibility, i.e. that the ratio vanishes, respectively, and diverges to ∞. Furthermore, I use the Landau symbols $o(a_n)$ for quantities whose ratio with a_n vanishes asymptotically and $O(a_n)$ for positive quantities whose ratio with a_n stays bounded as $n \to \infty$. When I do not want to specify the sense of the asymptotic approximation, then I use the symbol '$\approx$', but often I indicate in words what I would like to mean by that. For expressing convergence, I often use the arrow $\to$, or $\stackrel{t\to\infty}{\longrightarrow}$, if I want to indicate the limiting parameter. Convergence of random variables in distribution or weak convergence of measures is written using $\Longrightarrow$.

For integrals and inner products both on $\mathbb{R}^d$ and on $\mathbb{Z}^d$, I use the brackets $\langle\cdot,\cdot\rangle$, e.g. $\langle\mu,f\rangle = \langle f,\mu\rangle = \int f\,\mathrm{d}\mu$ for functions f and measures μ or $\langle f,g\rangle = \int f(x)g(x)\,\mathrm{d}x$ for two functions f and g, which I sometimes also abbreviate as $\int fg$ if the domain is $\mathbb{R}^d$ and $\langle f,g\rangle = \sum_{z\in\mathbb{Z}^d} f(z)g(z)$ if it is $\mathbb{Z}^d$. For $p \in [1,\infty)$ and $B \subset \mathbb{Z}^d$, we denote by $\ell^p(B)$ the vector space of functions $f\colon B \to \mathbb{R}$ such that $\|f\|_p^p = \sum_{z\in B} |f(z)|^p$ is finite and $\|f\|_p$ is the norm of f.

For the parameter of some frequently used functions or processes, I use both the index notation and the bracket notation, depending on how much space the parameter requires. Hence a scale function $\alpha(t)$ may be also written α_t, and the simple random walk at time t is denoted both by X_t and by $X(t)$. Likewise, I write both $\mathbb{1}_A$ and $\mathbb{1}A$ for the indicator function on an event A.

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