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Quantum Independent Increment Processes I

From Classical Probability to
Quantum Stochastic Calculus

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Dedicated to the memory of Paul-André Meyer

Preface

This volume is the first of two volumes containing the lectures given at the School “Quantum Independent Increment Processes: Structure and Applications to Physics”. This school was held at the Alfred Krupp Wissenschafts}kolleg in Greifswald during the period March 9 – 22, 2003. We thank the lecturers for all the hard work they accomplished. Their lectures give an introduction to current research in their domains that is accessible to Ph. D. students. We hope that the two volumes will help to bring researchers from the areas of classical and quantum probability, operator algebras and mathematical physics together and contribute to developing the subject of quantum independent increment processes.

We are greatly indebted to the Volkswagen Foundation for their financial support, without which the school would not have been possible.

Special thanks go to Mrs. Zeidler who helped with the preparation and organisation of the school and who took care of the logistics.

Finally, we would like to thank the students for coming to Greifswald and helping to make the school a success.

Greifswald,
February 2005

Michael Schürmann
Uwe Franz

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Introduction

Random variables and stochastic processes are used to describe the behaviour of systems in a vast range of areas including statistics, finance, actuarial mathematics and computer science, as well as engineering, biology and physics. Due to an unavoidable lack of information about the state of the system concerned at a given moment in time, it is often impossible to predict these fluctuations with certainty — think of meteorology, for example. The unpredictable behaviour may be due to more fundamental reasons, as is the case in quantum mechanics. Here Heisenberg uncertainty limits the accuracy of simultaneous predictions of so-called complementary observables such as the position and momentum of a particle.

If the random fluctuations do not depend on time or position, then they should be described by stochastic processes which are homogeneous in space and time. In Euclidean space this leads to the important class of stochastic processes called Lévy processes, which have independent and stationary increments ([Lév65]). These processes have been attracting increasing interest over the last decade or so (see [Sko91], [Ber96], [Sat99], [BNMR01] and [App04]).

In quantum mechanics complete knowledge of the state is still insufficient to predict with certainty the outcomes of all possible measurements. Therefore its statistical interpretation has to be an essential part of the theory. Quantum probability starts from von Neumann's formulation of quantum mechanics ([vN96]) and studies quantum theory from a probabilistic point of view. Two key papers in the field are [AFL82] and [HP84].

A typical situation where quantum noise plays a role is in the description of a 'small' quantum system interacting with its 'large' environment. The state of the environment, also called heat bath or reservoir, cannot be measured or controlled completely. However it is reasonable to assume, at least as a first approximation, that it is homogeneous in time and space, and that the influence of the system on the heat bath can be neglected.

In concrete models the heat bath is generally described by a Fock space. The Hilbert space for the joint 'system plus heat bath' is then the tensor product of the Hilbert space representing the system with this Fock space. The

separate time evolutions of the heat bath and system are coupled through their interaction to yield a unitary evolution of the system plus heat bath which is a cocycle with respect to the free evolution of the heat bath. Thus, through interaction (in other words, considered as an open system), the evolution of the system becomes non-unitary. In the Heisenberg picture this is given by a quantum dynamical semigroup, that is a one-parameter semigroup of completely positive maps (rather than $*$ -automorphisms) on the system observables, see *Quantum Markov processes and applications to physics*, by Burkhard Kümmerer, in volume two of these notes. In the physics literature the dual Schrödinger picture is usually preferred; this is adopted in the influential monograph [Dav76].

Fock spaces arose in quantum field theory and in representation theory as continuous tensor products. The close connection between independent increment processes on the one hand, and current representations and Fock space on the other, was realised in the late sixties and early seventies ([Ara70], [PS72] and [Gui72], see also the survey article [Str00]). The development of a quantum stochastic calculus was a natural sequel to this discovery. This calculus involves the integration of operator ‘processes’, that is time-indexed families of operators adapted to a Fock-space filtration, with respect to the so-called creation, preservation and annihilation processes. It is modelled on the Itô integral, but in fact may be based on the nonadapted stochastic calculus of Hitsuda and Skorohod, see part three of this volume, *Quantum stochastic analysis — an introduction*, by Martin Lindsay. The relationship between classical and quantum stochastic calculus is also the subject of the final lecture of part one, *Lévy processes in Euclidean spaces and groups*, by David Applebaum.

Part four of this volume, *Dilations, cocycles and product systems* by Rajarama Bhat, concerns the relation between the unitary evolution of the *closed* system plus heat bath and the quantum dynamical semigroup which is the evolution of the *open* system itself. It addresses the question of which unitary evolutions correspond to a given quantum dynamical semigroup.

Formally, quantum groups arise from groups in a similar way to how quantum probability arises from classical probability, and to how C^* -algebra theory is now commonly viewed as noncommutative topology. Namely, one casts the axioms for a group (or probability space, or topological space) in terms of the appropriate class of functions on the group (respectively, probability, or topological space). This yields a commutative algebra with extra structure, and the quantum object is then defined by dropping the commutativity axiom. This procedure has been successfully applied to differential geometry ([Con94]).

For example taking the algebra of representative functions on a group (i.e. those functions which can be written as matrix elements of a finite-dimensional representation of the group), one obtains the axioms of a commutative Hopf algebra ([Swe69]). Dropping commutativity, one arrives at one definition of a Hopf algebra. At least in finite dimension, the Hopf algebra axioms give a satisfactory definition of a (finite) quantum group.

Similarly the essentially bounded measurable functions on a probability space, with functions equal almost everywhere identified, form a commutative von Neumann algebra on which the expectation functional yields a state which is faithful and normal. Conversely, every commutative von Neumann algebra with faithful normal state is isomorphic to such an algebra of (measure equivalence classes of) random variables on a probability space with state given by the expectation functional.

Thus the axioms depend on the choice of functions. For example *all* functions on a group form a Hopf algebra only if the group is finite. The guiding principle for finding the ‘right’ set of axioms is that it should yield a rich theory which incorporates a good measure of the classical theory. In the case of quantum probability there is a straightforward choice. A unital $*$ -algebra with a state is called an algebraic noncommutative probability space, and simply a noncommutative probability space when the algebra is a von Neumann algebra and the state is normal. In the latter case the state is often, but not always, assumed to be faithful. In fact recent progress in the understanding of noncommutative stochastic independence has benefitted from a loosening of the axioms to allow noninvolutive algebras, see *Lévy processes on quantum groups and dual groups*, by Uwe Franz in volume two of these notes.

In what is now known as topological quantum group theory, the search for the ‘right’ foundations has a long history. Only recently have Kustermans and Vaes obtained a relatively simple set of axioms that is both rich enough to contain all the examples one would want to consider as quantum groups whilst still having a satisfactory duality theory, see part two of this volume, *Locally compact quantum groups*, by Johan Kustermans.

References

- [AFL82] L. Accardi, A. Frigerio, and J.T. Lewis. Quantum stochastic processes. *Publ. RIMS*, 18:97–133, 1982.
- [App04] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, 2004.
- [Ara70] H. Araki. Factorizable representations of current algebra. *Publ. RIMS Kyoto University*, 5:361–422, 1970.
- [Ber96] J. Bertoin. *Lévy Processes*. Cambridge University Press, Cambridge, 1996.
- [BNMR01] O. E. Barndorff-Nielsen, T. Mikosch, and S. I. Resnick, editors. *Lévy Processes*. Birkhäuser Boston Inc., Boston, MA, 2001. Theory and applications.
- [Con94] A. Connes. *Noncommutative Geometry*. Academic Press, San Diego, 1994.
- [Dav76] E. B. Davies. *Quantum Theory of open Systems*. Academic Press, London, 1976.
- [Gui72] A. Guichardet. *Symmetric Hilbert spaces and Related Topics*, volume 261 of *Lecture Notes in Math*. Springer-Verlag, Berlin, 1972.
- [HP84] R. L. Hudson and K. R. Parthasarathy. Quantum Ito’s formula and stochastic evolutions. *Comm. Math. Phys.*, 93(3):301–323, 1984.
- [Lév65] Paul Lévy. *Processus Stochastiques et Mowement Brownien*. Gauthier-Villars & Cie, Paris, 1965.

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- [PS72] K.R. Parthasarathy and K. Schmidt. *Positive Definite Kernels, Continuous Tensor Products, and Central Limit Theorems of Probability Theory*, volume 272 of *Lecture Notes in Math*. Springer-Verlag, Berlin, 1972.
- [Sat99] Ken-iti Sato. *Lévy processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.
- [Sko91] A. V. Skorohod. *Random Processes with Independent Increments*. Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the second Russian edition by P. V. Malyshev.
- [Str00] R. F. Streater. Classical and quantum probability. *J. Math. Phys.*, 41(6):3556–3603, 2000.
- [Swe69] M. E. Sweedler. *Hopf Algebras*. Benjamin, New York, 1969.
- [vN96] J. von Neumann. *Mathematical foundations of quantum mechanics*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, 1996. Translated from the German, with preface by R.T. Beyer.